ON $w^*$-SEQUENTIAL CONVERGENCE AND QUASI-REFLEXIVITY

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This paper characterizes quasi-reflexive Banach spaces in terms of certain properties of the $w^*$-sequential closure of subspaces. A real Banach space $X$ is quasi-reflexive of order $n$, where $n$ is a nonnegative integer, if and only if the canonical image $J_X X$ of $X$ has algebraic codimension $n$ in the second dual space $X^{**}$. The space $X$ will be said to have property $P_n$ if and only if every norm-closed subspace $S$ of $X^*$ has codimension $\leq n$ in its $w^*$-sequential closure $K_X(S)$. By use of a theorem of Singer it is proved that $X$ is quasi-reflexive of order $\leq n$ if and only if every norm-closed separable subspace of $X$ has property $P_n$. A certain parameter $Q^{\infty}(X)$ is shown to have value 1 if $X$ has property $P_n$ and to be infinite if $X$ does not have $P_n$. The space $X$ has $P_0$ if and only if $w$-sequential convergence and $w^*$-sequential convergence coincide in $X^*$. These results generalize a theorem of Fleming, Retherford, and the author.

2. If $X$ is a real Banach space, $S$ a subspace of $X^*$, and $K_X(S)$ the $w^*$-sequential closure of $S$ in $X^*$, then $K_X(S)$ is a Banach space under the norm $\varphi_S$ defined by

$$\varphi_S(f) = \inf \left\{ \sup_{x \in \omega} \| f_n \| : \{f_n\} \subset S, f_n \overset{w^*}{\longrightarrow} f \right\}$$

for $f \in K_X(S)$ [5]. If $S \subseteq T \subseteq K_X(S)$, let

$$C_X(S, T) = \sup \{ \varphi_S(f) : f \in T, \| f \| \leq 1 \} .$$

Thus, $K_X(S)$ is norm-closed in $(X^*, \| \cdot \|)$ if and only if $C_X(S, K_X(S))$ is finite [5]. For each integer $n \geq 0$ let $\mathcal{T}_n(S)$ be the family of all subspaces $T$ of $X^*$ such that $S \subseteq T \subseteq K_X(S)$ and such that $K_X(S)$ is the algebraic direct sum of $T$ and a subspace of dimension $\leq n$. Let

$$C^{\infty}_n(S) = \inf \{ C_X(S, T) : T \in \mathcal{T}_n(S) \} ,$$

and let

$$Q^{\infty}(X) = \sup \{ C^{\infty}_n(S) : S \text{ a subspace of } X^* \} .$$

It will be said that $X$ has property $P_n$ if and only if $S \in \mathcal{T}_n(S)$ for every norm-closed subspace $S$ of $(X^*, \| \cdot \|)$.

3. Theorem 1. Let $X$ be a real Banach space and $n$ a non-
negative integer. If $X$ has property $P_n$, then $Q^n(X) = 1$. If $X$ does not have property $P_n$, then $Q^n(X) = \infty$.

**Proof.** If $X$ has property $P_n$ and $S_i$ is a norm-closed subspace of $X^*$, then $S_i \in \mathcal{T}_n(S_i)$ and hence $C^\omega_x(S_i) = 1$. If $S$ is an arbitrary subspace of $X^*$ and $S_i$ the norm-closure of $S$, then $C^\omega_x(S) = C^\omega_x(S_i)$ and therefore $Q^n(x)(X) = 1$.

If $X$ does not have property $P_n$, then $X^*$ has a norm-closed subspace $S$ such that $K_x(S)$ contains an $(n+1)$-dimensional subspace $V$ such that $S \cap V = \{0\}$. Now $V$ has a basis $\{e_1, \ldots, e_{n+1}\}$ of vectors with $\|e_i\| = 1$, and there exist $F_1, \ldots, F_{n+1} \in X^{**}$ such that for each $j \in \{1, \ldots, n+1\}$, $F_j(f) = 0$ for every $f \in S$ and $F_j(f_i) = \delta_{ij}$ for each $i \in \{1, \ldots, n+1\}$ [7, p. 186]. Let $\alpha = \max \{\|F_j\| : 1 \leq j \leq n+1\}$. Further, there exist vectors $x_1, \ldots, x_{n+1} \in X$ such that $f_i(x_j) = \delta_{ij}$ for $1 \leq i, j \leq n+1$ [7, p. 138].

Since $f_1, \ldots, f_{n+1} \in K_x(S)$, the restrictions of $J_x x_1, \ldots, J_x x_{n+1}$ to $S$ must be linearly independent on $S$, and hence for each

\[i \in \{1, \ldots, n+1\}\]

there exists $g_i \in S$ such that $g_i(x_j) = \delta_{ij}$ for each $j$ [7, p. 138]. Now for each $i = 1, \ldots, n+1$ there is a sequence $\{p_{ih}\} \subset S$ such that

\[p_{ih} \underset{h}{\rightharpoonup} f_i\]

The sequence $\{p_{ih}\}$ may be chosen so that

\[\|p_{ih}(x_j) - \delta_{ij}\| < \frac{2^{-h}}{(n+1)\|g_j\|}\]

for each $j$. If we let $F_{ih} = p_{ih} + \sum_{j \neq i}^{n+1} \left[\delta_{ij} - p_{ih}(x_j)\right]g_j$, then $f_{ih}(x_j) = \delta_{ij}$ for all $i, h, j$, and $\|F_{ih} - p_{ih}\| < 2^{-h}$, so that $F_{ih} \underset{h}{\rightharpoonup} f_i$; clearly $\{f_{ih}\} \subset S$.

For each $i \in \{1, \ldots, n+1\}$ and $h \in \omega$, let $g_{ih} = f_{ih} - f_i$. Thus $g_{ih}(x_j) = 0$ and $F_j(g_{ih}) = -\delta_{ij}$ for all $i, h, j$, and $g_{ih} \underset{h}{\rightharpoonup} 0$ for each $i$.

Generalizing a method of Fleming [3], for each positive number $N$ we let $R_N$ be the linear span and $S_N$ the norm-closed linear span of $\{f_{ih} + Ng_{ih} : 1 \leq i \leq n+1; h \in \omega\}$. Note that for each

\[i \in \{1, \ldots, n+1\}, f_{ih} + Ng_{ih} \underset{h}{\rightharpoonup} f_i \]

thus $V \subseteq K_x(R_N)$. Now let $f$ be a nonzero element of $V$ and $\{v_m\}$ a sequence in $R_N$ such that $v_m \underset{w^*}{\rightharpoonup} f$. Clearly $f$ has the form

\[f = \sum_{i=1}^{n+1} \alpha_i f_i\]
and each \( v_m \) has the form
\[
v_m = \sum_{i=1}^{n+1} \sum_{h=1}^m \alpha_{m,ih} (f_{ih} + Ng_{ih}) .
\]
For every \( j \in \{1, \ldots, n+1\} \),
\[
\alpha_j = f(x_j) = \lim_m v_m(x_j) = \lim_m \sum_{h=1}^m \alpha_{m,ih} ,
\]
and since \( F_j(f_{ih} + Ng_{ih}) = -N\delta_{ij} \), it follows that
\[
F_j(v_m) = -N \sum_{h=1}^m \alpha_{m,ih} .
\]
Thus \( \lim_m F_j(v_m) \) exists and is equal to \(-N\alpha_j \). Now
\[
\| v_m \| \geq \frac{|F_j(v_m)|}{\| F_j \|} ,
\]
and hence \( \lim \inf_m \| v_m \| \geq N |\alpha_j| /\| F_j \| \geq N |\alpha_j| /\alpha \). Since \( j \) is arbitrary, \( \lim \inf_m \| v_m \| \geq (N/\alpha) \max |\alpha_j| \). From the definition of \( \varphi_{R(n)} \), it follows that \( \varphi_{R(n)}(f) \geq N/\alpha \max_j |\alpha_j| \geq N \| f \| /\alpha(n+1) \). If \( T \in \mathcal{T}(S_n) \), then \( T \) must contain some nonzero \( f \in V \) since \( V \) is \((n+1)\)-dimensional, and hence \( C_X(S_n, T) \geq N/\alpha(n+1) \). Therefore \( C_X^n(S_n) \geq N/\alpha(n+1) \). Since \( N \) is arbitrary and \( \alpha(n+1) \) is independent of \( N \), it follows that \( Q^n(X) = +\infty \).

**Theorem 2.** Let \( X \) be a real Banach space and \( n \) a nonnegative integer. If \( X \) is quasi-reflexive of order \( \leq n \), then \( X \) has property \( P_n \). If \( X \) is separable and has property \( P_n \), then \( X \) is quasi-reflexive of order \( \leq n \).

**Proof.** If \( X \) is quasi-reflexive of order \( m \leq n \) and \( S \) is a norm-closed subspace of \( X^* \), then it can be seen from the proofs of Theorems 5 and 6 of [4] that \( K_x(S) \) is the direct sum of \( S \) with a subspace of \( X^* \) of dimension \( \leq m \). Hence \( S \in \mathcal{T}(S) \), and consequently \( X \) has property \( P_n \).

On the other hand, let \( X \) be separable and suppose that \( X \) has property \( P_n \). Let \( F_1, \ldots, F_{n+1} \) be linearly independent elements of \( X^{**} \) and \( S = \bigcap_{i=1}^{n+1} \{ f \in X^* : F_i(f) = 0 \} \). Thus \( S \) is a norm-closed subspace of \( X^* \) of codimension \( n+1 \), and hence, by property \( P_n \), \( K_x(S) \) has codimension \( m \) for some \( m \in \{1, \ldots, n+1\} \). There exists a subspace \( U \) of \( X^* \) of codimension 1 such that \( K_x(S) \subseteq U \). Thus \( U = S \oplus V \) for some subspace \( V \) of \( X^* \) of dimension \( n \). Now \( U = K_x(U) \). Indeed, if \( \{g_i\} \subset U \) and \( g_i \xrightarrow{w^*} g \), and if \( P \) is the projection of \( U \) onto
V along S, then as in the proof of Theorem 5 of [4], P is bounded and \{a_i\} is bounded, so that \{Pg_i\} is bounded and hence has a subsequence \{Pg_{ij}\} which converges inner m to some v in the finite-dimensional subspace V. It follows that \( g_i - Pg_i \xrightarrow{w^*} g - v \in K_x(S) \) and hence that \( g \in K_x(S) + V = U \).

Since \( U = K_x(U) \) and \( X \) is separable, it follows, by an argument involving the \( bw^* \)-topology of \( X^* \) [3], that \( U \) is \( w^* \)-closed. If \( n = 0 \), let \( F = F_1 \). If \( n > 0 \), there exist linearly independent vectors \( f_1, \ldots, f_n \) spanning \( V \), and there exist scalars \( \alpha_1, \ldots, \alpha_n, \) not all of which are zero, such that \( \sum_{i=1}^{n+1} \alpha_i F_i(f_j) = 0 \) for \( 1 \leq j \leq n \); indeed, the \((n + 1)\) vectors

\[
\begin{bmatrix}
F_i(f_1) \\
\vdots \\
F_i(f_n)
\end{bmatrix} \\
(i = 1, \ldots, n + 1)
\]

in \( n \)-dimensional Euclidean space must be linearly dependent. Let \( F = \sum_{i=1}^{n+1} \alpha_i F_i \). Thus, for \( n \geq 0 \), \( F \neq 0 \) and \( U = \{ f \in X^* : F(f) = 0 \} \).

Since \( U \) is \( w^* \)-closed, \( F \) is \( w^* \)-continuous on \( X^* \) [7, p. 139], and hence \( F \in J_xX \). Thus every \((n + 1)\)-dimensional subspace of \( X^{**} \) contains a nonzero element of \( J_xX \), which means that \( X \) is quasi-reflexive of order \( \leq n \).

REMARK. Theorems 1 and 2 contain a generalization of Fleming's theorem [3] that if \( X \) is a separable Banach space, then \( X \) is reflexive if and only if \( Q(X) = 1 \). The following theorem generalizes a theorem of [3] and [4].

**Theorem 3.** A real Banach space \( X \) is quasi-reflexive of order \( \leq n \), where \( n \geq 0 \), if and only if every norm-closed separable subspace \( Y \) of \( X \) has the property \( P_n \).

**Proof.** If \( X \) is quasi-reflexive of order \( \leq n \) and \( Y \) is a closed subspace of \( X \), then \( Y \) is also quasi-reflexive of order \( \leq n \) [1] and hence \( Y \) has property \( P_n \) by Theorem 2. Conversely, if every norm-closed separable subspace \( Y \) of \( X \) has property \( P_n \), then every such \( Y \) is quasi-reflexive of order \( \leq n \) by Theorem 2, and hence \( X \) is quasi-reflexive of order \( \leq n \) by a theorem of Singer [6].

**Remark.** In Theorem 3 the word "separable" can be deleted. By virtue of Theorem 1, Theorem 3 is also true if "property \( P_n \)" is replaced with "property that \( Q^{(n)}(Y) = 1 \)". Since a space \( X \) is quasi-reflexive of order \( n \) if and only if \( X \) is quasi-reflexive of order \( \leq n \) but not of order \( \leq (n - 1) \), Theorem 3 can easily be rewarded in such
a way as to give a necessary and sufficient condition that $X$ be quasi-reflexive of order exactly $n$.

4. **Theorem 4.** If $X$ is a real Banach space, then $Q(0)(X) = 1$ if and only if $w$-sequential convergence and $w^*$-sequential convergence coincide in $X^*$.

**Proof.** Suppose the two kinds of sequential convergence coincide and $S$ is a subspace of $X^*$. If $\{f_i\} \subset S$ and $f_i \rightharpoonup f$, then $f_i \rightharpoonup f$ and hence some sequence of averages far out in $\{f_i\}$ converges in norm to $f$ [2, p. 40]; thus $f \in S$, the norm-closure of $S$, and hence $\varphi_S(f) = \|f\|$. Therefore, $C^*(S) = 1$ and $Q(0)(X) = 1$.

Conversely, suppose there are a sequence $\{f_i\}$ in $X^*$ and an $f_0 \in X^*$ such that $f_i \rightharpoonup f_0$ but $f_i \not\rightharpoonup f_0$. Then there exists an $F \in X^{**}$ such that $F(f_i) \neq F(f_0)$. The sequence $\{F(f_i)\}$ is bounded and hence contains a subsequence $\{F(f_{i_j})\}$ such that the limit $\alpha = \lim\inf F(f_{i_j})$ exists, but $\alpha \neq F(f_0)$. Since $F \neq 0$, there exists $g \in X^*$ such that $F(g) \neq 0$. Let $g_j = f_{i_j} - (F(f_{i_j})/F(g))g$ for each $j \in \omega$ and

$$g_0 = f_0 - \frac{\alpha}{F(g)}g.$$ 

Then $F(g_j) = 0$ for each $j \in \omega$, but $F(g_0) \neq 0$. For every $x \in X$,

$$g_j(x) \rightharpoonup f_0(x) - \frac{\alpha}{F(g)} g(x) = g_0(x),$$

so that $g_j \rightharpoonup g_0$. Let $S$ be the norm-closed subspace of $X^*$ spanned by $\{g_j : j \in \omega\}$. Then $g_0 \in K_X(S)$, but $g_0 \not\in S$, since $F(g_0) \neq 0$ whereas $F(f) = 0$ for all $f \in S$. Thus $S \not\in \mathcal{I}_\sigma(S)$, and hence $X$ does not have property $P_0$, so that $Q(0)(X) = \infty$ by Theorem 1.

**References**


Received June 15, 1965. Supported by National Science Foundation Grant GP-2179.

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