LINEAR TRANSFORMATIONS WHICH PRESERVE HERMITIAN AND POSITIVE SEMIDEFINITE OPERATORS

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Let $\mathfrak{A}$ and $\mathfrak{B}$ represent the full algebras of linear operators on the finite-dimensional unitary spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. The symbol $\mathcal{L}(\mathfrak{A}, \mathfrak{B})$ will denote the complex space of all linear maps from $\mathfrak{A}$ to $\mathfrak{B}$. This paper concerns itself with the study of the following two cones in $\mathcal{L}(\mathfrak{A}, \mathfrak{B})$:

(i) the cone $\mathcal{E}$ of all $T \in \mathcal{L}(\mathfrak{A}, \mathfrak{B})$ which send hermitian operators in $\mathfrak{A}$ to hermitian operators in $\mathfrak{B}$, and

(ii) the subcone $\mathcal{E}^+$ (of $\mathcal{E}$) of all $T \in \mathcal{L}(\mathfrak{A}, \mathfrak{B})$ which send positive semidefinite operators in $\mathfrak{A}$ to positive semidefinite operators in $\mathfrak{B}$.

In our main results, we characterize the transformations in the cone $\mathcal{E}$ (Theorem 2.1) and present a structure theorem concerning the transformations in the cone $\mathcal{E}^+$ (Theorem 2.3). Identifying operators in the algebras $\mathfrak{A}$ and $\mathfrak{B}$ with appropriate square matrices, we may summarize Theorem 2.1 by saying that any and every linear transformation $T$ which preserves hermitian matrices is of the form $T: A \rightarrow \sum \alpha_i X_i^* A X_i$, where each $\alpha_i$ is a real scaler, and each $X_i$ is a certain rectangular matrix depending on $T$; $X_i^*$ and $A^t$ represent the conjugate transpose and the transpose of matrices $X_i$ and $A$, respectively. Theorem 2.3 says that the cone of positive semidefinite-preserving transformations $\mathcal{E}^+$ "generates" or spans all of $\mathcal{L}(\mathfrak{A}, \mathfrak{B})$ in the sense that any $T$ in $\mathcal{L}(\mathfrak{A}, \mathfrak{B})$ can be written

$$T = (K_i - K_i') + i(K_i - K_i'),$$

where $i^2 = -1$, and each $K_i$ is an element of $\mathcal{E}^+$.

1. Preliminaries. $L(\mathcal{H}, \mathcal{K})$ denotes the space of linear transformations from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{K}$. We define:

1 (a). $(x \times y)$—the dyad transformation, an element of $L(\mathcal{H}, \mathcal{K})$, is defined for fixed $x \in \mathcal{H}$ and $y \in \mathcal{H}$ by: $(x \times y)(z) = (z, y)x$ for all $z \in \mathcal{H}$, where $(z, y)$ is the inner product of $z$ with $y$. As it turns out, $(x, y) = \text{tr}((x \times y))$, the trace of $(x \times y)$. If $A \in \mathfrak{A}(=L(\mathcal{H}, \mathcal{K}))$ and $B \in \mathfrak{B}(=L(\mathcal{H}, \mathcal{K}^*))$, then $(A(x) \times B(y)) = A(x \times y)B^*.$

1 (b). $P_x$—denotes the orthogonal projection onto the subspace spanned by $x$, i.e., for $(x, x) = 1$, we have $P_x = (x \times x)$.  

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1 (c). \([A, B]\) is the inner product defined on \( \mathcal{H} \) (resp. \( \mathcal{K} \)) by setting \([A, B] = \text{tr}(B^*A)\) for all \( A, B \in \mathcal{H} \) (resp. \( \mathcal{K} \)) where \( B^* \) is the Hilbert space adjoint of \( B \), and \( \text{tr}(\cdot) \) is the trace functional on \( \mathcal{H} \) (resp. \( \mathcal{K} \)). More generally, \( L(\mathcal{H}, \mathcal{K}) \) becomes a Hilbert space once we define the inner product \([A, B] = \text{tr}(B^*A)\) for all \( A, B \in L(\mathcal{H}, \mathcal{K}) \). Consequently, for \( w_1, w_2 \in \mathcal{H} \), and \( u_1, u_2 \in \mathcal{K} \), so that \((w_1 \times u_1)\) and \((w_2 \times u_2)\) belong to \( L(\mathcal{H}, \mathcal{K}) \), we have

\[
[(w_1 \times u_1), (w_2 \times u_2)] = \text{tr}((w_2 \times u_2)^*(w_1 \times u_1)) = \text{tr}((u_2 \times w_2)(w_1 \times u_1)) = \text{tr}((w_1, w_2)(u_2 \times u_1)) = (w_1, w_2)(u_2, u_1).
\]

1 (d). \((A|B)\)—the dyad transformation, an element of \( \mathcal{L}(\mathcal{K}, \mathcal{H}) \), is defined for fixed transformations \( A \in \mathcal{H} \) and \( B \in \mathcal{K} \) by \((A|B)C = [C, B]A\), for all \( C \) in \( B \). As in 1 (a)., \([A, B] = \text{tr}((A|B))\), the trace of \((A|B)\).

1 (e). \( \mathcal{H} \otimes \mathcal{K} \)—the tensor product of algebras \( \mathcal{H} \) and \( \mathcal{K} \), consists of sums of elements of the form \( A \otimes B \), where \( A \in \mathcal{H} \) and \( B \in \mathcal{K} \) [2, Chapter 16]. The symbol \((A \otimes B)^*\) will denote the element \( B \otimes A \), and can be linearly extended to any element of \( \mathcal{H} \otimes \mathcal{K} \).

1 (f). \([A_1 \otimes B_1, A_2 \otimes B_2]\)—the inner product which gives the algebra \( \mathcal{H} \otimes \mathcal{K} \) a Hilbert space structure, is defined by

\[
[A_1 \otimes B_1, A_2 \otimes B_2] = [A_1, A_2] \cdot [B_1, B_2]
\]

for all \( A_1, A_2 \in \mathcal{H} \), and all \( B_1, B_2 \in \mathcal{K} \).

1 (g). \( \mathcal{F}(T)\)—the element of \( \mathcal{H} \otimes \mathcal{K} \) which is defined for each \( T \) in \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) by \([\mathcal{F}(T), A^* \otimes B] = [T(A), B]\), for all \( A \in \mathcal{H}, B \in \mathcal{K} \). This equation also defines \( \mathcal{F} \) as a linear transformation, sending the space \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) to the algebra \( \mathcal{H} \otimes \mathcal{K} \).

1 (h). \( \mathcal{H}^* \)—the space of all linear functionals on \( \mathcal{H} \). For each \( x \in \mathcal{H} \), we define the functional \( \bar{x} \in \mathcal{H}^* \) by \( \bar{x}(y) = (y, x) \) for all \( y \in \mathcal{H} \). Moreover, these are the only elements of \( \mathcal{H}^* \). An inner product is defined on \( \mathcal{H}^* \) by setting \((\bar{x}, \bar{y}) = (y, x)\) for all \( \bar{x}, \bar{y} \in \mathcal{H}^* \). Thus, \((\bar{x}, \bar{y}) = (\bar{y}, \bar{x})\), the complex conjugate of \((y, x)\).

1 (i). \( A^t \)—the transpose of the operator \( A \), is the linear operator on \( \mathcal{H} \) defined by \( A^t(\bar{y})(x) = \bar{y}(A(x)) \), for all \( \bar{y} \in \mathcal{H}^* \), and all \( x \in \mathcal{H} \).
[1, p. 103]. From this it follows that \((x \times y)^t = (\bar{y} \times \bar{x})\). If \(\bar{A}\) is defined to be \((A^*)^t\), then \((x \times y) = (\bar{x} \times \bar{y})\) and \(\bar{A}(\bar{x}) = \bar{A}(x)\). From this we see that for all \(A \in \mathcal{H}, \bar{A}^* = A^t\). In fact, set \(A = (x \times y)\) for \(x, y \in \mathcal{H}\). Then
\[
\bar{A}^* = (x \times y)^* = (\bar{x} \times \bar{y})^* = (\bar{y} \times \bar{x}) = (y \times x) = (x \times y)^t = A^t.
\]
Hence, by linear extension, \(\bar{A}^* = A^t\) for all \(A \in \mathcal{H}\).

1 (j). \(L(\mathcal{H}, \mathcal{H})\) is spanned by the dyads \((x \times y)\), where \(x \in \mathcal{H}\) and \(\bar{y} \in \mathcal{H}\). In this context, we identify the transformation \(A \otimes B\) with the transformation \(C \rightarrow ACB^t\) for all \(C \in L(\mathcal{H}, \mathcal{H})\), where \(A \in \mathcal{H}(= L(\mathcal{H}, \mathcal{H}))\) and \(B \in \mathcal{B}(= L(\mathcal{H}, \mathcal{H}))\). Behind this identification is the isomorphism \(\phi: \mathcal{H} \otimes \mathcal{H} \rightarrow L(\mathcal{H}, \mathcal{H})\) defined by \(\phi(x \otimes y) = (x \times \bar{y})\) for all \(x \in \mathcal{H}, y \in \mathcal{H}\). If for each \(A \in \mathcal{H}, B \in \mathcal{B}\) we define the linear transformation \(O_{A,B}: L(\mathcal{H}, \mathcal{H}) \rightarrow L(\mathcal{H}, \mathcal{H})\) by \(O_{A,B}(C) = ACB^t\) for all \(C \in L(\mathcal{H}, \mathcal{H})\), then \(A \otimes B\) corresponds to \(O_{A,B}\) in the sense that \(\phi \circ (A \otimes B) \circ \phi^{-1} = O_{A,B}\). In fact, we have
\[
(\phi \circ (A \otimes B) \circ \phi^{-1}(x \times \bar{y})) = \phi(A \otimes B(x \otimes y)) \quad \text{definition of } \phi^{-1}
\]
\[
= \phi(A(x) \otimes B(y)) \quad \text{definition of } A \otimes B
\]
\[
= (A(x) \times \bar{B}(y)) \quad \text{definition of } \phi
\]
\[
= (A(x) \times \bar{B}(\bar{y})) \quad \text{from 1 (i)}.
\]
\[
= A(x \times \bar{y})\bar{B}^* \quad \text{from 1 (a)}.
\]
\[
= A(x \times \bar{y})B^t \quad \text{since } \bar{B}^* = B^t, \text{ see 1 (i)}.
\]
\[
= O_{A,B}((x \times \bar{y})) \quad \text{definition of } O_{A,B}.
\]

For convenience, however, we shall treat \(A \otimes B\) as though it were actually equal to the concrete linear transformation \(O_{A,B} = A(\cdot)B^t\). In so doing, we have
\[
(x \times y)^t[(u \times v) = (x \times u) \otimes (\bar{y} \times \bar{v})
\]
for vectors \(x, y, u, v\) in (not necessarily the same) Hilbert space.

The linear transformation \(\mathcal{S}\) (see 1(g).) will prove to be of fundamental importance. For this reason, we isolate some of its properties in

**Proposition 1.1.**

1. \(\mathcal{S}(B \| A) = A^* \otimes B\) for all \(A \in \mathcal{H}, B \in \mathcal{B}\).
2. \(\mathcal{S}(T) = \sum_i E_i^* \otimes T(E_i)\) for any and every orthonormal basis \(\{E_i\}\) for \(\mathcal{H}\).
3. If \(T(A^*) = T(A)^*\) for all \(A \in \mathcal{H}\) (i.e., if \(T \in \mathcal{S}\)), then \(\mathcal{S}(T) = \sum_i T^*(F_i) \otimes F_i^*\) for any orthonormal basis \(\{F_i\}\) for \(\mathcal{B}\).
4. If \(T(A^*) = T(A)^*\) for all \(A \in \mathcal{H}\), then \(\mathcal{S}(T^*) = \mathcal{S}(T)^0\).
is an isometric isomorphism from the Hilbert space \( \mathcal{L}(\mathfrak{A}, \mathfrak{B}) \) onto the Hilbert algebra \( \mathfrak{A} \otimes \mathfrak{B} \).

**Proof.** From the definition 1(g) of \( \mathcal{F} \), we have

\[
[\mathcal{F}(B)[A], C \otimes D] = [(B)[A](C^*), D]
\]

\[
= [C^*, A][B, D]
\]

\[
= [A^*, C][B, D]
\]

\[
= [A^* \otimes B, C \otimes D]
\]

for all \( A, C \in \mathfrak{A} \) and all \( B, D \in \mathfrak{B} \). This implies Part (1).

Now let \( \{E_i\} \) be any orthonormal (o.n.) basis for \( \mathfrak{A} \). If \( T = (B)[A] \) for \( A \in \mathfrak{A} \) and \( B \in \mathfrak{B} \), then

\[
\sum_i E_i^* \otimes T(E_i) = \sum_i E_i^* \otimes (B)[A](E_i)
\]

\[
= \sum_i [E_i, A]E_i^* \otimes B
\]

\[
= \sum_i [A^*, E_i^*]E_i^* \otimes B
\]

\[
= A^* \otimes B
\]

which, from Part (1)

\[
= \mathcal{F}(B)[A].
\]

The dyads \( (B)[A], A \in \mathfrak{A}, B \in \mathfrak{B}, \) span the space \( \mathcal{L}(\mathfrak{A}, \mathfrak{B}) \), so that (using linearity of \( \mathcal{F} \)) for all \( T \in \mathcal{L}(\mathfrak{A}, \mathfrak{B}), \mathcal{F}(T) = \sum_i E_i^* \otimes T(E_i) \), which establishes Part (2).

Part (3) follows from (2) and (4) inasmuch as if \( \mathcal{F}(T^*) = \mathcal{F}(T)^0 \), then \( \sum T^*(F_i) \otimes F_i^* = (\sum F_i^* \otimes T^*(F_i))^0 = \mathcal{F}(T^*)^0 = \mathcal{F}(T)^0 = \mathcal{F}(T) \)

But Part (4) obtains, since for all \( A \in \mathfrak{A}, B \in \mathfrak{B}, \)

\[
[\mathcal{F}(T^*), A \otimes B] = [T^*(A^*), B]
\]

\[
= [T(B^*), A]
\]

\[
= [T(B^*), A]
\]

\[
= [\mathcal{F}(T), B \otimes A]
\]

\[
= [\mathcal{F}(T)^0, A \otimes B].
\]

That is, \( \mathcal{F}(T^*) = \mathcal{F}(T)^0 \) and Part (4) is proven.

As for demonstrating Part (5), observe that for all \( A_1, A_2 \in \mathfrak{A}, \) and \( B_1, B_2 \in \mathfrak{B}, \)

\[
[\mathcal{F}(B_1)[A_1], \mathcal{F}(B_2)[A_2]] = [A_1^* \otimes B_1, A_2^* \otimes B_2]
\]

\[
= [A_1^*, A_2^*] \text{ tr } ((B_1)[B_2])
\]

\[
= \text{ tr } ((B_1)[A_1] \cdot (B_2)[A_2]^*)
\]

\[
= [(B_1)[A_1], (B_2)[A_2]].
\]
By linear extension on each argument of the inner product, we have that for all \( T_1, T_2 \in \mathcal{H}(\mathfrak{A}, \mathfrak{B}) \),

\[
[\mathcal{F}(T_1), \mathcal{F}(T_2)] = [T_1, T_2]
\]

so that \( \mathcal{F} \) is an isometry from \( \mathcal{H}(\mathfrak{A}, \mathfrak{B}) \) to \( \mathfrak{A} \otimes \mathfrak{B} \). From Part (1) it is easy to see that \( \mathcal{F} \) is also an onto transformation as well, since the algebra \( \mathfrak{A} \otimes \mathfrak{B} \) is spanned by elements of the form \( A^* \otimes B \). This completes the proof of Proposition 1.1.

Our next result establishes a necessary and sufficient condition for a transformation in \( \mathcal{H}(\mathfrak{A}, \mathfrak{B}) \) to be in the cone \( \mathfrak{C} \).

**Proposition 1.2.** A transformation \( T \in \mathcal{H}(\mathfrak{A}, \mathfrak{B}) \) is in \( \mathfrak{C} \) if and only if \( \mathcal{F}(T) \) is hermitian.

**Proof.** Recall that \( \mathcal{F} \) maps \( \mathcal{H}(\mathfrak{A}, \mathfrak{B}) \) (isometrically) onto \( \mathfrak{A} \otimes \mathfrak{B} \), which has been identified as the algebra of linear operators on the Hilbert space \( L(\mathfrak{H}, \mathfrak{K}) \) (see 1(j)). Now for all \( A \in \mathfrak{A}, B \in \mathfrak{B} \),

\[
\begin{align*}
(a) \quad [\mathcal{F}(T)^*, A \otimes B] &= [\mathcal{F}(T), A^* \otimes B^*] \\
(b) \quad &= [T(A), B^*] \quad \text{definition 1(g) of \( \mathcal{F} \)}
\end{align*}
\]

where (a) and (c) follow from the properties of the inner product, viz., \( [Y, Z] = [Y^*, Z^*] \) for all operators \( Y \) and \( Z \). Now,

\[
[T(A)^*, B] = [T(A^*), B]
\]

for all \( A \in \mathfrak{A}, B \in \mathfrak{B} \), if and only if \( T(A)^* = T(A^*) \) for all \( A \in \mathfrak{A} \). Finally, \( [T(A^*), B] \) is equal to \( [\mathcal{F}(T), A \otimes B] \), so that for all \( A \in \mathfrak{A}, B \in \mathfrak{B} \),

\[
[\mathcal{F}(T) - \mathcal{F}(T)^*, A \otimes B] = 0
\]

if and only if \( T(A^*) = T(A)^* \). This completes the proof.

**Remark.** We have just shown that \( T \in \mathcal{H}(\mathfrak{A}, \mathfrak{B}) \) preserves hermitian operators \( (T \in \mathfrak{C}) \) if and only if \( \mathcal{F}(T) \) is hermitian. It is not unreasonable to suspect that \( T \) preserves positive semidefinite (psd) operators \( (T \in \mathfrak{C}^+) \) if and only if \( \mathcal{F}(T) \) is psd. However, this conjecture is false, for if \( \mathfrak{A} = L(\mathfrak{K}, \mathfrak{K}) \), and if \( \mathfrak{B} = L(\mathfrak{K}, \mathfrak{K}) \), then for any multiplicative transformation \( T \in \mathcal{H}(\mathfrak{A}, \mathfrak{B}) \) \( (T(AB) = T(A)T(B)) \), we have \( T \in \mathfrak{C}^+ \); but \( \mathcal{F}(T) \) will always have some negative eigenvalues. For a specific example choose \( \mathfrak{A} = \mathfrak{B} = L(\mathfrak{K}, \mathfrak{K}) \), the algebra of operators on \( \mathfrak{K} \). Let \( T \in \mathcal{H}(\mathfrak{A}, \mathfrak{B}) \) be the identity transformation \( T(A) = A \) for all \( A \in \mathfrak{A} \). Surely \( T \in \mathfrak{C}^+ \). Now choose the o.n. basis \( \{e_1, e_2, \ldots, e_n\} \) for \( \mathfrak{K} \); then \( \{(e_i \times e_j): i, j = 1, 2, \ldots, n\} \) is an o.n. basis for \( \mathfrak{A} \) so that from Proposition 1.1 Part (2), we have
\[ \mathcal{I}(T) = \sum (e_i \times e_j)^* \otimes (e_i \times e_j) = \sum (e_i \times e_i) \otimes (e_i \times e_j). \]

The situation may be represented by the following diagram:

\[ \begin{array}{ccc}
\mathcal{A} = L(\mathcal{H}, \mathcal{H}) & T = \text{identity} & \mathcal{A} = L(\mathcal{H}, \mathcal{H}) \\
(e_i \times e_j) & \otimes & (e_i \times e_j) \\
L(\mathcal{H}, \mathcal{H}) & \text{transpose} & L(\mathcal{H}, \mathcal{H}) \\
(e_p \times \bar{e}_q) & \otimes & (e_q \times \bar{e}_p). 
\end{array} \]

From 1(i) and 1(j) we conclude that \( \mathcal{I}(T)((e_p \times \bar{e}_q)) = (e_q \times \bar{e}_p) \) for \((e_p \times \bar{e}_q), p, q = 1, 2, \cdots, n, \) in the space \( L(\mathcal{H}, \mathcal{H}). \) That is, if \( T \) is the identity operator on the Hilbert algebra \( L(\mathcal{H}, \mathcal{H}), \) then \( \mathcal{I}(T) \) is the transpose operator on the Hilbert space \( L(\mathcal{H}, \mathcal{H}). \) It is easy to see that vectors of the form \((e_p \times \bar{e}_q) - (e_q \times \bar{e}_p) \) in \( L(\mathcal{H}, \mathcal{H}) \) are eigenvectors for \( \mathcal{I}(T) \) corresponding to the eigenvalue \(-1.\) \( \mathcal{I}(T) \) (which is hermitian due to Proposition 1.2) is therefore not a psd operator on the Hilbert space \( L(\mathcal{H}, \mathcal{H}). \)

**2. The main results.** We present a structure theorem which characterizes elements of the cone \( \mathcal{C}. \)

**Theorem 2.1.** Suppose that \( T \in \mathcal{C} \subset \mathcal{L}(\mathcal{A}, \mathcal{B}). \) \( \mathcal{I}(T) \) is self-adjoint by Proposition 1.2, with spectral resolution \( \sum_i \alpha_i \mathcal{P}(X_i), \) where \( \alpha_i \) is real, \( \mathcal{P}(X_i) = (X_i)[X_i] \) is the orthogonal one-dimensional projection on the unit vector \( X_i \in L(\mathcal{H}, \mathcal{H}), \) and the \( X_i \)'s form an o.n. basis for \( L(\mathcal{H}, \mathcal{H}). \) Let \( A \in \mathcal{A}; \) then

\[ T(A)^t = \sum_i \alpha_i X_i^*AX_i. \]

**Proof.** For any \( x \in \mathcal{H} \) and \( y \in \mathcal{H}, \)

1. \[ [T(P_x), P_y] = [\mathcal{I}(T), P_x \otimes P_y] \]
2. \[ = \sum_i [\alpha_i(X_i)[X_i], (x \times x) \otimes (y \times y)] \quad \text{from 1(b)} \]
3. \[ = \sum_i [\alpha_i(X_i)[X_i], (x \times \bar{y})][(x \times \bar{y})] \quad \text{from 1(j)} \]
4. \[ = \sum \alpha_i \text{tr} ((x \times \bar{y})[x \times \bar{y}](X_i)[X_i]) \quad \text{from 1(c)} \]
5. \[ = \sum \alpha_i[X_i, (x \times \bar{y})][(x \times \bar{y}), X_i] \]
6. \[ = \sum \alpha_i \text{tr} ((\bar{y} \times x)X_i) \text{tr}(X_i^*(x \times \bar{y})) \]
7. \[ = \sum \alpha_i \text{tr} ((\bar{e} \times X_i^*(x)) \text{tr}(X_i^*(x \times \bar{y})) \quad \text{since} \]

\((\bar{y} \times x)X_i = \bar{y} \times X_i^*(x); \quad \text{see 1(a)}\)
Now for $w_1, w_2 \in \mathcal{H}$ and $u_1, u_2 \in \mathcal{K}$, we have that

$$(u_2, u_1)(w_1, w_2) = [(w_1 \times u_1), (w_2 \times u_2)]$$

(see 1(e)).

so (8) becomes

$$(9) = \sum \alpha_i[(X_i^*(x) \times X_i^*(x)), (\bar{y} \times \bar{y})]$$

Since the transpose is a self-adjoint operator, equation (10) becomes

$$(11) = \sum [\alpha_i(X_i^*P_xX_i)^t, P_y].$$

Thus, for every $x \in \mathcal{H}$ and every $y \in \mathcal{K}$,

$$\left[T(P_x) - \left(\sum \alpha_iX_i^*P_xX_i\right)^t, P_y \right] = 0.$$
if \( T \) preserves hermitian matrices. Equivalently,

\[
T(A) = \left( \sum_i \alpha_i X_i^* A X_i \right)'
\]

\[
= \sum_i \alpha_i X_i' A' (X_i)^t
\]

\[
= \sum_i \alpha_i Y_i^* A' Y_i
\]

setting \( Y_i = (X_i)^t \)

for certain real scalars \( \alpha_i \) and certain \( n \times m \) matrices \( Y_i \) depending on \( T \), characterizes those transformations \( T: \mathbb{A} \rightarrow \mathbb{B} \) which preserve hermitian matrices.

**Corollary 2.2.** Let \( T \in \mathcal{L}(\mathbb{A}, \mathbb{B}) \) where \( \mathcal{I}(T) \) is psd in \( \mathbb{A} \otimes \mathbb{B} \). Then \( T \in \mathcal{E}^+ \subset \mathcal{L}(\mathbb{A}, \mathbb{B}) \).

**Proof.** Since \( \mathcal{I}(T) \) is psd in \( \mathbb{A} \otimes \mathbb{B} \), \( \mathcal{I}(T) \) has spectral resolution \( \sum \alpha_i \mathcal{P}(X_i) \) where the scalars \( \alpha_i \) are nonnegative, \( \mathcal{P}(X_i) \) is the orthogonal one-dimensional projection onto \( X_i \in L(\mathcal{H}, \mathcal{H}) \) and the \( X_i \)'s form an o.n. basis for \( L(\mathcal{H}, \mathcal{H}) \). Since \( \mathcal{I}(T) \) is psd, it is, a fortiori, self-adjoint, so that \( T \) is at least an element of the cone \( \mathcal{C} \) (Proposition 1.2). But this gives us sufficient leverage to employ the representation of Theorem 2.1. Hence, \( T(\cdot)' = \sum \alpha_i X_i^*(\cdot) X_i \) where the \( \alpha_i \)'s are nonnegative scalars. In order to show that \( T \) sends psd operators to psd operators (i.e., \( T \in \mathcal{E}^+ \)), it is (necessary and) sufficient to show that \( T \) sends one-dimensional orthogonal projections \( P_x \) to psd operators; to do this, it is (necessary and) sufficient to show that the operator \( T(\cdot)' \) sends these projections \( P_x \) to psd operators. But

\[
T(P_x)' = \sum \alpha_i (X_i^* P_x X_i)
\]

from Theorem 2.1. Observe that each term \( X_i^* P_x X_i = (P_x X_i)^*(P_x X_i) \) is psd, and hence, so is \( \sum_i \alpha_i X_i^* P_x X_i \), the sum of nonnegative multiples of these psd terms. The proof is done.

We come to our final theorem which tells us that the cone \( \mathcal{E}^+ \) "generates" the space \( \mathcal{L}(\mathbb{A}, \mathbb{B}) \) in much the same way that the cone of psd operators (in \( \mathbb{A} \), say) "generates" \( \mathbb{A} \).

**Theorem 2.3.** Suppose \( T \in \mathcal{L}(\mathbb{A}, \mathbb{B}) \). Then for some \( K_1, K_2, K_3, K_i \in \mathcal{E}^+ \),

\[
T = (K_1 - K_2) + i(K_3 - K_i)
\]

where \( i^2 = -1 \)

**Proof.** \( \mathcal{I}(T) \), an element of the algebra \( \mathbb{A} \otimes \mathbb{B} \) can be decomposed as follows:
\[(*) \quad \mathcal{F}(T) = (U_1 - U_2) + i(U_3 - U_4),\]

where each of the $U_i$'s is psd in $\mathbb{A} \otimes \mathbb{B}$. Proposition 1.1, Part (5), tells us that $\mathcal{F}: \mathcal{L}(\mathbb{A}, \mathbb{B}) \to \mathbb{A} \otimes \mathbb{B}$ is an isometry. Since the (vector space) dimensions of $\mathcal{L}(\mathbb{A}, \mathbb{B})$ and $\mathbb{A} \otimes \mathbb{B}$ agree, $\mathcal{F}$ is, in fact, one-to-one and onto; thus, $\mathcal{F}^{-1}$ exists as a well-defined linear operator. Applying $\mathcal{F}^{-1}$ to $(*)$ yields

\[T = \left[\mathcal{F}^{-1}(U_1) - \mathcal{F}^{-1}(U_2)\right] + i\left[\mathcal{F}^{-1}(U_3) - \mathcal{F}^{-1}(U_4)\right].\]

Now let $K_i = \mathcal{F}^{-1}(U_i), i = 1, 2, 3, 4$. Corollary 2.2 forces us to conclude that $K_i \in \mathbb{C}^+$ since $\mathcal{F}(K_i) = U_i$ is psd. Thus, for any $T \in \mathcal{L}(\mathbb{A}, \mathbb{B})$

\[T = (K_1 - K_2) + i(K_3 - K_4)\]

where each $K_i \in \mathbb{C}^+ \subset \mathcal{L}(\mathbb{A}, \mathbb{B})$.

**Bibliography**


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