FIXED POINTS IN A CLASS OF SETS

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THEOREM. A set of the form $X = A \cup \bigcup_{i \in J} B_i$ has the fixed point property if

(i) $A$ is a closed simplex and each $B_i$ is a closed simplex;
(ii) $A \cap B_i$ is a single point $p_i$ for each $i$;
(iii) any arc in $X$ joining a point in some $B_i$ to a point in $X - B_i$ must pass through $p_i$.

($J$ can be any index set. The topology on $X$ can be given by any metric satisfying (i) and (iii).)

The statement that $X$ has the fixed point property means that each continuous mapping of $X$ into $X$ has a fixed point. The theorem applies to many sets which are not locally connected so that even Lefschetz's fixed point theorem is inapplicable. Instead of assuming that the subsets $A$ and $B_i$ are simplices we could merely assume that each of these subsets is locally arcwise connected and has the fixed point property. The result should still be true if each point $p_i$ is replaced by a simplex $P_i$ but this generalization would require altogether different methods.

Proof of the theorem. Let $T$ be a continuous mapping of $X$ into $X$. We distinguish three cases.

Case 1. Suppose $T p_i \in B_i - \{p_i\}$ for some $i$. Then we will show that $T$ has a fixed point in $B_i$.

Define $S : B_i \rightarrow B_i$ by

$S_x = T x$ if $T x \in B_i$
$S_x = p_i$ if $T x \not\in B_i$.

Then $S$ is continuous by Lemma 2 below.

Since $B_i$ has the fixed point property, $S x = x$ for some $x$ in $B_i$. Now $x \neq p_i$ (for $x = p_i$ would give $S p_i = p_i$, impossible since $S p_i = T p_i \in B_i - \{p_i\}$). Thus $S x \neq p_i$ so that $T x = S x = x$.

Case 2. Suppose $T p_i = p_i$ for some $i$. Then $p_i$ is a fixed point.

Case 3. Suppose $T p_i \in X - B_i$ for all $i$. Then we will show that $T$ has a fixed point in $A$. Define $R : A \rightarrow A$ by

$R x = T x$ if $T x \in A$. 

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Then $R$ is continuous by Lemma 2. Since $A$ has the fixed point property, $R$ has a fixed point in $A$. The fixed point $\xi$ cannot be a point $p_i$ since $Rx = p_i$ only if $Tx \in B_i$; and $T_{p_i} \not\in B_i$. Since the fixed point is not $p_i$, $T\xi \in B_i$. Thus $T\xi \in A$ so that $T\xi = R\xi = \xi$.

Thus in each case $T$ has a fixed point, which proves the theorem. The above proof depends on two lemmas.

**Lemma 1.** If $z(t)$ is a continuous function on $[0,1]$ to a metric space and either

(i) $w(t)$ is a constant, or

(ii) $w(t) = z(t)$ except on a non-overlapping sequence of intervals $[t_{2n-1}, t_{2n}]$ ($n \geq 1$) such that

\[
t_1 = 0 \text{ and } w(t) = z(t_1) \text{ on } [t_1, t_2]
\]

\[
t_2 = 1 \text{ and } w(t) = z(t_2) \text{ on } [t_2, t_4]
\]

and for $n > 2$, $z(t_{2n-1}) = z(t_{2n})$ and $w(t) = z(t_{2n})$ on $[t_{2n-1}, t_{2n}]$. Then $w(t)$ is continuous on $[0,1]$.

**Proof.** Obvious. (One proof is: if $z_n$ is the function obtained from $z$ by changing its value to that of $w$ on the first $n$ intervals, then $z_n$ is continuous. Also $z_n \rightarrow w$ uniformly on $[0,1]$ since the length of $[t_{2n-1}, t_{2n}]$ must tend to 0.

**Lemma 2.** Let $Y$ be a closed simplex contained in a metric space $X$. Suppose that $X - Y$ is the union of disjoint sets $Z_i$, that $Z_i \cap Y$ is a one-point set $\{q_i\}$, and that any path from a point in a $Z_i$ to a point in $X - Z_i$ must pass through $q_i$. Let $U$ be continuous on $Y \times X$. Define $T$ by

\[
Ty = Uy \text{ if } Uy \in Y
\]

\[
Ty = q_i \text{ if } Uy \in Z_i.
\]

Then $T$ is continuous.

**Proof.** If $y_n \rightarrow y$ in $Y$ we must show that $Ty_n \rightarrow Ty$. Consider a path $g(t)$ in $Y$ ($0 \leq t \leq 1$) such that $g(0) = y$ and $g(1/n) = y_n$. Writing $Ug(t) = z(t)$ and $Tg(t) = w(t)$ the conditions of Lemma 1 are satisfied. For if $w(t)$ differs from $z(t)$ the possibilities are: $z(t)$ could be in some $Z_i$ for all $t$, in which case $w(t)$ is a constant; otherwise, there is an initial interval $[0, t_1]$ where $z(t)$ is in some $Z_i$, and/or some intermediate intervals $[t_{2n-1}, t_{2n}]$ where $z(t)$ is in some $Z_{i(n)}$ and/or a final interval $[t_3, 1]$ where $z(t)$ is in some $Z_j$. By Lemma 1, $w(t)$ is continuous. Thus $Tg(1/n) \rightarrow Tg(0)$ as required.
The theorem can be used to establish some pathological examples. (It seems that all of these are already known.)

I. There exists a noncompact set having the fixed point property.

Take

\[ A = \{(x, y) : 0 \leq x \leq 1, y = 0\} \]

\[ B_n = \left\{(x, y) : x = \frac{1}{n}, 0 \leq y \leq 1\right\}. \]

(In this case \( X \) also has the fixed point property.)

II. There exists an unbounded set having the fixed point property.

Take \( A \) as above.

\[ B_n = \left\{(x, y) : x = \frac{1}{n}, 0 \leq y \leq n\right\}. \]

III. There exists a set with the fixed point property whose closure lacks this property. Take \( X \) as in II.

IV. There exists a precompact set with the fixed point property, whose closure lacks this property.

Take

\[ A = \left\{e^{i\theta} : \frac{\pi}{2} \leq \theta \leq 2\pi\right\} \]

\[ B_n = \left\{(1 + \frac{\theta}{n})e^{i\theta} : 0 \leq \theta \leq \frac{\pi}{2}\right\}. \]

Several sets which have some interest in other contexts have the fixed point property in consequence of our theorem:–

V. The set

\[ A \cup \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{n=1}^{\infty} C_n \]

where \( A \) is the unit interval, \( B_n \) is a unit line segment sloping up from (0,0) with slope \( 1/n \), and \( C_n \) is a unit line segment sloping up to (0,1) with slope \( 1/n \). (This is a non contractible set.)

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