LINEAR TRANSFORMATIONS WHICH PRESERVE HERMITIAN AND POSITIVE SEMIDEFINITE OPERATORS

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Let $\mathcal{A}$ and $\mathcal{B}$ represent the full algebras of linear operators on the finite-dimensional unitary spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. The symbol $\mathcal{L}(\mathcal{A}, \mathcal{B})$ will denote the complex space of all linear maps from $\mathcal{A}$ to $\mathcal{B}$. This paper concerns itself with the study of the following two cones in $\mathcal{L}(\mathcal{A}, \mathcal{B})$:

(i) the cone $\mathcal{C}$ of all $T \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ which send hermitian operators in $\mathcal{A}$ to hermitian operators in $\mathcal{B}$, and

(ii) the subcone $\mathcal{C}^+$ (of $\mathcal{C}$) of all $T \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ which send positive semidefinite operators in $\mathcal{A}$ to positive semidefinite operators in $\mathcal{B}$.

In our main results, we characterize the transformations in the cone $\mathcal{C}$ (Theorem 2.1) and present a structure theorem concerning the transformations in the cone $\mathcal{C}^+$ (Theorem 2.3). Identifying operators in the algebras $\mathcal{A}$ and $\mathcal{B}$ with appropriate square matrices, we may summarize Theorem 2.1 by saying that any and every linear transformation $T$ which preserves hermitian matrices is of the form

$$T: A \rightarrow \sum \alpha_i X_i^* A^i X_i,$$

where each $\alpha_i$ is a real scalar, and each $X_i$ is a certain rectangular matrix depending on $T$; $X_i^*$ and $A^i$ represent the conjugate transpose and the transpose of matrices $X_i$ and $A$, respectively. Theorem 2.3 says that the cone of positive semidefinite-preserving transformations $\mathcal{C}^+$ “generates” or spans all of $\mathcal{L}(\mathcal{A}, \mathcal{B})$ in the sense that any $T$ in $\mathcal{L}(\mathcal{A}, \mathcal{B})$ can be written

$$T = (K_1 - K_2) + i(K_3 - K_4),$$

where $i^2 = -1$, and each $K_i$ is an element of $\mathcal{C}^+$.

1. Preliminaries. $L(\mathcal{H}, \mathcal{K})$ denotes the space of linear transformations from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{K}$. We define:

1 (a). $(x \times y)$—the dyad transformation, an element of $L(\mathcal{H}, \mathcal{K})$, is defined for fixed $x \in \mathcal{H}$ and $y \in \mathcal{K}$ by: $(x \times y)(z) = (z, y)x$ for all $z \in \mathcal{H}$, where $(z, y)$ is the inner product of $z$ with $y$. As it turns out, $(x, y) = \text{tr}((x \times y))$, the trace of $(x \times y)$. If $A \in \mathcal{A} (= L(\mathcal{K}, \mathcal{K}))$ and $B \in \mathcal{B} (= L(\mathcal{K}, \mathcal{K}))$, then $(A(x) \times B(y)) = A(x \times y)B^*$.

1 (b). $P_x$—denotes the orthogonal projection onto the subspace spanned by $x$, i.e., for $(x, x) = 1$, we have $P_x = (x \times x)$. 

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1 (c). \([A, B]\)—is the inner product defined on \(\mathfrak{A}\) (resp. \(\mathfrak{B}\)) by setting \([A, B] = \text{tr}(B^* A)\) for all \(A, B \in \mathfrak{A}\) (resp. \(\mathfrak{B}\)) where \(B^*\) is the Hilbert space adjoint of \(B\), and \(\text{tr}(\cdot)\) is the trace functional on \(\mathfrak{A}\) (resp. \(\mathfrak{B}\)). More generally, \(L(\mathcal{H}, \mathcal{H})\) becomes a Hilbert space once we define the inner product \([A, B] = \text{tr}(B^* A)\) for all \(A, B \in L(\mathcal{H}, \mathcal{H})\). Consequently, for \(w_1, w_2 \in \mathcal{H}\), and \(u_1, u_2 \in \mathcal{H}\), so that \((w_1 \times u_1)\) and \((w_2 \times u_2)\) belong to \(L(\mathcal{H}, \mathcal{H})\), we have

\[
[(w_1 \times u_1), (w_2 \times u_2)] = \text{tr}((w_2 \times u_2)(w_1 \times u_1)) = \text{tr}((w_1, w_2)(u_2, u_1)).
\]

1 (d). \((A\rangle\langle B)\)—the dyad transformation, an element of \(\mathcal{L}(\mathfrak{B}, \mathfrak{A})\), is defined for fixed transformations \(A \in \mathfrak{A}\) and \(B \in \mathfrak{B}\) by \((A\rangle\langle B)\cdot C = [C, B]\cdot A\), for all \(C\) in \(B\). As in 1 (a)., \([A, B] = \text{tr}((A\rangle\langle B))\), the trace of \((A\rangle\langle B)\).

1 (e). \(\mathfrak{A} \otimes \mathfrak{B}\)—the tensor product of algebras \(\mathfrak{A}\) and \(\mathfrak{B}\), consists of sums of elements of the form \(A \otimes B\), where \(A \in \mathfrak{A}\) and \(B \in \mathfrak{B}\) \([2, \text{Chapter 16}]\). The symbol \((A \otimes B)^0\) will denote the element \(B \otimes A\), and can be linearly extended to any element of \(\mathfrak{A} \otimes \mathfrak{B}\).

1 (f). \([A_1 \otimes B_1, A_2 \otimes B_2]\)—the inner product which gives the algebra \(\mathfrak{A} \otimes \mathfrak{B}\) a Hilbert space structure, is defined by

\[
[A_1 \otimes B_1, A_2 \otimes B_2] = [A_1, A_2] \cdot [B_1, B_2]
\]

for all \(A_1, A_2 \in \mathfrak{A}\), and all \(B_1, B_2 \in \mathfrak{B}\).

1 (g). \(\mathcal{J}(T)\)—the element of \(\mathfrak{A} \otimes \mathfrak{B}\) which is defined for each \(T\) in \(\mathcal{L}(\mathfrak{A}, \mathfrak{B})\) by \(\mathcal{J}(T), A^* \otimes B\) = \([T(A), B]\), for all \(A \in \mathfrak{A}, B \in \mathfrak{B}\). This equation also defines \(\mathcal{J}\) as a linear transformation, sending the space \(\mathcal{L}(\mathfrak{A}, \mathfrak{B})\) to the algebra \(\mathfrak{A} \otimes \mathfrak{B}\).

1 (h). \(\mathcal{H}\)—the space of all linear functionals on \(\mathcal{H}\). For each \(x \in \mathcal{H}\), we define the functional \(\bar{x} \in \mathcal{H}\) by \(\bar{x}(y) = (y, x)\) for all \(y \in \mathcal{H}\). Moreover, these are the only elements of \(\mathcal{H}\). An inner product is defined on \(\mathcal{H}\) by setting \((\bar{x}, \bar{y}) = (y, x)\) for all \(\bar{x}, \bar{y} \in \mathcal{H}\). Thus, \((\bar{x}, \bar{y}) = (x, y)\), the complex conjugate of \((y, x)\).

1 (i). \(A^t\)—the transpose of the operator \(A\), is the linear operator on \(\mathcal{H}\) defined by \(A^t(\bar{y})(x) = \bar{y}(A(x))\), for all \(\bar{y} \in \mathcal{H}\), and all \(x \in \mathcal{H}\).
From this it follows that \((x \times y)^t = (\bar{y} \times \bar{x})\). If \(\bar{A}\) is
defined to be \((A^*)^t\), then \((\bar{x} \times \bar{y}) = (\bar{x} \times \bar{y})\) and \(\bar{A}(\bar{x}) = \bar{A}(x)\). From
this we see that for all \(A \in \mathcal{A}\), \(A^* = A^t\). In fact, set \(A = (x \times y)\) for
\(x, y \in \mathcal{A}\). Then
\[
A^* = (\bar{x} \times \bar{y})^t = (\bar{x} \times \bar{y})^* = (\bar{y} \times \bar{x}) = (x \times y)^t = A^t.
\]
Hence, by linear extension, \(A^* = A^t\) for all \(A \in \mathcal{A}\).

1 (j). \(L(\mathcal{H}, \mathcal{H})\)—is spanned by the dyads \((x \times \bar{y})\), where \(x \in \mathcal{H}\)
and \(\bar{y} \in \mathcal{H}\). In this context, we identify the transformation \(A \otimes B\)
with the transformation \(C \mapsto A B^t\) for all \(C \in L(\mathcal{H}, \mathcal{H})\), where
\(A \in \mathcal{A}(= L(\mathcal{H}, \mathcal{H}))\) and \(B \in \mathcal{B}(= L(\mathcal{H}, \mathcal{H}))\). Behind this identification
is the isomorphism \(\phi: \mathcal{H} \otimes \mathcal{H} \to L(\mathcal{H}, \mathcal{H})\) defined by \(\phi(x \otimes y) = (x \times \bar{y})\)
for all \(x \in \mathcal{H}, y \in \mathcal{H}\). If for each \(A \in \mathcal{A}, B \in \mathcal{B}\) we define
the linear transformation \(O_{A,B}: L(\mathcal{H}, \mathcal{H}) \to L(\mathcal{H}, \mathcal{H})\) by \(O_{A,B}(C) =
ACB^t\) for all \(C \in L(\mathcal{H}, \mathcal{H})\), then \(A \otimes B\) corresponds to \(O_{A,B}\) in the
sense that \(\phi \circ (A \otimes B) \circ \phi^{-1} = O_{A,B}\). In fact, we have
\[
(\phi \circ (A \otimes B) \circ \phi^{-1})(x \times \bar{y}) = \phi(A \otimes B(x \otimes y)) \quad \text{definition of } \phi^{-1}
\]
\[
= \phi(A(x) \otimes B(y)) \quad \text{definition of } A \otimes B
\]
\[
= (A(x) \times \bar{B}(y)) \quad \text{definition of } \phi
\]
\[
= (A(x) \times \bar{B}(\bar{y})) \quad \text{from 1 (i)}.
\]
\[
= A(x \times \bar{y}) \bar{B}^* \quad \text{from 1 (a)}.
\]
\[
= A(x \times \bar{y}) B^t \quad \text{since } \bar{B}^* = B^t, \text{see 1 (i)}.
\]
\[
= O_{A,B}(x \times \bar{y}) \quad \text{definition of } O_{A,B}.
\]
For convenience, however, we shall treat \(A \otimes B\) as though it were
actually equal to the concrete linear transformation \(O_{A,B} = A(\cdot)B^t\).
In so doing, we have
\[
(x \times y)[(u \times v) = (x \times u) \otimes (\bar{y} \times \bar{v})
\]
for vectors \(x, y, u, v\) in (not necessarily the same) Hilbert space.

The linear transformation \(\mathcal{F}\) (see 1(g).) will prove to be of funda-
mental importance. For this reason, we isolate some of its properties in

**Proposition 1.1.**
(1) \(\mathcal{F}(B)[A] = A^* \otimes B\) for all \(A \in \mathcal{A}, B \in \mathcal{B}\).
(2) \(\mathcal{F}(T) = \sum_i E_i^* \otimes T(E_i)\) for any and every orthonormal basis
\(\{E_i\}\) for \(\mathcal{A}\).
(3) If \(T(A^*) = T(A)^*\) for all \(A \in \mathcal{A}\) (i.e., if \(T \in \mathcal{C}\)), then
\(\mathcal{F}(T) = \sum_i T^*(F_i) \otimes F_i^*\) for any orthonormal basis \(\{F_i\}\)
for \(\mathcal{B}\).
(4) If \(T(A^*) = T(A)^*\) for all \(A \in \mathcal{A}\), then \(\mathcal{F}(T^*) = \mathcal{F}(T)^t\).
(5) \( \mathcal{F} \) is an isometric isomorphism from the Hilbert space \( \mathcal{L}(\mathcal{H}, \mathcal{B}) \) onto the Hilbert algebra \( \mathcal{A} \otimes \mathcal{B} \).

**Proof.** From the definition 1 (g). of \( \mathcal{F} \), we have

\[
[\mathcal{F}(A), C \otimes D] = [(B)[A](C^*), D]
= [C^*, A][B, D]
= [A^*, C][B, D]
= [A^* \otimes B, C \otimes D]
\]

for all \( A, C \in \mathcal{A} \) and all \( B, D \in \mathcal{B} \). This implies Part (1).

Now let \( \{E_i\} \) be any orthonormal (o.n.) basis for \( \mathcal{A} \). If \( T = (B)[A] \) for \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), then

\[
\sum E_i^* \otimes T(E_i) = \sum E_i^* \otimes (B)[A](E_i)
= \sum [E_i, A]E_i^* \otimes B
= \sum [A^*, E_i^*]E_i^* \otimes B
= A^* \otimes B
\]

which, from Part (1)

\[
\sum T^*(F_i) \otimes F_i^* = (\sum F_i^* \otimes T^*(F_i))^\circ = \mathcal{F}(T^*)^\circ = \mathcal{F}(T)
\]

Part (3) follows from (2) and (4) inasmuch as if \( \mathcal{F}(T^*) = \mathcal{F}(T)^\circ \), then \( \sum T^*(F_i) \otimes F_i^* = (\sum F_i^* \otimes T^*(F_i))^\circ = \mathcal{F}(T^*)^\circ = \mathcal{F}(T) \)

But Part (4) obtains, since for all \( A \in \mathcal{A}, B \in \mathcal{B}, \)

\[
[\mathcal{F}(T^*), A \otimes B] = [T^*(A^*), B] \quad \text{definition 1 (g). of } \mathcal{F}
= [T(B^*), A]
= [T(B^*), A] \quad \text{if and only if } T(B^*) = T(B)^*
= [\mathcal{F}(T), B \otimes A] \quad \text{definition 1 (g). of } \mathcal{F}
= [\mathcal{F}(T)^\circ, A \otimes B].
\]

That is, \( \mathcal{F}(T^*) = \mathcal{F}(T)^\circ \) and Part (4) is proven.

As for demonstrating Part (5), observe that for all \( A_1, A_2 \in \mathcal{A}, \) and \( B_1, B_2 \in \mathcal{B}, \)

\[
[\mathcal{F}(A_1), \mathcal{F}(A_2)] = [A_1^* \otimes B_1, A_2^* \otimes B_2] \quad \text{from Part (1)}
= [A_1^*, A_2^*] \text{ tr } ((B_1)[B_2]) \quad \text{from 1 (d). and 1 (f).}
= \text{ tr } ((B_1)[A_1^*] \cdot (B_2)[A_2^*])
= [(B_1)[A_1], (B_2)[A_2]].
\]
By linear extension on each argument of the inner product, we have that for all \( T, T_2 \in \mathcal{L}(\mathcal{A}, \mathcal{B}) \),

\[
[\mathcal{J}(T_1), \mathcal{J}(T_2)] = [T_1, T_2]
\]

so that \( \mathcal{J} \) is an isometry from \( \mathcal{L}(\mathcal{A}, \mathcal{B}) \) to \( \mathcal{A} \otimes \mathcal{B} \). From Part (1) it is easy to see that \( \mathcal{J} \) is also an onto transformation as well, since the algebra \( \mathcal{A} \otimes \mathcal{B} \) is spanned by elements of the form \( A^* \otimes B \). This completes the proof of Proposition 1.1.

Our next result establishes a necessary and sufficient condition for a transformation in \( \mathcal{L}(\mathcal{A}, \mathcal{B}) \) to be in the cone \( \mathcal{C} \).

**Proposition 1.2.** A transformation \( T \in \mathcal{L}(\mathcal{A}, \mathcal{B}) \) is in \( \mathcal{C} \) if and only if \( \mathcal{J}(T) \) is hermitian.

**Proof.** Recall that \( \mathcal{J} \) maps \( \mathcal{L}(\mathcal{A}, \mathcal{B}) \) (isometrically) onto \( \mathcal{A} \otimes \mathcal{B} \), which has been identified as the algebra of linear operators on the Hilbert space \( L(\mathcal{H}, \mathcal{H}) \) (see 1(j)). Now for all \( A \in \mathcal{A}, B \in \mathcal{B} \),

\[
\begin{align*}
(a) \quad [\mathcal{J}(T)^*, A \otimes B] &= [\mathcal{J}(T), A^* \otimes B^*] \\
(b) \quad &= [T(A), B^*] \quad \text{definition 1(g) of } \mathcal{J} \\
(c) \quad &= [T(A)^*, B]
\end{align*}
\]

where (a) and (c) follow from the properties of the inner product, viz., \([Y, Z] = [Y^*, Z^*]\) for all operators \( Y \) and \( Z \). Now,

\[
[T(A)^*, B] = [T(A^*), B] \quad \text{for all } A \in \mathcal{A}, B \in \mathcal{B},
\]

if and only if \( T(A)^* = T(A^*) \) for all \( A \in \mathcal{A} \). Finally, \([T(A^*), B]\) is equal to \([\mathcal{J}(T), A \otimes B]\), so that for all \( A \in \mathcal{A}, B \in \mathcal{B} \),

\[
[\mathcal{J}(T) - \mathcal{J}(T)^*, A \otimes B] = 0
\]

if and only if \( T(A^*) = T(A)^* \). This completes the proof.

**Remark.** We have just shown that \( T \in \mathcal{L}(\mathcal{A}, \mathcal{B}) \) preserves hermitian operators \( (T \in \mathcal{C}) \) if and only if \( \mathcal{J}(T) \) is hermitian. It is not unreasonable to suspect that \( T \) preserves positive semidefinite (psd) operators \( (T \in \mathcal{C}^+) \) if and only if \( \mathcal{J}(T) \) is psd. However, this conjecture is false, for if \( \mathcal{A} = L(\mathcal{H}, \mathcal{H}) \), and if \( \mathcal{B} = L(\mathcal{K}, \mathcal{K}) \), then for any multiplicative transformation \( T \in \mathcal{L}(\mathcal{A}, \mathcal{B}) \) \((T(AB) = T(A)T(B))\), we have \( T \in \mathcal{C}^+ \); but \( \mathcal{J}(T) \) will always have some negative eigenvalues. For a specific example choose \( \mathcal{A} = \mathcal{B} = L(\mathcal{H}, \mathcal{H}) \), the algebra of operators on \( \mathcal{H} \). Let \( T \in \mathcal{L}(\mathcal{A}, \mathcal{B}) \) be the identity transformation \( T(A) = A \) for all \( A \in \mathcal{A} \). Surely \( T \in \mathcal{C}^+ \). Now choose the o.n. basis \( \{e_1, e_2, \ldots, e_n\} \) for \( \mathcal{H}^* \); then \( \{(e_i \times e_j): i, j = 1, 2, \ldots, n\} \) is an o.n. basis for \( \mathcal{A} \) so that from Proposition 1.1 Part (2), we have
\[ \mathcal{F}(T) = \sum (e_i \times e_j)^* \otimes (e_i \times e_j) = \sum (e_j \times e_i) \otimes (e_i \times e_j). \]

The situation may be represented by the following diagram:

\[ \begin{array}{ccc}
\mathcal{A} = L(H, H) & \xrightarrow{T = \text{identity}} & \mathcal{A} = L(H, H) \\
(e_i \times e_j) & \xrightarrow{\mathcal{F}(T) = \text{transpose}} & (e_i \times e_j) \\
L(H, H) & \xrightarrow{\mathcal{F}(T)} & L(H, H) \\
(e_p \times e_q) & \xrightarrow{} & (e_q \times e_p) .
\end{array} \]

From 1(i) and 1(j) we conclude that \( \mathcal{F}(T)((e_p \times e_q)) = (e_q \times e_p) \) for \((e_p \times e_q), p, q = 1, 2, \ldots, n, \) in the space \( L(H, H). \) That is, if \( T \) is the identity operator on the Hilbert algebra \( L(H, H), \) then \( \mathcal{F}(T) \) is the transpose operator on the Hilbert space \( L(H, H). \) It is easy to see that vectors of the form \((e_p \times e_q) - (e_q \times e_p)\) in \( L(H, H) \) are eigenvectors for \( \mathcal{F}(T) \) corresponding to the eigenvalue \(-1.\) \( \mathcal{F}(T) \) (which is hermitian due to Proposition 1.2), is therefore not a psd operator on the Hilbert space \( L(H, H). \)

2. The main results. We present a structure theorem which characterizes elements of the cone \( \mathcal{C}. \)

**Theorem 2.1.** Suppose that \( T \in \mathcal{C} \subset L(\mathcal{H}, \mathcal{B}). \) \( \mathcal{F}(T) \) is self-adjoint by Proposition 1.2, with spectral resolution \( \sum_i \alpha_i \mathcal{P}(X_i), \) where \( \alpha_i \) is real, \( \mathcal{P}(X_i) = (X_i)[X_i] \) is the orthogonal one-dimensional projection on the unit vector \( X_i \in L(H, H), \) and the \( X_i \)'s form an o.n. basis for \( L(H, H). \) Let \( A \in \mathcal{A}: \) then

\[ T(A)^* = \sum_i \alpha_i X_i^* AX_i. \]

**Proof.** For any \( x \in H \) and \( y \in H, \)

\[
\begin{align*}
(1) & \quad [T(P_x), P_y] = [\mathcal{F}(T), P_x \otimes P_y] \\
(2) & \quad = \sum_i [\alpha_i (X_i)[X_i], (x \times x) \otimes (y \times y)] \quad \text{from 1(b)} \\
(3) & \quad = \sum_i [\alpha_i (X_i)[X_i], (x \times \bar{y})][(x \times \bar{y})] \quad \text{from 1(j)} \\
(4) & \quad = \sum_i \alpha_i \text{ tr } ((x \times \bar{y})[(x \times \bar{y}):(X_i)[X_i])) \quad \text{from 1(c)} \\
(5) & \quad = \sum_i \alpha_i [X_i, (x \times \bar{y})][(x \times \bar{y}), X_i] \\
(6) & \quad = \sum_i \alpha_i \text{ tr } ((\bar{y} \times x)X_i) \text{ tr } (X_i^*(x \times \bar{y})) \\
(7) & \quad = \sum_i \alpha_i \text{ tr } ((\bar{y} \times x)X_i^*) \text{ tr } (X_i^*(x \times \bar{y})) \quad \text{since} \\
(\bar{y} \times x)X_i & = \bar{y} \times X_i^*(x) ; \quad \text{see 1(a)}
\end{align*}
\]
Now for \( w_1, w_2 \in \mathcal{H} \) and \( u_1, u_2 \in \mathcal{H} \), we have that
\[
(u_2, u_1)(w_1, w_2) = [(w_1 \times u_1), (w_2 \times u_2)]
\]
(see 1 (c)).

so (8) becomes
\[
(9) = \sum \alpha_i(X_i^*(x) \times X_i^*(x), (\bar{y} \times \bar{y})]
\]
\[
(10) = \sum [\alpha_iX_i^*(x \times x)X_i, (P_y)']
\]
Since the transpose is a self-adjoint operator, equation (10) becomes
\[
(11) = \sum [\alpha_i(X_i^*P_xX_i)', P_y].
\]
Thus, for every \( x \in \mathcal{H} \) and every \( y \in \mathcal{H} \),
\[
\begin{bmatrix}
T(P_x) - \left(\sum \alpha_iX_i^*P_xX_i\right)', P_y
\end{bmatrix} = 0.
\]
But then,
\[
T(P_x) = \left(\sum \alpha_iX_i^*P_xX_i\right)'
\]
for all \( P_x \in \mathfrak{A} \). Since the transpose operator squared is the identity, we may apply it to both sides of the last equation to obtain
\[
(12) T(P_x)' = \sum \alpha_iX_i^*P_xX_i
\]
for all \( P_x \in \mathfrak{A} \). This result extends from the set of one dimensional orthogonal projections \( P_x \) to hermitian operators; this, in turn, extends to arbitrary operators of \( \mathfrak{A} \). Thus, the theorem is proved.

**Remark.** Suppose the dimension of \( \mathcal{H} = n \) and the dimension of \( \mathcal{H}' = m \), where \( \mathcal{H} \) and \( \mathcal{H}' \) are the underlying Hilbert spaces for the operator algebras \( \mathfrak{A} \) and \( \mathfrak{B} \), respectively. Relative to certain ordered bases for \( \mathcal{H} \) and \( \mathcal{H}' \), each operator of \( \mathfrak{A} \) and \( \mathfrak{B} \) is identified with a certain square matrix. The o.n. basis vectors \( X_i \) of \( L(\mathcal{H}', \mathcal{H}) \) are then realized as certain \( n \times m \) matrices; the operator \( X_i^* \) is identified with the \( m \times n \) conjugate transpose matrix of \( X_i \). Thus, Theorem 2.1 may be interpreted as saying that any linear transformation \( T \), sending the full matrix algebra \( \mathfrak{A} \) to the full matrix algebra \( \mathfrak{B} \) is of the form
\[
T(A) = \left(\sum \alpha_iX_i^*AX_i\right)'
\]
for certain real scalars \( \alpha_i \) and certain \( n \times m \) matrices \( X_i \), if and only
if $T$ preserves hermitian matrices. Equivalently,

$$T(A) = \left( \sum_i \alpha_i X_i^* A X_i \right)'$$

$$= \sum_i \alpha_i X_i^* A' (X_i^*)'$$

$$= \sum_i \alpha_i Y_i^* A' Y_i$$

setting $Y_i = (X_i^*)'$

for certain real scalars $\alpha_i$ and certain $n \times m$ matrices $Y_i$ depending on $T$, characterizes those transformations $T: \mathcal{A} \rightarrow \mathcal{B}$ which preserve hermitian matrices.

**Corollary 2.2.** Let $T \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ where $\mathcal{J}(T)$ is psd in $\mathcal{A} \otimes \mathcal{B}$. Then $T \in \mathcal{C}^+ \subset \mathcal{L}(\mathcal{A}, \mathcal{B})$.

**Proof.** Since $\mathcal{J}(T)$ is psd in $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{J}(T)$ has spectral resolution $\sum \alpha_i \mathcal{P}(X_i)$ where the scalars $\alpha_i$ are nonnegative, $\mathcal{P}(X_i)$ is the orthogonal one-dimensional projection onto $X_i \in L(\mathcal{H}, \mathcal{H})$ and the $X_i$'s form an o.n. basis for $L(\mathcal{H}, \mathcal{H})$. Since $\mathcal{J}(T)$ is psd, it is, a fortiori, self-adjoint, so that $T$ is at least an element of the cone $\mathcal{C}$ (Proposition 1.2). But this gives us sufficient leverage to employ the representation of Theorem 2.1. Hence, $T(\cdot)' = \sum \alpha_i X_i^*(\cdot) X_i$ where the $\alpha_i$'s are nonnegative scalars. In order to show that $T$ sends psd operators to psd operators (i.e., $T \in \mathcal{C}^+$), it is (necessary and) sufficient to show that $T$ sends one-dimensional orthogonal projections $P_z$ to psd operators; to do this, it is (necessary and) sufficient to show that the operator $T(\cdot)'$ sends these projections $P_z$ to psd operators. But

$$T(P_z)' = \sum \alpha_i (X_i^* P_z X_i)$$

from Theorem 2.1. Observe that each term $X_i^* P_z X_i = (P_z X_i)^*(P_z X_i)$ is psd, and hence, so is $\sum \alpha_i X_i^* P_z X_i$, the sum of nonnegative multiples of these psd terms. The proof is done.

We come to our final theorem which tells us that the cone $\mathcal{C}^+$ "generates" the space $\mathcal{L}(\mathcal{A}, \mathcal{B})$ in much the same way that the cone of psd operators (in $\mathcal{A}$, say) "generates" $\mathcal{A}$.

**Theorem 2.3.** Suppose $T \in \mathcal{L}(\mathcal{A}, \mathcal{B})$. Then for some $K_1, K_2, K_3, K_i \in \mathcal{C}^+$,

$$T = (K_1 - K_2) + i(K_3 - K_i)$$

where $i^2 = -1$

**Proof.** $\mathcal{J}(T)$, an element of the algebra $\mathcal{A} \otimes \mathcal{B}$ can be decomposed as follows:
where each of the $U_i$'s is psd in $\mathbb{A} \otimes \mathbb{B}$. Proposition 1.1, Part (5), tells us that $\mathcal{F}: \mathcal{L}(\mathbb{A}, \mathbb{B}) \rightarrow \mathbb{A} \otimes \mathbb{B}$ is an isometry. Since the (vector space) dimensions of $\mathcal{L}(\mathbb{A}, \mathbb{B})$ and $\mathbb{A} \otimes \mathbb{B}$ agree, $\mathcal{F}$ is, in fact, one-to-one and onto; thus, $\mathcal{F}^{-1}$ exists as a well-defined linear operator. Applying $\mathcal{F}^{-1}$ to (*) yields

$$T = [\mathcal{F}^{-1}(U_1) - \mathcal{F}^{-1}(U_2)] + i[\mathcal{F}^{-1}(U_3) - \mathcal{F}^{-1}(U_4)].$$

Now let $K_i = \mathcal{F}^{-1}(U_i)$, $i = 1, 2, 3, 4$. Corollary 2.2 forces us to conclude that $K_i \in \mathcal{C}^+$ since $\mathcal{F}(K_i) = U_i$ is psd. Thus, for any $T \in \mathcal{L}(\mathbb{A}, \mathbb{B})$

$$T = (K_1 - K_2) + i(K_3 - K_4)$$

where each $K_i \in \mathcal{C}^+ \subset \mathcal{L}(\mathbb{A}, \mathbb{B})$.

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