If $S$ is a locally compact and Hausdorff space and $A$ is a continuous linear operator from $C_c(S)$ into the space $C(T)$ with the supremum norm topology then the Riesz Representation Theorem yields the formula $\mathcal{A}_f(x) = \int_S f(y)\lambda(x, dy)$, where for each $x \in T$, $\lambda(x, \cdot)$ is a complex-valued regular Borel measure on $S$. More generally a study is made of kernel functions $\lambda$ such that $\int_S f(y)\lambda(\cdot, dy) \in C(T)$ for $f$ of compact support on $S$. It is shown that $\lambda(\cdot, E)$ is measurable for each Borel set $E$ and that $\mu(E) = \int_T \lambda(x, E)v(dx)$ is a regular measure on $S$ yielding the adjoint formula $A^*v = \mu$. Necessary and sufficient conditions are given on $\lambda$ so that $A^{**}(C(S)) \subseteq C(T)$ and that $A^{**}$ be continuous from $C(S)_b$ to $C(T)_b$ when $S$ is paracompact. Furthermore, kernel representations of $\beta$-continuous operators are studied with applications to semi-groups of operators in $C_c(S)$ and $C(S)_\beta$ when $S$ is locally compact.

We point out that as a consequence of our work the condition (1.7) in the paper by Foguel [7] follows from (1.6) when the space is locally compact and Hausdorff. Further the regularity of the above measure yields the more specific vector-valued measure representation of $A$, $\mu(E) = \lambda(\cdot, E)$ in the sense of [5, Th. 2, p. 492].

DEFINITION AND NOTATION. If $X$ is a locally compact Hausdorff space we denote by $C(X), C_0(X)$ and $C_c(X)$ the collection of all bounded continuous complex-valued functions on $X$, those vanishing at infinity, and those nonnegative functions of compact support, respectively. The $\sigma$-algebra of Borel sets is the $\sigma$-algebra generated by the open subsets of $X$. We denote by $M(X)$ the space of bounded regular Borel measures on $X$ with variation norm and by $B(X)$ the space of bounded Borel measurable functions on $X$. Let $M(X)_+$ denote the nonnegative measures in $M(X)$. We give $B(X), C_0(X)$ and $C(X)$ the supremum norm topology and $||f|| = \sup \{ |f(x)| : x \in X \}$.

We wish to consider two further topologies on the space $C(X)$. We denote by $C(X)_\beta$ the space $C(X)$ with the locally convex topology defined by the collection of seminorms $P_\phi(f) = ||\phi f||$, $\phi \in C_0(X)$. Buck [1] has shown that $C(X)_\beta$ has adjoint or dual space $M(X)$. We denote by $C(X)_\beta$ the space $C(X)$ with the locally convex topology whose base of neighborhoods at the origin consists of all convex, balanced,
absorbent sets \( V \) such that for each \( r > 0 \) there is a \( \beta \) neighborhood of the origin, \( W \), such that \( W \cap B_r \subset V \) where \( B_r = \{ f \in C(X) : ||f|| \leq r \} \).

In a recently submitted paper Dorroh [4] introduces this topology and shows that \( C(X)_\beta \) has dual \( M(X) \) and that \( \beta = \beta' \) for \( X \) a paracompact space. Further results on \( C(X)_\beta \) and \( C(X)_\beta' \) have been recently obtained by Collins and Dorroh in [2]. A set \( H \subset M(X) \) is \( \beta \)-equicontinuous (\( \beta' \)-equicontinuous) if there is a \( \beta(\beta') \) neighborhood of 0, \( W \), such that \( \left| \int_X f d\mu \right| \leq 1 \) for all \( f \in W \) and \( \mu \in H \). The \( \beta \)-equicontinuous sets of \( M(X) \) have been characterized by Conway [3] who has shown that \( H \) is \( \beta \)-equicontinuous if and only if \( H \) is uniformly bounded and for each \( \varepsilon > 0 \) there is a compact set \( K \subset X \) such that the variation of \( \mu \) on \( X - K \) is less than \( \varepsilon \) for all \( \mu \in H \). Since \( \beta' \) is a finer topology than \( \beta \) any \( \beta \)-equicontinuous set is \( \beta' \)-equicontinuous and these are the same when \( X \) is paracompact.

Suppose \( S \) and \( T \) are locally compact and Hausdorff. Let \( \Delta \) denote the collection of open sets in \( S \) and \( \sigma(\Delta) \) the collection of Borel sets. We consider complex-valued functions \( \lambda \) defined on \( T \times \sigma(\Delta) \) such that \( \lambda(x) = \lambda(x, \cdot) \in M(S) \). For brevity we will denote this by \( \lambda : T \to M(S) \).

We denote the norm of the measure \( \lambda(x) \) by \( ||\lambda(x)|| \) and set \( ||\lambda|| = \sup \{ ||\lambda(x)|| : x \in T \} \). If \( f \in B(S) \) we write \( \lambda(f) \) for the function defined by \( \lambda(f)(x) = \int_S f(y)\lambda(x, dy) \) and \( \lambda(\cdot, E) \) is the function whose value at \( x \) is \( \lambda(x, E) \) for \( E \in \sigma(\Delta) \). We let \( |\lambda|(x, E) \) be the variation of the measure \( \lambda(x, \cdot) \) on the set \( E \). We will say that the kernel \( \lambda \) satisfies condition \( E(E') \) if \( \{ \lambda(x) : x \in K \} \) is \( \beta \)-equicontinuous (\( \beta' \)-continuous) for each compact set \( K \subset T \).

Finally we take our topology from [8] and topological vector space terminology from [9]. We make use of the Riesz Representation theorem throughout and in particular its corollary:

\[
|\mu|(U) = \sup \left\{ \left| \int_X fd\mu \right| : f \in C_\Delta(S), ||f|| \leq 1, \text{ support } (f) \subset U \right\}
\]

for each open set \( U \).

We prove the following theorems.

THEOREM 1. (1) If \( \lambda : T \to M(S)^+ \) and \( \lambda(f) \) is lower semi-continuous for each \( f \in C_\Delta(S)^+ \) then \( \lambda(\cdot, E) \) is Borel measurable for each \( E \in \sigma(\Delta) \).

(2) If \( \lambda : T \to M(S) \) and \( \lambda(f) \in C(T) \) for all \( f \in C_\Delta(S) \) then \( \lambda(\cdot, E) \) and \( |\lambda|(\cdot, E) \) are measurable for each \( E \in \sigma(\Delta) \).

(3) If \( \lambda \) satisfies (1) or (2) and \( ||\lambda|| < \infty \) then \( \lambda(f) \in B(T) \) for \( f \in B(S) \).

THEOREM 2. If \( \lambda \) satisfies (3) of Theorem 1 then for each \( \nu \in M(T) \)
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the formula \( \mu(E) = \int \lambda(x, E)\nu(dx) \) defines a regular Borel measure on \( S \) such that \( |\mu|_1(E) = \int |\lambda|_1(x, E)|\nu|(dx) \) and for \( f \in B(S) \) we have \( \int f d\mu = \int \lambda(f) d\nu \).

**Theorem 3.** Suppose \( A \) is a continuous linear operator from the space \( X \) to the space \( Y \) where \( X \) denotes \( C_0(S), C(S)_\beta \) or \( C(S)_{\beta'} \) and \( Y \) denotes \( C(T), C(T)_\beta \) or \( C(S)_{\beta'} \). Then there is a unique mapping \( \lambda: T \to M(S) \) such that

1. \( Af = \lambda(f) \) for all \( f \in X \) and
   \[ ||\lambda|| = \sup \{ ||Af||: f \in X, ||f|| \leq 1 \} < \infty . \]

2. The adjoint of \( A, A^* \), takes \( M(T) \) into \( M(S) \) and is given by
   \[ (A^*\mu)(E) = \int \lambda(x, E)\mu(dx) . \]

3. Under the natural imbeddings of \( B(S) \) and \( B(T) \) into \( M(S)^* \) and \( M(T)^* \) respectively we have for \( f \in B(S) \)
   \[ \lambda(f) = A^{**}f \text{ where } A^{**} \text{ is the adjoint of } A^* \text{ restricted to } M(T) \]
   Hence \( A^{**}(B(S)) \subseteq B(T) \) and \( A^{**} \) defines a continuous extension of \( A \) to \( B(S) \) into \( B(T) \).

**Theorem 4.** Let \( \lambda: T \to M(S) \). If \( \lambda(f) \in C(T) \) for all \( f \in C_c(S) \) and \( \lambda \) satisfies condition \( E' \) then \( \lambda(f) \) is a continuous function on \( T \) for \( f \in C(S) \). Conversely, if \( S \) is paracompact and \( \lambda(f) \) is continuous for \( f \in C(S) \) then \( \lambda \) satisfies condition \( E \).

**Theorem 5.** Let \( \lambda: T \to M(S) \) and \( A \) the linear operator on \( C(S) \) defined by \( Af = \lambda(f) \). Then \( A \) is a continuous operator from \( C(S)_{\beta'} \) into \( C(T)_{\beta'} \) or \( C(T)_\beta \) if and only if \( ||\lambda|| < \infty, \lambda(f) \in C(T) \) for \( f \in C_c(S) \) and \( \lambda \) satisfies condition \( E' \).

**Corollary 1.** Let \( A: C_0(S) \to Y \) where \( Y \) is as in Theorem 3. Then \( A^{**} \) is a continuous operator from \( C(S)_{\beta'} \) into \( C(T)_\beta \) if and only if the kernel \( \lambda \) satisfies condition \( E' \). Moreover \( A^{**} \) is the only extension of \( A \) to \( C(S) \) given by a kernel and consequently is the only \( \beta \) or \( \beta' \) continuous extension of \( A \) to \( C(S) \).

**Proof of Theorem 1.** Let \( U \) be an open subset of \( S \) and let \( \chi \) denote its characteristic function. Since \( \lambda(x) \) is regular it follows that \( \lambda(x, U) = \sup \{ \lambda(f)(x): 0 \leq f \leq \chi, f \in C_c(S)^+ \} \). Since \( \lambda(f) \) is lower semi-continuous for each \( f \in C_c(S)^+ \), then \( \lambda(\cdot, U) \) is lower semi-continuous and hence Borel-measurable. Let \( \Sigma \) denote the class of Borel sets \( E \).
for which \(\lambda(\cdot, E)\) is measurable. Then \(\Sigma\) contains all open sets and is closed under countable unions of mutually disjoint sets \(E \in \Sigma\) and, if \(A, B \in \Sigma\) and \(A \supset B\) then \(A - B \in \Sigma\). It now follows from [6, p. 2] that \(\Sigma = \sigma(\Delta)\) and (1) is proven.

We now prove (2). If \(U\) is an open set then as a consequence of the Riesz Representation Theorem we have

\[
|\lambda| (x, U) = \sup \{ |\lambda(f)(x)| : f \in C_*(S), \|f\| = 1 \text{ and support } (f) \subseteq U \}
\]

for each \(x \in T\). This means that \(|\lambda| (\cdot, U)\) is lower semi-continuous and as in the proof of (1) that \(|\lambda| (\cdot, E)\) is measurable for each Borel set \(E\).

We can suppose for the remainder of the proof that \(\lambda(x)\) is a real signed measure for each \(x \in T\) and we then have [5, p. 123] that \(\lambda(x) = \lambda(x)^+ - \lambda(x)^-\) where \(\lambda(x)^+, \lambda(x)^- \in M(S)^+\) and \(|\lambda(x)| = \lambda(x)^+ + \lambda(x)^-\) for all \(x \in T\). We show that \(\lambda^+, \lambda^-\) satisfy condition (1).

Let \(f \in C_*(S)^+\) and set \(\mu(x, E) = \int_E f(y) \lambda(x, dy)\). Then for each \(x, \mu(x) \in M(S)\) and for

\[
g \in C_*(S), \mu(g) = \int_S g(y)f(y)\lambda(x, dy) = \lambda(gf).
\]

Hence \(\mu(g)\) is continuous for each \(g \in C_*(S)\) and therefore from what we have just shown \(|\mu| (\cdot, S)\) is lower-semicontinuous since \(S\) is open. But \(|\mu| (x, S) = \int_S f(y)|\lambda| (x, dy)\) and therefore \(|\lambda| (f)\) is lower semi-continuous for each \(f \in C_*(S)^+\). Since \(|\lambda| (x) = \lambda^+(x) + \lambda^-(x)\) and \(\lambda(x) = \lambda^+(x) - \lambda^-(x)\) it now follows that for \(f \in C_*(S)^+, \lambda^+(f)\) and \(\lambda^-(f)\) are lower semi-continuous. But then it follows from (1) that \(\lambda^+(\cdot, E), \lambda^-(\cdot, E)\) and hence \(\lambda(\cdot, E)\) are measurable for each Borel set \(E\).

Condition (3) easily follows for we can approximate \(\lambda(f)\) uniformly by means of measurable functions of the form \(\sum_{i=1}^n a_i \lambda(\cdot, E_i)\).

**Remark 1.** \(T\) need not be Hausdorff or locally compact in Theorem 1.

**Proof of Theorem 2.** It is well known that \(\mu(E) = \int_S \lambda(x, E)\nu(dx)\) defines a measure on \(S\) such that \(\int_S f d\mu = \int_S \lambda(f) d\nu\) for \(f \in B(S)\). Hence we will only show that \(\mu\) is regular.

We can assume that \(\nu\) is real and \(\|\nu\| = 1\). Further we can suppose that \(\lambda(x) \in M(S)^+\) for each \(x \in T\). For we can first assume that \(\lambda(x)\) is a real signed measure, and writing \(\lambda(x) = \lambda(x)^+ - \lambda(x)^-\), the proof of Theorem 1 shows that for \(f \in C_*(S)^+, \lambda^+(f)\) and \(\lambda^-(f)\) are lower semi-continuous. Hence we have the condition (1) of Theorem 1 and additionally, \(\|\lambda\| = \sup \{|\lambda(x)| : x \in S\} < \infty\).
**Lemma 1.** Let $U$ be an open set in $S$, $\chi$ its characteristic function. Let $X = \{ f \in C_c(S) : 0 \leq f \leq \chi \}$, $Y = \{ g \in C_c(T) : 0 \leq g \leq \lambda(\cdot, U) \}$. Then

$$\sup \left\{ \int_X g \, d\nu : g \in Y \right\} \leq \sup \left\{ \int_T \lambda(f) \, d\nu : f \in X \right\}.$$ 

**Proof.** Let $g \in Y$, $\varepsilon > 0$ and let $g$ vanish outside the compact set $K$ and fix $x \in K$.

Since $g \in Y$ then $g(x) - \varepsilon/2 < \lambda(x, U)$ and hence there is a function $f \in X$ such that $g(x) - \varepsilon/2 < \lambda(f)(x)$. Since $\lambda(f)$ is lower semicontinuous there is a neighborhood $V$ of $x$ such that for $t \in V$ one has $g(x) - \varepsilon/2 < \lambda(f)(t)$. But also there is a neighborhood $V'$ of $x$ such that if $t \in V'$ then $g(t) - \varepsilon < g(x) - \varepsilon/2$. Hence there is a neighborhood $W$ of $x$ such that for $t \in W$, $g(t) - \varepsilon < \lambda(f)(t)$. We extract a finite cover of sets $W$ of $K$ with associated functions $f \in X$. If we let $h$ be the pointwise maximum of the corresponding functions $f$ then $h \in X$ and for $t \in K$ we have

$$g(t) - \varepsilon < \lambda(h)(t).$$

Hence $\int_T g \, d\nu - \varepsilon < \int_T \lambda(h) \, d\nu$ and the proof is complete.

**Lemma 2.** $\int_T \lambda(x, U) \nu(dx) \leq \sup \left\{ \int_X g \, d\nu : g \in Y \right\}$.

**Proof.** Let $\varepsilon > 0$ and $n$ be an integer such that $n\varepsilon > \|\lambda\| \geq (n-1)\varepsilon$. Then set

$$E_k = \{ x \in T : k\varepsilon < \lambda(x, U) \leq (k+1)\varepsilon \} \quad \text{for } k = 0, 1, \ldots, n-1.$$ 

Then $\{E_k\}$ is a partition of $T$ by Borel sets and

$$(1) \quad 0 \leq \int_T \lambda(x, U) \nu(dx) - \sum_{k=0}^{n-1} k\varepsilon \nu(E_k) < \varepsilon.$$ 

Let

$$U_k = \{ x : \lambda(x, U) > k\varepsilon \}.$$ 

Then $U_k$ is an open set and $E_k = U_k - U_{k+1}$. Since $\nu$ is regular then for each $k$ there is a compact set $K_k \subset E_k$ such that $\nu(E_k - K_k) < \varepsilon/n^2$. We can then find for each $k$ an open set $V_k$ with compact closure contained in $U_k$ and containing $K_k$. Further there exist functions $f_k \in C_c(T)^+$ for $k = 0, \ldots, n-1$ such that $f_k(x) = k\varepsilon$ for $x \in K_k$, $f_k(x) = 0$ for $x \in T - V_k$ and $0 \leq f_k(x) \leq k\varepsilon$ for all $x \in T$. Therefore $f_k(x) \leq k\varepsilon < \lambda(x, U)$ for $x \in U_k$ and hence $f_k \in Y$. We let
\[ f(x) = \max \{ f_k(x): 0 \leq k \leq n - 1 \}. \]

It follows that \( f \in Y \) and
\[
\int_{T} \sum_{k=0}^{n-1} k \varepsilon \chi_k(x) \, d\nu - \int_{T} f \, d\nu \\
\leq \sum_{k=0}^{n-1} \int_{E_k} (k \varepsilon - f_k) \, d\nu \\
= \sum_{k=0}^{n-1} \int_{E_k-K_k} (k \varepsilon - f_k) \, d\nu \\
\leq \sum_{k=0}^{n-1} \int_{E_k-K_k} k \varepsilon \, d\nu \\
\leq \sum_{k=0}^{n-1} k \varepsilon \varepsilon / n^2 \leq \varepsilon^2.
\]

But
\[
\int_{T} \sum_{k=0}^{n-1} k \varepsilon \chi_k \, d\nu = \int_{0}^{n-1} k \varepsilon \nu(E_k)
\]
and applying (1) we have
\[
0 \leq \int_{T} \lambda(x, U) \nu(dx) - \int_{T} f \, d\nu \leq \varepsilon^2 + \varepsilon
\]
completing the proof.

**Lemma 3.** \( \mu(U) = \sup \left\{ \int_{S} f \, d\mu: f \in X \right\} \) and \( \mu \) is regular.

**Proof.** Combining Lemma 1 and Lemma 2 we have
\[
\mu(U) \leq \sup \left\{ \int_{T} \lambda(f) \, d\nu: f \in X \right\}.
\]
But \( \int_{S} f \, d\mu = \int_{T} \lambda(f) \, d\nu \) and therefore
\[
\mu(U) \leq \sup \left\{ \int_{S} f \, d\mu; f \in X \right\} \leq \mu(U).
\]
Now the mapping \( f \to \int_{S} f \, d\mu \) defines a bounded linear form on the space \( C_\circ(S) \) and hence there is a measure \( \omega \in M(S)^+ \) such that \( \int_{S} f \, d\mu = \int_{S} f \, d\omega \) for all \( f \in C_\circ(S) \) and since \( \omega \) is regular.
\[
\omega(U) = \sup \left\{ \int_s f \, d\omega : f \in X \right\} = \mu(U).
\]

This means the collection \( \Sigma \) of all Borel sets \( E \) for which \( \omega(E) = \mu(E) \) contains all open sets and it follows from [6, p. 2] as in the proof of (1) Theorem 1 that \( \Sigma \) is the class of all Borel sets. Hence \( \mu \) is the regular measure \( \omega \). It is easily seen that \( |u|(E) \leq \int_T |\lambda|(x, E)|\nu|(dx) \) and the proof is complete.

**Proof of Theorem 3.** From [1], [4] and the Riesz Representation Theorem, \( X^* = M(S) \) and \( Y^* \supset M(T) \). From [9, pp. 38–39]

\[ A^*(M(T)) \supset M(S) \]

and the formula \( \lambda(x) = A^*\beta \), where \( \beta(E) = 1 \) if \( x \in E \), 0 if \( x \notin E \), defines a map \( \lambda : T \to M(S) \) satisfying (3) of Theorem 1 since \( \|\lambda\| = \sup \{\|Af\| : \|f\| \leq 1, f \in C_0(S)\} < \infty \) because the norm, \( \beta \) and \( \beta' \) bounded sets are the same (see [1] and [4]) and from [9, p. 45] \( A \) takes bounded sets into bounded sets. Furthermore \( Af = \lambda(f) \) for \( f \in X \) and if \( \nu(E) = \int_T \lambda(x, E)\mu(dx) \) then

\[ \int_s f \, d\nu = \int_T \lambda(f) \, d\mu = \int_T Af \, d\mu = \int_s f \, d(A^*\mu) \]

for all \( f \in X \) and consequently \( A^*\mu = \nu \) since \( \nu \) is regular. Finally if \( A^{**} \) is the adjoint of \( A^* \) restricted to \( M(T) \) then for \( \mu \in M(T) \) and

\[ f \in B(S) \implies [A^{**}f](\mu) = f(A^*\mu) = \int_s f \, d(A^*\mu) = \int_T \lambda(f) \, d\mu = [\lambda(f)](\mu) \]

since \( \lambda(f) \in B(T) \). This holds for all \( u \in M(T) \) and consequently \( A^{**}f = \lambda(f) \). Hence \( A^{**}(B(S)) \subset B(T) \) and \( \|A^{**}\| = \|\lambda\| \).

**Remark 2.** If for each \( t \in [0, \infty) \), \( T(t) \) is a continuous operator from \( X \) to \( X \) and \( T(t + u) = T(t)T(u) \) then \( T(t + u)^{**} = T(t)^{*}T(u)^{*} \). If we then write \( [T(t)f](x) = \int_s f(y)\lambda_t(x, dy) \), then by the above theorem \( \lambda_t(f) = T(t)^{*}f \) for \( f \in B(S) \). If \( \chi \) is the characteristic function of the Borel set \( E \) we have

\[ \lambda_{t+u}(\chi) = \chi(\lambda_u(\chi)) \]

or the Chapmann-Kolmogorov equation

\[ \lambda_{t+u}(x, E) = \int_s \lambda_u(y, E)\lambda_t(x, dy) \, d\nu. \]

Consequently a transition function \( \lambda_t(x, \cdot) \) can be obtained for a semi-
group of \( \beta \) or \( \beta' \) continuous operators on the space \( C(S) \) when \( S \) is locally compact.

**Remark 3.** One can obtain a kernel \( \lambda \) satisfying (1) under the weaker condition that \( A \) have range \( B(T) \) and domain \( C_0(S) \). For the set of linear mappings \( f \to \lambda(f)(x) \) for \( x \in T \) is pointwise bounded and hence uniformly bounded since \( C_0(S) \) is a Banach space.

**Proof of Theorem 4.** For each compact set \( K \subset S \) there is a function \( \varphi_K \in C_\beta(S) \) such that \( \varphi_K \equiv 1 \) on \( K \). If \( f \in C(S) \) then the net \( \{\varphi_K f\} \subset C_\beta(S) \) converges \( \beta' \) to \( f \) since it is uniformly bounded and \( \beta \) convergent to \( f \). Consequently \( C_\beta(S) \) is \( \beta' \) dense in \( C(S) \). If \( x \in T \) and \( U \) is a neighborhood of \( x \) with compact closure then \( \{\lambda(x_n): x_n \in U\} \) is a \( \beta' \)-equicontinuous set of linear functionals on \( C(S) \) for any net \( \{x_n\} \subset U \) converging to \( x \). By hypothesis \( \lambda(x_n) \to \lambda(x) \) on \( C_\beta(S) \). Since \( C_\beta(S) \) is \( \beta' \) dense and \( \{\lambda(x_n)\} \) is \( \beta' \)-equicontinuous, \( \lambda(x_n) \to \lambda(x) \) on \( C(S) \). Hence \( \lambda(f) \) is continuous at \( x \) for all \( f \in C(S) \).

Conversely if \( \lambda(f) \in C(T) \) for \( f \in C(S) \) then for any compact set \( K \subset T \{\lambda(x): x \in K\} \) is weak-* compact as an subset of the dual of \( C(S)_\beta \) and, as Conway [3] has shown, must be \( \beta' \)-equicontinuous.

**Proof of Theorem 5.** Suppose that \( A \) is continuous from \( C(S)_{\beta'} \) to \( C(T)_{\beta'} \) or \( C(T)_\beta \). Then \( || \lambda || < \infty \) by Theorem 3 and if \( K \) is a compact set in \( T \) and \( V \) is the \( \beta \) neighborhood of 0 defined by some function \( \varphi \in C_0(T) \) identically 1 on \( K \) there is a \( \beta' \) neighborhood of 0, \( U \), such that \( A(U) \subset V \). That is, \( || \lambda(f)(x) || \leq 1 \) for all \( f \in U \) and \( x \in K \). Consequently \( \lambda \) satisfies condition \( E' \).

Conversely, let us show \( A \) is continuous from \( C(S)_{\beta'} \) into \( C(T)_{\beta'} \). Let \( V \) be a \( \beta' \) neighborhood of 0 in \( C(T) \) and \( r > 0 \). We show there is a \( \beta \) neighborhood \( U \) of 0 in \( C(S) \) such that \( A^{-1}(V) \supset B_r \cup U \) thus showing that \( A^{-1}(V) \) is a \( \beta' \) neighborhood.

Let \( p = r \ || \lambda || \). There is a \( \phi \in C_\beta(T) \) such that

\[ V \supset B_p \cap \{g: P_p(g) \leq 1\} \quad \text{and} \quad \phi \not\equiv 0. \]

Let \( K = \{t: || \phi(t) || \geq 1/(p + 1)\} \). Since \( \lambda \) satisfies condition \( E' \) there is a \( \beta' \) neighborhood \( U_0 \) in \( C(S) \) such that \( || \lambda(f)(x) || \leq 1 \) for all \( f \in U_0 \) and \( x \in K \). Let \( W = \{f \in C(S): || \phi \ || f \in U\} \). Then \( A^{-1}(V) \supset B_r \cap W \) for if \( f \in B_r \cap W \) then \( Af \in B_p \) and \( || \phi(x)[Af](x) || < p/(p + 1) \) for \( x \in K \) while for \( x \in K \), \( || \phi(x)[Af](x) || \leq || \phi || || [Af](x) || \leq 1 \) since \( || \phi || f \in U_0 \).

Hence

\[ A^{-1}(V) \supset A^{-1}(B_p) \cap A^{-1}(g: P_p(g) \leq 1) \supset B_r \cap (B_r \cap W) = B_r \cap W. \]

We then choose a \( \beta \) neighborhood \( U \) such that \( W \supset B_r \cap U \) completing the proof.
REMARK 4. If $A$ is continuous from $C(S)_\beta$ into $C(T)_\beta$, it follows that $\lambda$ satisfies $E$.

The proof of Corollary 1 is almost immediate. As a consequence of Theorem 3 and Theorem 5 continuity from $C(S)_\beta$ to $C(T)_\beta$ is equivalent to condition $E'$. If $A'$ is an extension of $A$ to $C(S)$ into $C(T)$ given by a kernel $\mu$ then $\mu = \lambda$ on $C(S)$ and consequently $\mu = \lambda$ on $C(S)$ and $A = A'$. Since by Theorem 3 any $\beta$ or $\beta'$ continuous extension is given by a kernel this shows that $A^{**}$ is unique.

It should be noted that if $S$ is paracompact and $A$ is any operator on $C(S)$ into $C(T)$ given by a bounded kernel $\lambda$ then by Theorems 4 and 5, $A$ is continuous from $C(S)_\beta$ to $C(T)_\beta$.

We conclude with a brief remark on operators from $M(T)$ into $M(S)$. Suppose $B$ is such a linear operator and $B^*$ its adjoint on $B(S)$. Define $\lambda: T \to M(S)$ by $\lambda(x) = B\hat{\delta}$ where $\hat{\delta}$ is the measure defined in the proof of Theorem 3. If $B$ is bounded and $B^*(C_c(S)) \subset C(T)$ then $B^*(B(S)) \subset B(T)$ by Theorem 1. By Theorem 2,

$$(B\mu)(E) = \int_T (B\hat{\delta})(E) \mu(dx).$$

If $\lambda$ satisfies condition $E'$ then by Theorem 5 $B$ is the adjoint of the continuous operator $B^*$ from $C(S)_\beta$ to $C(T)_\beta$. Thus $B$ is completely determined by its action on the point measures $\{\hat{\delta}: x \in T\}$.

REMARK 5 (added January 13, 1967). One can amplify Remark 4 by observing that if, moreover, $\lambda$ satisfies $E$ then Theorem 5 remains true with $\beta'$ replaced by $\beta$. For then $A$ is continuous from $C(S)_\beta$ to $C(T)_\beta$ and using condition $E$, [3], part (2) of Theorem 3 and [9, p. 39] it follows that $A^*$ takes $\beta$-equicontinuous sets of $M(T)$ into $\beta$-equicontinuous sets of $M(S)$ making $A$ continuous on $C(S)_\beta$ into $C(T)_\beta$.

REMARK 6. It has recently come to the author's attention that a version of Theorem 2 can be found on page 176 of the recent book by P. A. Meyer, Probability and Potentials, Blaisdell, Waltham, Massachusetts, 1966, under the conditions that $S$ be $\sigma$-compact, $\lambda: S \to M(S)^+$, $\lambda(f)$ be continuous for all $f \in C_c(S)^+$ and that $\nu$ have compact support.

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