

# Pacific Journal of Mathematics

## **DISJOINT BASIC SUBGROUPS**

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## DISJOINT BASIC SUBGROUPS

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**This paper arose from consideration of the following questions. First, what characterizes those infinite Abelian reduced  $p$ -groups which possess disjoint basic subgroups? Second, are there properties that a basic subgroup must possess to insure the existence of a basic subgroup disjoint from it?**

**We show that a necessary and sufficient condition for an infinite Abelian reduced  $p$ -group  $G$  to contain disjoint basic subgroups is that  $|G| = \text{final rank } G$ . Furthermore, in such a group a necessary and sufficient condition for a basic subgroup  $B$  to have a basic subgroup disjoint from it is that  $B$  is a lower basic subgroup of  $G$ .**

Throughout this paper the word "group" will mean "Abelian group" and the notation used will be that of L. Fuchs in [1] with the exception that  $A \oplus B$  will denote the direct sum of the groups  $A$  and  $B$ , and  $A + B$  will be the, not necessarily direct, sum.

We will use the following theorem:

**THEOREM A.** (*Mitchell and Mitchell in [4]*) *Let  $G$  be an infinite reduced Abelian  $p$ -group and  $B$  a basic subgroup of  $G$  such that  $G/B = \sum_{\alpha \in I} (G_\alpha/B)$  where  $G_\alpha/B \cong Z(p^\infty)$  for all  $\alpha \in I$ . Then  $G = H \oplus K$  and  $B = H \oplus L$  where  $L$  is a basic subgroup of  $K$  such that  $r(K/L) = r(G/B) = |I|$  and  $|K| = \text{maximum } \{\aleph_0, |I|\}$ .*

We first prove the following lemmas:

**LEMMA 1.** *Let  $G$  be a  $p$ -group without elements of infinite height, and such that final rank  $(G) = |G|$ . Let  $B$  be a lower basic subgroup of  $G$ . Then there exists a basic subgroup,  $B'$ , of  $G$  which is disjoint from  $B$ .*

*Proof.* Let  $B = \sum_{\alpha \in I} \langle y_\alpha \rangle$ , and let  $G/B = \sum_{\beta \in J} C_\beta$  where each  $C_\beta \cong Z(p^\infty)$ . Let  $\{\{y_\alpha \mid \alpha \in I\}, \{c_{\beta,n} \mid \beta \in J, n = 1, 2, \dots\}\}$  be a quasibasis for  $G$ . Since  $B$  is a lower basic subgroup of  $G$  and final rank  $(G) = |G|$ , we have  $|J| = r(G/B) = |G| \geq |B| \geq |I|$ . If indeed we have  $|J| > |I|$  a pure subgroup  $H$  of  $G$  can be chosen such that  $H \supset B$ ,  $H^1 = 0$ , and final rank  $(H) = |H| = |I|$ . We can then prove that there is a basic subgroup  $B'$  of  $H$  which is disjoint from  $B$  and  $H$  being pure in  $G$  will insure  $B'$  is a basic subgroup of  $H$ . Thus it suffices to complete the proof when  $|J| = |I|$  and we will assume moreover  $I = J$ .

Now for each  $\alpha \in I$  choose from  $\{c_{\alpha, n}\}_{n=1}^{\infty}$  the element  $c_{\alpha, 2E(y_\alpha)}$ . Define  $B' = \langle \{y_\alpha - p^{E(y_\alpha)} c_{\alpha, 2E(y_\alpha)}\}_{\alpha \in I} \rangle$ . We now claim that  $B'$  is the desired basic subgroup of  $G$  which is disjoint from  $B$ . To see this we prove the following:

(i) First claim that

$$B' = \sum_{\alpha \in I} \langle y_\alpha - p^{E(y_\alpha)} c_{\alpha, 2E(y_\alpha)} \rangle .$$

Suppose that

$$0 = \sum_{i=1}^n a_i (y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})}) ,$$

then

$$\sum_{i=1}^n a_i y_{\alpha_i} = \sum_{i=1}^n a_i p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})} .$$

Since the

$$E(y_\alpha - p^{E(y_\alpha)} c_{\alpha, 2E(y_\alpha)}) = E(y_\alpha) ,$$

we would be finished if  $\sum_{i=1}^n a_i y_{\alpha_i} = 0$ , so we can assume that  $\sum_{i=1}^n a_i y_{\alpha_i} \neq 0$ , and without loss of generality  $a_i$  is not equal to 0 mod  $o(y_{\alpha_i})$ . Now the height  $h_{\mathfrak{a}}(\sum_{i=1}^n a_i y_{\alpha_i}) = r$ , where  $r$  is the largest positive integer such that  $p^r$  divides each  $a_i$ , since  $\sum_{i=1}^n a_i y_{\alpha_i}$  is an element of  $B = \sum_{\alpha \in I} \langle y_\alpha \rangle$ . But,

$$h_{\mathfrak{a}}\left(\sum_{i=1}^n a_i p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})}\right) \geq \underset{i=1, 2, \dots, n}{\text{minimum}} \{r + E(y_{\alpha_i})\} > r ,$$

which contradicts the equality

$$\sum_{i=1}^n a_i y_{\alpha_i} = \sum_{i=1}^n a_i p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})} .$$

Therefore, we must have that

$$B' = \sum_{\alpha \in I} \langle y_\alpha - p^{E(y_\alpha)} c_{\alpha, 2E(y_\alpha)} \rangle .$$

(ii) Next we will show that  $B'$  is a pure subgroup of  $G$ . Let  $z \in B'[p]$ , and write

$$z = \sum_{i=1}^n a_i p^{E(y_{\alpha_i})-1} (y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})}) ,$$

where each  $a_i$  is relatively prime to  $p$ . Now we have that

$$\begin{aligned} h_{B'}(z) &= \underset{i=1, \dots, n}{\text{minimum}} \{h_{B'}[a_i p^{E(y_{\alpha_i})-1} (y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})})]\} \\ &= \underset{i=1, \dots, n}{\text{minimum}} \{E(y_{\alpha_i}) - 1\} . \end{aligned}$$

But

$$h_G(z) = h_G \left[ \left( \sum_{i=1}^n a_i p^{E(y_{\alpha_i})-1} y_{\alpha_i} \right) - \left( \sum_{i=1}^n a_i p^{2E(y_{\alpha_i})-1} c_{\alpha_i, 2E(y_{\alpha_i})} \right) \right],$$

and

$$\begin{aligned} h_G \left( \sum_{i=1}^n a_i p^{2E(y_{\alpha_i})-1} c_{\alpha_i, 2E(y_{\alpha_i})} \right) &\geq \text{minimum} \{2E(y_{\alpha_i}) - 1\} \\ &> \text{minimum}_{i=1, \dots, n} \{E(y_{\alpha_i}) - 1\}, \end{aligned}$$

and

$$h_G \left( \sum_{i=1}^n a_i p^{E(y_{\alpha_i})-1} y_{\alpha_i} \right) = \text{minimum}_{i=1, \dots, n} \{E(y_{\alpha_i}) - 1\}.$$

Since the height of the sum of two elements with different heights is just the height of the smaller, we have that

$$h_G(z) = \text{minimum}_{i=1, \dots, n} \{E(y_{\alpha_i}) - 1\}.$$

Therefore  $h_G(z) = h_{B'}(z)$  for each element  $z \in B'[p]$ , and hence by Lemma 7, page 20, in [3], we have that  $B'$  is pure.

(iii) We will now complete the proof that  $B'$  is a basic subgroup of  $G$  by showing that  $B'$  cannot be extended to a larger pure direct sum of cyclic groups. Suppose that  $B' \oplus \langle z \rangle$  is a pure direct sum of cyclic groups. Since  $\{\{y_\alpha \mid \alpha \in I\}, \{c_{\alpha, n} \mid \alpha \in I, n = 1, 2, \dots\}\}$  is a quasi-basis for  $G$ , we can write

$$z = \sum_{i=1}^n a_i y_{\alpha_i} + \sum_{i=1}^k s_j c_{\alpha_j, r_j}.$$

Now we also know that  $c_{\alpha_j, r_j} = p c_{\alpha_j, r_j+1} + b_j$  where  $b_j \in B = \sum_{\alpha \in I} \langle y_\alpha \rangle$ , hence we can write

$$z = \sum_{j=1}^k s_j p c_{\alpha_j, r_j+1} + \sum_{i=1}^m t_i y_{\alpha_i}.$$

Now write

$$z = \sum_{j=1}^k s_j p c_{\alpha_j, r_j+1} + \sum_{i=1}^{m_1} t'_i y_{\alpha_i} + \sum_{i=1}^{m_2} t''_i y_{\alpha_i}$$

where each  $t'_i$  is divisible by  $p$ , and each  $t''_i$ , is relatively prime to  $p$ . We are assuming that  $H = B' \oplus \langle z \rangle$  is pure in  $G$ , and hence,  $h_G(z) = h_H(z) = 0$ , and since  $H$  is a direct sum of cyclic groups we must also have that  $h_H(b' + z) = 0$ , for any  $b' \in B$ . Consider the following element of  $B'$ ,

$$\sum_{i=1}^{m_2} t''_i (y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})}).$$

Now we have

$$\begin{aligned} z - \sum_{i=1}^{m_2} t_i''(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})}) \\ = \sum_{j=1}^k s_j p c_{\alpha_j, r_{j+1}} + \sum_{i=1}^{m_1} t_i' y_{\alpha_i} + \sum_{i=1}^{m_2} t_i'' p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})} , \end{aligned}$$

thus

$$h_G \left( z - \sum_{i=1}^{m_2} t_i''(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})}) \right) \geq 1 ,$$

but this contradicts the assumption that  $H$  is a pure subgroup of  $G$ . Thus  $B'$  is a basic subgroup of  $G$ .

To complete the proof of the theorem, we need only show that  $B \cap B' = 0$ . To see this suppose that

$$\sum_{j=1}^k s_j (y_{\alpha_j} - p^{E(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})}) = \sum_{i=1}^n a_i y_{\alpha_i} .$$

Consider

$$\begin{aligned} \sum_{i=1}^n a_i y_{\alpha_i} + B = 0 + B &= \sum_{j=1}^k s_j (y_{\alpha_j} - p^{E(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})}) + B \\ &= \sum_{j=1}^k s_j p^{E(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})} + B \\ &= \sum_{j=1}^k s_j c_{\alpha_j, E(y_{\alpha_j})} + B , \end{aligned}$$

so that  $s_j c_{\alpha_j, E(y_{\alpha_j})} + B = 0 + B$  for  $j = 1, \dots, k$  since each is from a different summand of  $G/B$ . But this means that  $s_j$  is divisible by  $p^{E(y_{\alpha_j})}$  for  $j = 1, \dots, k$ . Thus

$$\sum_{j=1}^k s_j (y_{\alpha_j} - p^{E(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})}) = 0 ,$$

and so  $B \cap B' = 0$ .

**LEMMA 2.** *Let  $G$  be a reduced  $p$ -group such that final rank  $(G) = |G|$ . Let  $B$  be a lower basic subgroup of  $G$ . Then there exists a basic subgroup  $B'$  of  $G$  which is disjoint from  $B$ .*

*Proof.* Let  $H$  be a high subgroup of  $G$  which contains  $B$ . By Theorem 5 in [2]  $H$  is pure and a basic subgroup of  $H$  is a basic of  $G$ . Thus  $\text{rank}(G/B) = \text{rank}(H/B) + \text{rank}(G/H)$ , and we consider the following cases:

*Case (i).* Suppose that  $\text{rank}(H/B) = \text{rank}(G/B)$ , then we know that final rank  $(H) \geq \text{rank}(H/B) = \text{rank}(G/B) = \text{final rank}(G) = |G| \geq |H|$ . Thus final rank  $(H) = |H|$ , and Lemma 1 completes the proof.

*Case (ii).* Suppose that  $\text{rank}(G/B) > \text{rank}(H/B)$ . Since  $|G| = \text{final rank}(G)$ , and  $\text{final rank}(G) = \text{rank}(G/B)$ , we know that  $\text{rank}(G/B)$  is infinite. But if  $\text{rank}(G/B) > \text{rank}(H/B)$ , and is infinite, then the  $|G^1[p]|$  is infinite, and hence we have  $|G^1[p]| = \text{rank}(G/H) > \text{rank}(H/B)$ . Now  $\text{rank}(G/B) = \text{rank}(H/B) + \text{rank}(G/H) = \text{rank}(H/B) + |G^1[p]|$ , and thus  $|G^1[p]| = \text{rank}(G/B) = |G|$ . So that  $|G^1[p]| \geq |B|$ , and for the purposes of this proof we can assume that  $|G^1[p]| = |B|$ . Let  $G^1[p] = \sum_{\alpha \in I} \langle y_\alpha \rangle$ , and let  $B = \sum_{\alpha \in I} \langle x_\alpha \rangle$ . For each  $\alpha \in I$  choose  $z_\alpha$  such that  $y_\alpha = p^{\mathbb{E}(x_\alpha)-1} z_\alpha$ , which can be done since each  $y_\alpha$  has infinite height. Now consider the subgroup  $B' = \langle \{x_\alpha + z_\alpha\}_{\alpha \in I} \rangle$ . We claim that  $B'$  is a basic subgroup of  $G$  which is disjoint from  $B$ . To see this we prove:

(i) First we must show that  $B' = \sum_{\alpha \in I} \langle x_\alpha + z_\alpha \rangle$ . Suppose

$$\sum_{i=1}^n a_i(x_{\alpha_i} + z_{\alpha_i}) = 0,$$

where  $a_i \not\equiv 0 \pmod{o(x_{\alpha_i})}$ , and  $a_i < o(x_{\alpha_i})$ . Notice that  $o(x_\alpha) = o(x_\alpha + z_\alpha)$  since  $p(y_\alpha) = 0 = p^{\mathbb{E}(x_\alpha)} z_\alpha$ . Let  $k_i$  be the largest positive integer such that  $p^{k_i}$  divides  $a_i$ . Let  $r = \text{maximum}_{i=1, \dots, n} \{E(x_{\alpha_i}) - k_i\}$ , and consider

$$\begin{aligned} 0 &= p^{r-1} \left( \sum_{i=1}^n a_i(x_{\alpha_i} + z_{\alpha_i}) \right) = \sum_{i=1}^n a_i p^{r-1} x_{\alpha_i} + \sum_{i=1}^n a_i p^{r-1} z_{\alpha_i} \\ &= \sum_{i=1}^n a_i p^{r-1} x_{\alpha_i} + \sum_{i=1}^n a'_i y_{\alpha_i}. \end{aligned}$$

Hence we have

$$\sum_{i=1}^n a'_i y_{\alpha_i} = - \sum_{i=1}^n a_i p^{r-1} x_{\alpha_i},$$

but this means that an element of infinite height is equal to an element of finite height which is contradiction. Thus  $B' = \sum_{\alpha \in I} \langle x_\alpha + z_\alpha \rangle$ .

(ii) We must show that  $B'$  is pure. Let  $s \in B'[p]$ , and write

$$s = \sum_{i=1}^n a_i p^{\mathbb{E}(x_{\alpha_i})-1} (x_{\alpha_i} + z_{\alpha_i})$$

where  $a_i$  is relatively prime to  $p$  for each  $i$ . Since  $B' = \sum_{\alpha \in I} \langle x_\alpha + z_\alpha \rangle$ , we know that  $h_{B'}(s) = \text{minimum}_{i=1, \dots, n} \{E(x_{\alpha_i}) - 1\}$ . Now consider

$$\begin{aligned} h_G(s) &= h_G \left( \sum_{i=1}^n a_i p^{\mathbb{E}(x_{\alpha_i})-1} x_{\alpha_i} + \sum_{i=1}^n a_i p^{\mathbb{E}(x_{\alpha_i})-1} z_{\alpha_i} \right) \\ &= h_G \left( \sum_{i=1}^n a_i p^{\mathbb{E}(x_{\alpha_i})-1} x_{\alpha_i} + \sum_{i=1}^n a_i y_{\alpha_i} \right) \\ &= h_G \left( \sum_{i=1}^n a_i p^{\mathbb{E}(x_{\alpha_i})-1} x_{\alpha_i} \right) \\ &= \text{minimum}_{i=1, \dots, n} \{E(x_{\alpha_i}) - 1\}. \end{aligned}$$

Thus  $B'$  is a pure subgroup of  $G$ .

(iii) To complete the proof that  $B'$  is a basic subgroup of  $G$ , we need only prove that the quotient  $G/B'$  is divisible. If every element  $s + B' \in (G/B')[p]$  has infinite height then  $G/B'$  is divisible. Thus we can assume  $h_{G/B'}(s + B') = n$ , a finite integer, and we can assume that  $o(s) = o(s + B')$ . Now since  $G/B$  is divisible we know that  $s + B$  has infinite height in  $G/B$ . Consider the following cases:

*Case (a).* Suppose that  $s \in B$ , then

$$s = \sum_{i=1}^m a_i p^{E(x_{\alpha_i})-1} x_{\alpha_i}$$

where  $a_i$  is relatively prime to  $p$  for each  $i$ . Now define the following element of  $B'$ , let

$$b' = \sum_{i=1}^m a_i p^{E(x_{\alpha_i})-1} (x_{\alpha_i} + z_{\alpha_i}).$$

But

$$s - b' = \sum_{i=1}^m a_i y_{\alpha_i}, \quad \text{and} \quad \sum_{i=1}^m a_i y_{\alpha_i}$$

has infinite height in  $G$  so that  $h_{G/B'}(s + B')$  is infinite. Therefore  $G/B'$  must be divisible.

*Case (b).* Suppose that  $s \notin B$ , then there exists an element  $\sum_{i=1}^m a_i x_{\alpha_i} \in B$ , such that

$$s + \sum_{i=1}^m a_i x_{\alpha_i} = p^{n+1}g$$

since  $h_{G/B}(s + B)$  is infinite. Now write

$$\sum_{i=1}^m a_i x_{\alpha_i} = \sum_{j=1}^r c_j x_{\alpha_j} + \sum_{k=1}^t d_k x_{\alpha_k},$$

where  $c_j$  is divisible by  $p^{E(x_{\alpha_j})-1}$ , and  $d_k$  is not divisible by  $p^{E(x_{\alpha_k})-1}$ . Thus

$$s + \sum_{j=1}^r c_j x_{\alpha_j} + \sum_{k=1}^t d_k x_{\alpha_k} = p^{n+1}g$$

and so multiplication by  $p$  yields

$$\sum_{k=1}^t p d_k x_{\alpha_k} = p^{n+2}g.$$

By choice of the  $x_{\alpha_k}$ 's we know  $p d_k x_{\alpha_k} \neq 0$ . Thus we must have

$$h_G\left(\sum_{k=1}^t d_k x_{\alpha_k}\right) \geq n + 1.$$

Therefore by letting  $c'_j = c_j/p^{E(x_{\alpha_j})-1}$  we have that

$$s + \sum_{j=1}^r p^{E(x_{\alpha_j})-1} c'_j x_{\alpha_j} = p^{n+1} g' .$$

Consider the element  $b' \in B'$  such that

$$b' = \sum_{j=1}^r c'_j p^{E(x_{\alpha_j})-1} (x_{\alpha_j} + z_{\alpha_j}) = \sum_{j=1}^r c'_j p^{E(x_{\alpha_j})-1} x_{\alpha_j} + \sum_{j=1}^r c'_j y_{\alpha_j} .$$

Then

$$s - b' = p^{n+1} g' - \sum_{j=1}^r c'_j y_{\alpha_j} = p^{n+1} g''$$

since  $\sum_{j=1}^r c'_j y_{\alpha_j}$  has infinite height in  $G$ . But this implies that

$$h_{G/B'}(s + B') \geq n + 1$$

which contradicts the assumption that  $h_{G/B'}(s + B') = n$ . Thus  $G/B'$  must be divisible.

(iv) To complete the proof of the theorem, we need only show that  $B \cap B' = 0$ . To see this, suppose that

$$\sum_{i=1}^n a_i x_{\alpha_i} = \sum_{j=1}^k s_j (x_{\alpha_j} + z_{\alpha_j}) \neq 0$$

so

$$\sum_{i=1}^n a_i x_{\alpha_i} - \sum_{j=1}^k s_j x_{\alpha_j} = \sum_{j=1}^k s_j z_{\alpha_j} \neq 0 ,$$

and by multiplying both sides of this equation by an appropriate power of  $p$  we get an element of infinite height on one side and an element of finite height on the other side, which is a contradiction. Thus  $B \cap B' = 0$ , and the proof is finished.

The following theorem gives a sufficient condition for a group  $G$  to possess disjoint basic subgroups.

**THEOREM 3.** *Let  $G$  be a reduced Abelian  $p$ -group. If final rank  $(G) = |G|$ , then  $G$  contains two disjoint basic subgroups.*

*Proof.* Since every  $p$ -group has a lower basic subgroup, then Lemma 2 will complete the proof.

The next corollary shows that the restriction final rank  $(G) = |G|$ , can be removed if instead of disjoint basic subgroups, one is seeking two basic subgroups whose intersection is bounded.

**COROLLARY 4.** *Let  $G$  be a reduced Abelian  $p$ -group. Then there exists two basic subgroups of  $G$  whose intersection is bounded.*



*Proof.* By Theorem 31.5, page 106, in [1], we can write  $G = H \oplus K$ , where  $K$  is bounded direct sum of cyclic groups, and final rank  $(H) = |H|$ . Now by Theorem 3 there exists  $A$  and  $B$  which are disjoint basic subgroups of  $H$ . Now  $A \oplus K$  and  $B \oplus K$  are basic subgroups of  $G$  whose intersection is bounded.

**THEOREM 5.** *Let  $G$  be a reduced Abelian  $p$ -group, and suppose that  $A$  and  $B$  are two disjoint basic subgroups of  $G$ . Then  $\text{rank}(G/A) = \text{rank}(G/B) = |G|$ .*

*Proof.* Suppose that  $\text{rank}(G/A) < |G|$ , then by Lagrange's Theorem and since basic subgroups are isomorphic we know that  $|G| = |B| = |A|$ . By Theorem A we have  $G = L \oplus F$  and  $A = A' \oplus F$ , where  $|L| = \text{maximum}\{\aleph_0, \text{rank}(G/A)\}$ . Since  $A$  and  $B$  are disjoint basic subgroup of  $G$  we know  $G$  cannot be bounded. Now  $(G/A)[p] \supset [(A \oplus B)/A][p]$  and  $|[(A \oplus B)/A][p]| = |B|$  which must be at least  $\aleph_0$ . Thus  $\text{rank}(G/A) \geq \aleph_0$ , and therefore

$$|L| = \text{rank}(G/A) < |G|.$$

We can write each  $x \in B$  as  $x = y_x + f_x$ , where  $y_x \in L$  and  $f_x \in F$ . Since  $|B| = |A| = |G| > |L|$  and  $B$  is a subgroup, there must exist some  $y \in B$  such that  $y \in F$ , but  $F \subset A$  which contradicts  $A \cap B = 0$ . Thus  $\text{rank}(G/A) = |G|$ , and similarly  $\text{rank}(G/B) = |G|$ .

We are now in a position to state the results of the original questions in Theorem 6 and Theorem 7.

**THEOREM 6.** *A necessary and sufficient condition for a reduced Abelian  $p$ -group to possess disjoint basic subgroups is that final rank  $(G) = |G|$ .*

*Proof.* If final rank  $(G) = |G|$  then Theorem 3 completes the proof. If  $A$  and  $B$  are disjoint basic subgroups of  $G$  then by Theorem 5 we have  $r(G/A) = r(G/B) = |G|$ . But final rank  $(G) \geq \text{rank}(G/M)$  for any basic subgroup  $M$  of  $G$ . Thus final rank  $(G) \geq \text{rank}(G/A) = |G|$ , and since  $|G| \geq \text{final rank}(G)$  we have final rank  $(G) = |G|$ .

**THEOREM 7.** *If  $G$  is a reduced Abelian  $p$ -group such that final rank  $(G) = |G|$ , and  $A$  is a basic subgroup of  $G$ , then there is a basic subgroup of  $G$  which is disjoint from  $A$  if and only if  $A$  is a lower basic subgroup of  $G$ .*

*Proof.* If  $A$  is a lower basic subgroup then Lemma 2 assures the existence of a disjoint basic subgroup. If  $G$  possesses a basic

subgroup  $B$  disjoint from  $A$  then by Theorem 5 we have  $\text{rank}(G/A) = |G|$  and by hypothesis  $\text{final rank}(G) = |G|$  thus  $\text{rank}(G/A) = \text{final rank}(G)$  and  $A$  is a lower basic subgroup.

#### BIBLIOGRAPHY

1. L. Fuchs, *Abelian Groups*, Pergamon Press, New York, 1960.
2. John M. Irwin, *High subgroups of abelian torsion groups*, Pacific J. Math. **11**, (1961), 1375-1384.
3. Irving Kaplansky, *Infinite Abelian Groups*, The University of Michigan Press, Ann. Arbor, 1954.
4. A. Richard Mitchell, and Roger W. Mitchell *Some structure theorems for infinite abelian  $p$ -groups*. Journal of Algebra vol. 5, no. 3 (1967), 367-372.

Received July 14, 1966.



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AMERICAN MATHEMATICAL SOCIETY  
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TRW SYSTEMS  
NAVAL ORDNANCE TEST STATION

# Pacific Journal of Mathematics

Vol. 23, No. 1

March, 1967

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