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JOHN EMANUEL DE PILLIS

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Let $\mathfrak{A}$ and $\mathfrak{B}$ represent the full algebras of linear operators on the finite-dimensional unitary spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. The symbol $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ will denote the complex space of all linear maps from $\mathfrak{A}$ to $\mathfrak{B}$. This paper concerns itself with the study of the following two cones in $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$:

(i) the cone $\mathscr{C}$ of all $T \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ which send hermitian operators in $\mathfrak{A}$ to hermitian operators in $\mathfrak{B}$, and

(ii) the subcone $\mathscr{C}^+$ (of $\mathscr{C}$) of all $T \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ which send positive semidefinite operators in $\mathfrak{A}$ to positive semidefinite operators in $\mathfrak{B}$.

In our main results, we characterize the transformations in the cone $\mathscr{C}$ (Theorem 2.1) and present a structure theorem concerning the transformations in the cone $\mathscr{C}^+$ (Theorem 2.3). Identifying operators in the algebras $\mathfrak{A}$ and $\mathfrak{B}$ with appropriate square matrices, we may summarize Theorem 2.1 by saying that any and every linear transformation $\boldsymbol{T}$ which preserves hermitian matrices is of the form $\boldsymbol{T}: A \rightarrow \sum \alpha_i X_i^* A^t X_i$, where each α_i is a real scaler, and each X_i is a certain rectangular matrix depending on $\boldsymbol{T}$; X_i^* and A^t represent the conjugate transpose and the transpose of matrices X_i and A, respectively. Theorem 2.3 says that the cone of positive semidefinite-preserving transformations $\mathscr{C}^+$ "generates" or spans all of $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ in the sense that any $\boldsymbol{T}$ in $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ can be written

$$\boldsymbol{T} = (\boldsymbol{K}_1 - \boldsymbol{K}_2) + i(\boldsymbol{K}_3 - \boldsymbol{K}_4) ,$$

where $i^2 = -1$, and each $\boldsymbol{K}_i$ is an element of $\mathscr{C}^+$.

1. Preliminaries.

$L(\mathscr{K}, \mathscr{H})$ denotes the space of linear transformations from the Hilbert space $\mathscr{K}$ to the Hilbert space $\mathscr{H}$. We define:

1 (a). $(x \times y)$—the dyad transformation, an element of $L(\mathscr{K}, \mathscr{H})$, is defined for fixed $x \in \mathscr{H}$ and $y \in \mathscr{K}$ by: $(x \times y)(z) = (z, y)x$ for all $z \in \mathscr{K}$, where (z, y) is the inner product of z with y. As it turns out, $(x, y) = \operatorname{tr}((x \times y))$, the trace of $(x \times y)$. If $A \in \mathfrak{A}(=(L(\mathscr{H}, \mathscr{H}))$ and $B \in \mathfrak{B}(=L(\mathscr{K}, \mathscr{K}))$, then $(A(x) \times B(y)) = A(x \times y)B^*$.

1 (b). P_x—denotes the orthogonal projection onto the subspace spanned by x, i.e., for $(x, x) = 1$, we have $P_x = (x \times x)$.

1 (c). $[A, B]$—is the inner product defined on $\mathfrak{A}$ (resp. $\mathfrak{B}$) by setting $[A, B] = \operatorname{tr}(B^*A)$ for all $A, B \in \mathfrak{A}$ (resp. $\mathfrak{B}$) where B^* is the Hilbert space adjoint of B, and $\operatorname{tr}(\cdot)$ is the trace functional on $\mathfrak{A}$ (resp. $\mathfrak{B}$). More generally, $L(\mathscr{K}, \mathscr{H})$ becomes a Hilbert space once we define the inner product $[A, B] = \operatorname{tr}(B^*A)$ for all $A, B \in L(\mathscr{K}, \mathscr{H})$. Consequently, for $w_1, w_2 \in \mathscr{H}$, and $u_1, u_2 \in \mathscr{K}$, so that $(w_1 \times u_1)$ and $(w_2 \times u_2)$ belong to $L(\mathscr{K}, \mathscr{H})$, we have

$$\begin{aligned}[(w_1 \times u_1), (w_2 \times u_2)] &= \operatorname{tr}((w_2 \times u_2)^*(w_1 \times u_1)) \\ &= \operatorname{tr}((u_2 \times w_2)(w_1 \times u_1)) \\ &= \operatorname{tr}((w_1, w_2)(u_2 \times u_1)) \\ &= (w_1, w_2)(u_2, u_1) .\end{aligned}$$

1 (d). $(A][B)$—the dyad transformation, an element of $\mathscr{L}(\mathfrak{B}, \mathfrak{A})$, is defined for fixed transformations $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ by $(A][B)\cdot C = [C, B]A$, for all C in B. As in 1 (a)., $[A, B] = \operatorname{tr}((A][B))$, the trace of $(A][B)$.

1 (e). $\mathfrak{A} \otimes \mathfrak{B}$—the tensor product of algebras $\mathfrak{A}$ and $\mathfrak{B}$, consists of sums of elements of the form $A \otimes B$, where $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ [2, Chapter 16]. The symbol $(A \otimes B)^0$ will denote the element $B \otimes A$, and can be linearly extended to any element of $\mathfrak{A} \otimes \mathfrak{B}$.

1 (f). $[A_1 \otimes B_1, A_2 \otimes B_2]$—the inner product which gives the algebra $\mathfrak{A} \otimes \mathfrak{B}$ a Hilbert space structure, is defined by

$$[A_1 \otimes B_1, A_2 \otimes B_2] = [A_1, A_2]\cdot[B_1, B_2]$$

for all $A_1, A_2 \in \mathfrak{A}$, and all $B_1, B_2 \in \mathfrak{B}$.

1 (g). $\mathscr{I}(\boldsymbol{T})$—the element of $\mathfrak{A} \otimes \mathfrak{B}$ which is defined for each $\boldsymbol{T}$ in $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ by $[\mathscr{I}(\boldsymbol{T}), A^* \otimes B] = [\boldsymbol{T}(A), B]$, for all $A \in \mathfrak{A}, B \in \mathfrak{B}$. This equation also defines $\mathscr{I}$ as a linear transformation, sending the space $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ to the algebra $\mathfrak{A} \otimes \mathfrak{B}$.

1 (h). $\overline{\mathscr{H}}$—the space of all linear functionals on $\mathscr{H}$. For each $x \in \mathscr{H}$, we define the functional $\bar{x} \in \overline{\mathscr{H}}$ by $\bar{x}(y) = (y, x)$ for all $y \in \mathscr{H}$. Moreover, these are the only elements of $\overline{\mathscr{H}}$. An inner product is defined on $\overline{\mathscr{H}}$ by setting $(\bar{x}, \bar{y}) = (y, x)$ for all $\bar{x}, \bar{y} \in \overline{\mathscr{H}}$. Thus, $(\bar{x}, \bar{y}) = \overline{(x, y)}$, the complex conjugate of (y, x).

1 (i). A^t—the transpose of the operator A, is the linear operator on $\overline{\mathscr{H}}$ defined by $A^t(\bar{y})(x) = \bar{y}(A(x))$, for all $\bar{y} \in \overline{\mathscr{H}}$, and all $x \in \mathscr{H}$

[1, p. 103]. From this it follows that $(x \times y)^t = (\bar{y} \times \bar{x})$. If $\bar{A}$ is defined to be $(A^*)^t$, then $\overline{(x \times y)} = (\bar{x} \times \bar{y})$ and $\bar{A}(\bar{x}) = \overline{A(x)}$. From this we see that for all $A \in \mathfrak{A}$, $\bar{A}^* = A^t$. In fact, set $A = (x \times y)$ for $x, y \in \mathfrak{A}$. Then

$$\bar{A}^* = \overline{(x \times y)}^* = (\bar{x} \times \bar{y})^* = (\bar{y} \times \bar{x}) = \overline{(y \times x)} = (x \times y)^t = A^t .$$

Hence, by linear extension, $\bar{A}^* = A^t$ for all $A \in \mathfrak{A}$.

1 (j). $L(\overline{\mathscr{K}}, \mathscr{H})$—is spanned by the dyads $(x \times \bar{y})$, where $x \in \mathscr{H}$ and $\bar{y} \in \overline{\mathscr{K}}$. In this context, we identify the transformation $A \otimes B$ with the transformation $C \to ACB^t$ for all $C \in L(\overline{\mathscr{K}}, \mathscr{H})$, where $A \in \mathfrak{A}(=L(\mathscr{H}, \mathscr{H}))$ and $B \in \mathfrak{B}(=L(\mathscr{K}, \mathscr{K}))$. Behind this identification is the isomorphism ϕ: $\mathscr{H} \otimes \overline{\mathscr{K}} \to L(\overline{\mathscr{K}}, \mathscr{H})$ defined by $\phi(x \otimes y) = (x \times \bar{y})$ for all $x \in \mathscr{H}, y \in \mathscr{K}$. If for each $A \in \mathfrak{A}, B \in \mathfrak{B}$ we define the linear transformation $\boldsymbol{O}_{A,B}$: $L(\overline{\mathscr{K}}, \mathscr{H}) \to L(\overline{\mathscr{K}}, \mathscr{H})$ by $\boldsymbol{O}_{A,B}(C) = ACB^t$ for all $C \in L(\overline{\mathscr{K}}, \mathscr{H})$, then $A \otimes B$ corresponds to $\boldsymbol{O}_{A,B}$ in the sense that $\phi \circ (A \otimes B) \circ \phi^{-1} = \boldsymbol{O}_{A,B}$. In fact, we have

$$\begin{aligned}
(\phi \circ (A \otimes B) \circ \phi^{-1}(x \times \bar{y}) &= \phi(A \otimes B(x \otimes y)) && \text{definition of } \phi^{-1} \\
&= \phi(A(x) \otimes B(y)) && \text{definition of } A \otimes B \\
&= (A(x) \times \overline{B(y)}) && \text{definition of } \phi \\
&= (A(x) \times \bar{B}(\bar{y})) && \text{from 1 (i).} \\
&= A(x \times \bar{y})\bar{B}^* && \text{from 1 (a).} \\
&= A(x \times \bar{y})B^t && \text{since } \bar{B}^* = B^t\text{, see 1 (i).} \\
&= \boldsymbol{O}_{A,B}((x \times \bar{y})) && \text{definition of } \boldsymbol{O}_{A,B} .
\end{aligned}$$

For convenience, however, we shall treat $A \otimes B$ as though it were actually equal to the concrete linear transformation $\boldsymbol{O}_{A,B} = A(\cdot)B^t$. In so doing, we have

$$(x \times y)][(u \times v) = (x \times u) \otimes (\bar{y} \times \bar{v})$$

for vectors x, y, u, v in (not necessarily the same) Hilbert space.

The linear transformation $\mathscr{I}$ (see 1(g).) will prove to be of fundamental importance. For this reason, we isolate some of its properties in

PROPOSITION 1.1. (1) $\mathscr{I}(B][A) = A^* \otimes B$ for all $A \in \mathfrak{A}, B \in \mathfrak{B}$.

(2) $\mathscr{I}(\boldsymbol{T}) = \sum_i E_i^* \otimes \boldsymbol{T}(E_i)$ for any and every orthonormal basis $\{E_i\}$ for $\mathfrak{A}$.

(3) If $\boldsymbol{T}(A^*) = \boldsymbol{T}(A)^*$ for all $A \in \mathfrak{A}$ (i.e., if $T \in \mathscr{C}$), then $\mathscr{I}(\boldsymbol{T}) = \sum_i \boldsymbol{T}^*(F_i) \otimes F_i^*$ for any orthonormal basis $\{F_i\}$ for $\mathfrak{B}$.

(4) If $\boldsymbol{T}(A^*) = \boldsymbol{T}(A)^*$ for all $A \in \mathfrak{A}$, then $\mathscr{I}(\boldsymbol{T}^*) = \mathscr{I}(\boldsymbol{T})^0$.

(5) $\mathscr{I}$ is an isometric isomorphism from the Hilbert space $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ onto the Hilbert algebra $\mathfrak{A} \otimes \mathfrak{B}$.

Proof. From the definition 1 (*g*). of $\mathscr{I}$, we have

$$\begin{aligned}
[\mathscr{I}(B][A), C \otimes D] &= [(B][A)(C^*), D] \\
&= [C^*, A][B, D] && \text{from 1 (d).} \\
&= [A^*, C][B, D] \\
&= [A^* \otimes B, C \otimes D] && \text{from 1 (f).}
\end{aligned}$$

for all $A, C \in \mathfrak{A}$ and all $B, D \in \mathfrak{B}$. This implies Part (1).

Now let $\{E_i\}$ be any orthonormal (o.n.) basis for $\mathfrak{A}$. If $\boldsymbol{T} = (B][A)$ for $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, then

$$\begin{aligned}
\sum_i E_i^* \otimes \boldsymbol{T}(E_i) &= \sum_i E_i^* \otimes (B][A)(E_i) \\
&= \sum_i [E_i, A] E_i^* \otimes B && \text{from 1 (d).} \\
&= \sum_i [A^*, E_i^*] E_i^* \otimes B \\
&= A^* \otimes B && \text{which, from Part (1)} \\
&= \mathscr{I}(B][A) .
\end{aligned}$$

The dyads $(B][A), A \in \mathfrak{A}, B \in \mathfrak{B}$, span the space $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$, so that (using linearity of $\mathscr{I}$) for *all* $\boldsymbol{T} \in \mathscr{L}(\mathfrak{A}, \mathfrak{B}), \mathscr{I}(\boldsymbol{T}) = \sum_i E_i^* \otimes \boldsymbol{T}(E_i)$, which establishes Part (2).

Part (3) follows from (2) and (4) inasmuch as if $\mathscr{I}(\boldsymbol{T}^*) = \mathscr{I}(\boldsymbol{T})^0$, then $\sum \boldsymbol{T}^*(F_i) \otimes F_i^* = (\sum F_i^* \otimes \boldsymbol{T}^*(F_i))^0 = \mathscr{I}(\boldsymbol{T}^*)^0 = \mathscr{I}(\boldsymbol{T})$

But Part (4) obtains, since for all $A \in \mathfrak{A}, B \in \mathfrak{B}$,

$$\begin{aligned}
[\mathscr{I}(\boldsymbol{T}^*), A \otimes B] &= [\boldsymbol{T}^*(A^*), B] && \text{definition 1 (g). of } \mathscr{I} \\
&= [\boldsymbol{T}(B)^*, A] \\
&= [\boldsymbol{T}(B^*), A] && \text{if and only if } \boldsymbol{T}(B^*) = \boldsymbol{T}(B)^* \\
&= [\mathscr{I}(\boldsymbol{T}), B \otimes A] && \text{definition 1 (g). of } \mathscr{I} \\
&= [\mathscr{I}(\boldsymbol{T})^0, A \otimes B] .
\end{aligned}$$

That is, $\mathscr{I}(\boldsymbol{T}^*) = \mathscr{I}(\boldsymbol{T})^0$ and Part (4) is proven.

As for demonstrating Part (5), observe that for all $A_1, A_2 \in \mathfrak{A}$, and $B_1, B_2 \in \mathfrak{B}$,

$$\begin{aligned}
[\mathscr{I}(B_1][A_1), \mathscr{I}(B_2][A_2)] &= [A_1^* \otimes B_1, A_2^* \otimes B_2] && \text{from Part (1)} \\
&= [A_1^*, A_2^*] \operatorname{tr}((B_1][B_2)) && \text{from 1 (d). and 1 (f).} \\
&= \operatorname{tr}((B_1][A_1) \cdot (B_2][A_2)^*) \\
&= [(B_1][A_1), (B_2][A_2)] .
\end{aligned}$$

By linear extension on each argument of the inner product, we have that for *all* $\boldsymbol{T}_1, \boldsymbol{T}_2 \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$,

$$[\mathscr{I}(\boldsymbol{T}_1), \mathscr{I}(\boldsymbol{T}_2)] = [\boldsymbol{T}_1, \boldsymbol{T}_2]$$

so that $\mathscr{I}$ is an isometry from $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ to $\mathfrak{A} \otimes \mathfrak{B}$. From Part (1) it is easy to see that $\mathscr{I}$ is also an onto transformation as well, since the algebra $\mathfrak{A} \otimes \mathfrak{B}$ is spanned by elements of the form $A^* \otimes B$. This completes the proof of Proposition 1.1.

Our next result establishes a necessary and sufficient condition for a transformation in $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ to be in the cone $\mathscr{C}$.

PROPOSITION 1.2. A transformation $\boldsymbol{T} \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ is in $\mathscr{C}$ if and only if $\mathscr{I}(\boldsymbol{T})$ is hermitian.

Proof. Recall that $\mathscr{I}$ maps $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ (isometrically) onto $\mathfrak{A} \otimes \mathfrak{B}$, which has been identified as the algebra of linear operators on the Hilbert space $L(\mathscr{K}, \mathscr{H})$ (see 1(j)). Now for all $A \in \mathfrak{A}, B \in \mathfrak{B}$,

$$\begin{aligned} &(\text{a}) \quad [\mathscr{I}(\boldsymbol{T})^*, A \otimes B] = \overline{[\mathscr{I}(\boldsymbol{T}), A^* \otimes B^*]} \\ &(\text{b}) \quad = \overline{[\boldsymbol{T}(A), B^*]} \qquad \text{definition 1(g) of } \mathscr{I} \\ &(\text{c}) \quad = [\boldsymbol{T}(A)^*, B] \end{aligned}$$

where (a) and (c) follow from the properties of the inner product, viz., $\overline{[Y, Z]} = [Y^*, Z^*]$ for all operators Y and Z. Now,

$$[\boldsymbol{T}(A)^*, B] = [\boldsymbol{T}(A^*), B] \qquad \text{for all } A \in \mathfrak{A}, B \in \mathfrak{B}\ ,$$

if and only if $\boldsymbol{T}(A)^* = \boldsymbol{T}(A^*)$ for all $A \in \mathfrak{A}$. Finally, $[\boldsymbol{T}(A^*), B]$ is equal to $[\mathscr{I}(\boldsymbol{T}), A \otimes B]$, so that for all $A \in \mathfrak{A}, B \in \mathfrak{B}$,

$$[\mathscr{I}(\boldsymbol{T}) - \mathscr{I}(\boldsymbol{T})^*, A \otimes B] = 0$$

if and only if $\boldsymbol{T}(A^*) = \boldsymbol{T}(A)^*$. This completes the proof.

REMARK. We have just shown that $\boldsymbol{T} \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ preserves hermitian operators ($\boldsymbol{T} \in \mathscr{C}$) if and only if $\mathscr{I}(\boldsymbol{T})$ is hermitian. It is not unreasonable to suspect that $\boldsymbol{T}$ preserves positive semidefinite (psd) operators ($\boldsymbol{T} \in \mathscr{C}^+$) if and only if $\mathscr{I}(\boldsymbol{T})$ is psd. However, this conjecture is false, for if $\mathfrak{A} = L(\mathscr{H}, \mathscr{H})$, and if $\mathfrak{B} = L(\mathscr{K}, \mathscr{K})$, then for any multiplicative transformation $\boldsymbol{T} \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ ($\boldsymbol{T}(AB) = \boldsymbol{T}(A)\boldsymbol{T}(B)$), we have $\boldsymbol{T} \in \mathscr{C}^+$; but $\mathscr{I}(\boldsymbol{T})$ will always have some negative eigenvalues. For a specific example choose $\mathfrak{A} = \mathfrak{B} = L(\mathscr{H}, \mathscr{H})$, the algebra of operators on $\mathscr{H}$. Let $\boldsymbol{T} \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ be the identity transformation $\boldsymbol{T}(A) = A$ for all $A \in \mathfrak{A}$. Surely $\boldsymbol{T} \in \mathscr{C}^+$. Now choose the o.n. basis $\{e_1, e_2, \cdots, e_n\}$ for $\mathscr{H}$; then $\{(e_i \times e_j): i, j = 1, 2, \cdots, n\}$ is an o.n. basis for $\mathfrak{A}$ so that from Proposition 1.1 Part (2), we have

$$\mathscr{I}(T) = \sum (e_i \times e_j)^* \otimes (e_i \times e_j) = \sum (e_j \times e_i) \otimes (e_i \times e_j) .$$

The situation may be represented by the following diagram:

$$\begin{array}{ccc} \mathfrak{A} = L(\mathscr{H}, \mathscr{H}) & \xrightarrow{\quad T = \text{identity} \quad} & \mathfrak{A} = L(\mathscr{H}, \mathscr{H}) \\ (e_i \times e_j) & & (e_i \times e_j) \\ L(\overline{\mathscr{H}}, \mathscr{H}) & \xrightarrow{\mathscr{I}(T) = \text{transpose}} & L(\overline{\mathscr{H}}, \mathscr{H}) \\ (e_p \times \bar{e}_q) & & (e_q \times \bar{e}_p) . \end{array}$$

From 1(i) and 1(j) we conclude that $\mathscr{I}(T)((e_p \times \bar{e}_q)) = (e_q \times \bar{e}_p)$ for $(e_p \times \bar{e}_q)$, $p, q = 1, 2, \cdots, n$, in the space $L(\overline{\mathscr{H}}, \mathscr{H})$. That is, if T is the identity operator on the Hilbert algebra $L(\mathscr{H}, \mathscr{H})$, then $\mathscr{I}(T)$ is the transpose operator on the Hilbert space $L(\overline{\mathscr{H}}, \mathscr{H})$. It is easy to see that vectors of the form $(e_p \times \bar{e}_q) - (e_q \times \bar{e}_p)$ in $L(\overline{\mathscr{H}}, \mathscr{H})$ are eigenvectors for $\mathscr{I}(T)$ corresponding to the eigenvalue -1. $\mathscr{I}(T)$ (which is hermitian due to Proposition 1.2), is therefore not a psd operator on the Hilbert space $L(\overline{\mathscr{H}}, \mathscr{H})$.

2. The main results. We present a structure theorem which characterizes elements of the cone $\mathscr{C}$.

THEOREM 2.1. *Suppose that* $T \in \mathscr{C} \subset \mathscr{L}(\mathfrak{A}, \mathfrak{B})$. $\mathscr{I}(T)$ *is self-adjoint by Proposition* 1.2, *with spectral resolution* $\sum_i \alpha_i \mathscr{P}(X_i)$, *where* α_i *is real,* $\mathscr{P}(X_i) = (X_i][X_i)$ *is the orthogonal one-dimensional projection on the unit vector* $X_i \in L(\overline{\mathscr{H}}, \mathscr{H})$, *and the* X_i*'s form an* o.n. *basis for* $L(\overline{\mathscr{H}}, \mathscr{H})$. *Let* $A \in \mathfrak{A}$: *then*

$$T(A)^t = \sum_i \alpha_i X_i^* A X_i .$$

Proof. For any $x \in \mathscr{H}$ and $y \in \overline{\mathscr{H}}$,

(1) $$[T(P_x), P_y] = [\mathscr{I}(T), P_x \otimes P_y]$$

(2) $$= \sum_i [\alpha_i (X_i][X_i), (x \times x) \otimes (y \times y)] \qquad \text{from 1(b)}$$

(3) $$= \sum_i [\alpha_i (X_i][X_i), (x \times \bar{y})][(x \times \bar{y})] \qquad \text{from 1(j)}$$

(4) $$= \sum_i \alpha_i \operatorname{tr}((x \times \bar{y}][x \times \bar{y}) \cdot (X_i][X_i)) \qquad \text{from 1(c)}$$

(5) $$= \sum_i \alpha_i [X_i, (x \times \bar{y})][(x \times \bar{y}), X_i]$$

(6) $$= \sum_i \alpha_i \operatorname{tr}((\bar{y} \times x) X_i) \operatorname{tr}(X_i^*(x \times \bar{y}))$$

(7) $$= \sum_i \alpha_i \operatorname{tr}((\bar{e} \times X_i^*(x)) \operatorname{tr}(X_i^*(x \times \bar{y})) \qquad \text{since}$$

$(\bar{y} \times x) X_i = \bar{y} \times X_i^*(x)$; see 1(a)

$$(8) \qquad = \sum_i \alpha_i(\bar{y}, X_i^*(x))(X_i^*(x), \bar{y}) \qquad \text{from } 1(\text{a})$$

Now for $w_1, w_2 \in \mathscr{H}$ and $u_1, u_2 \in \mathscr{K}$, we have that

$$(u_2, u_1)(w_1, w_2) = [(w_1 \times u_1), (w_2 \times u_2)] \qquad (\text{see } 1\ (\text{c})).\ ,$$

so (8) becomes

$$(9) \qquad = \sum_i \alpha_i[(X_i^*(x) \times X_i^*(x)), (\bar{y} \times \bar{y})]$$

$$(10) \qquad = \sum_i [\alpha_i X_i^*(x \times x)X_i, (P_y)^t]\ .$$

Since the transpose is a self-adjoint operator, equation (10) becomes

$$(11) \qquad = \sum_i [\alpha_i(X_i^* P_x X_i)^t, P_y]\ .$$

Thus, for every $x \in \mathscr{H}$ and every $y \in \mathscr{K}$,

$$\left[T(P_x) - \left(\sum_i \alpha_i X_i^* P_x X_i\right)^t, P_y\right] = 0\ .$$

But then,

$$T(P_x) = (\sum \alpha_i X_i^* P_x X_i)^t$$

for all $P_x \in \mathfrak{A}$. Since the transpose operator squared is the identity, we may apply it to both sides of the last equation to obtain

$$(12) \qquad T(P_x)^t = \sum \alpha_i X_i^* P_x X_i$$

for all $P_x \in \mathfrak{A}$. This result extends from the set of one dimensional orthogonal projections P_x to hermitian operators; this, in turn, extends to arbitrary operators of $\mathfrak{A}$. Thus, the theorem is proved.

REMARK. Suppose the dimension of $\mathscr{H} = n$ and the dimension of $\mathscr{K} = m$, where $\mathscr{H}$ and $\mathscr{K}$ are the underlying Hilbert spaces for the operator algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Relative to certain ordered bases for $\mathscr{H}$ and $\mathscr{K}$, each operator of $\mathfrak{A}$ and $\mathfrak{B}$ is identified with a certain square matrix. The o.n. basis vectors X_i of $L(\overline{\mathscr{K}}, \mathscr{H})$ are then realized as certain $n \times m$ matrices; the operator X_i^* is identified with the $m \times n$ conjugate transpose matrix of X_i. Thus, Theorem 2.1 may be interpreted as saying that any linear transformation T, sending the full matrix algebra $\mathfrak{A}$ to the full matrix algebra $\mathfrak{B}$ is of the form

$$T(A) = \left(\sum_i \alpha_i X_i^* A X_i\right)^t$$

for certain real scalars α_i and certain $n \times m$ matrices X_i, if and only if $\boldsymbol{T}$ preserves hermitian matrices. Equivalently,

$$\begin{aligned}\boldsymbol{T}(A) &= \Big(\sum_i \alpha_i X_i^* A X_i\Big)^t \\ &= \sum_i \alpha_i X_i^t A^t (X_i^*)^t \\ &= \sum_i \alpha_i Y_i^* A^t Y_i \qquad\qquad \text{setting } Y_i = (X_i^*)^t\end{aligned}$$

for certain real scalars α_i and certain $n \times m$ matrices Y_i depending on $\boldsymbol{T}$, characterizes those transformations $\boldsymbol{T}\colon \mathfrak{A} \to \mathfrak{B}$ which preserve hermitian matrices.

COROLLARY 2.2. *Let* $\boldsymbol{T} \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ *where* $\mathscr{J}(\boldsymbol{T})$ *is* psd *in* $\mathfrak{A} \otimes \mathfrak{B}$. *Then* $\boldsymbol{T} \in \mathscr{C}^+ \subset \mathscr{L}(\mathfrak{A}, \mathfrak{B})$.

Proof. Since $\mathscr{J}(\boldsymbol{T})$ is psd in $\mathfrak{A} \otimes \mathfrak{B}$, $\mathscr{J}(\boldsymbol{T})$ has spectral resolution $\sum \alpha_i \mathscr{P}(X_i)$ where the scalars α_i are nonnegative, $\mathscr{P}(X_i)$ is the orthogonal one-dimensional projection onto $X_i \in L(\overline{\mathscr{K}}, \mathscr{H})$ and the X_i's form an o.n. basis for $L(\overline{\mathscr{K}}, \mathscr{H})$. Since $\mathscr{J}(\boldsymbol{T})$ is psd, it is, *a fortiori*, self-adjoint, so that $\boldsymbol{T}$ is at least an element of the cone $\mathscr{C}$ (Proposition 1.2). But this gives us sufficient leverage to employ the representation of Theorem 2.1. Hence, $\boldsymbol{T}(\cdot)^t = \sum \alpha_i X_i^*(\cdot) X_i$ where the α_i's are nonnegative scalars. In order to show that $\boldsymbol{T}$ sends psd operators to psd operators (i.e., $\boldsymbol{T} \in \mathscr{C}^+$), it is (necessary and) sufficient to show that $\boldsymbol{T}$ sends one-dimensional orthogonal projections P_x to psd operators; to do this, it is (necessary and) sufficient to show that the operator $\boldsymbol{T}(\cdot)^t$ sends these projections P_x to psd operators. But

$$\boldsymbol{T}(P_x)^t = \sum \alpha_i (X_i^* P_x X_i)$$

from Theorem 2.1. Observe that each term $X_i^* P_x X_i = (P_x X_i)^* (P_x X_i)$ is psd, and hence, so is $\sum_i \alpha_i X_i^* P_x X_i$, the sum of nonnegative multiples of these psd terms. The proof is done.

We come to our final theorem which tells us that the cone $\mathscr{C}^+$ "generates" the space $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ in much the same way that the cone of psd operators (in $\mathfrak{A}$, say) "generates" $\mathfrak{A}$.

THEOREM 2.3. *Suppose* $\boldsymbol{T} \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$. *Then for some* $\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, \boldsymbol{K}_4 \in \mathscr{C}^+$,

$$\boldsymbol{T} = (\boldsymbol{K}_1 - \boldsymbol{K}_2) + i(\boldsymbol{K}_3 - \boldsymbol{K}_4)$$

where $i^2 = -1$

Proof. $\mathscr{J}(\boldsymbol{T})$, an element of the algebra $\mathfrak{A} \otimes \mathfrak{B}$ can be decomposed as follows:

$$(*) \qquad \mathscr{I}(\boldsymbol{T}) = (\boldsymbol{U}_1 - \boldsymbol{U}_2) + i(\boldsymbol{U}_3 - \boldsymbol{U}_4) \,,$$

where each of the $\boldsymbol{U}_i$'s is psd in $\mathfrak{A} \otimes \mathfrak{B}$. Proposition 1.1, Part (5), tells us that $\mathscr{I}: \mathscr{L}(\mathfrak{A}, \mathfrak{B}) \rightarrow \mathfrak{A} \otimes \mathfrak{B}$ is an isometry. Since the (vector space) dimensions of $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{A} \otimes \mathfrak{B}$ agree, $\mathscr{I}$ is, in fact, one-to-one and onto; thus, $\mathscr{I}^{-1}$ exists as a well-defined linear operator. Applying $\mathscr{I}^{-1}$ to $(*)$ yields

$$\boldsymbol{T} = [\mathscr{I}^{-1}(\boldsymbol{U}_1) - \mathscr{I}^{-1}(\boldsymbol{U}_2)] + i[\mathscr{I}^{-1}(\boldsymbol{U}_3) - \mathscr{I}^{-1}(\boldsymbol{U}_4)] \,.$$

Now let $\boldsymbol{K}_i = \mathscr{I}^{-1}(\boldsymbol{U}_i)$, $i = 1, 2, 3, 4$. Corollary 2.2 forces us to conclude that $\boldsymbol{K}_i \in \mathscr{C}^+$ since $\mathscr{I}(\boldsymbol{K}_i) = \boldsymbol{U}_i$ is psd. Thus, for any $\boldsymbol{T} \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$

$$\boldsymbol{T} = (\boldsymbol{K}_1 - \boldsymbol{K}_2) + i(\boldsymbol{K}_3 - \boldsymbol{K}_4)$$

where each $\boldsymbol{K}_i \in \mathscr{C}^+ \subset \mathscr{L}(\mathfrak{A}, \mathfrak{B})$.

BIBLIOGRAPHY

1. Hoffman and Kunze, *Lineara algebra*, Prentice Hall, New York, 1961.

2. Mostow, Sampson and Meyer, *Fundamental structures of algebra*, McGraw Hill, New York, 1963.

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