

# Pacific Journal of Mathematics

**FIXED POINTS IN A CLASS OF SETS**

D. R. SMART

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**THEOREM.** A set of the form  $X = A \cup \bigcup_{i \in J} B_i$  has the fixed point property if

- (i)  $A$  is a closed simplex and each  $B_i$  is a closed simplex;
- (ii)  $A \cap B_i$  is a single point  $p_i$  for each  $i$ ;
- (iii) any arc in  $X$  joining a point in some  $B_i$  to a point in  $X - B_i$  must pass through  $p_i$ .

( $J$  can be any index set. The topology on  $X$  can be given by any metric satisfying (i) and (iii).)

The statement that  $X$  has the fixed point property means that each continuous mapping of  $X$  into  $X$  has a fixed point. The theorem applies to many sets which are not locally connected so that even Lefschetz's fixed point theorem is inapplicable. Instead of assuming that the subsets  $A$  and  $B_i$  are simplices we could merely assume that each of these subsets is locally arcwise connected and has the fixed point property. The result should still be true if each point  $p_i$  is replaced by a simplex  $P_i$  but this generalization would require altogether different methods.

*Proof of the theorem.* Let  $T$  be a continuous mapping of  $X$  into  $X$ . We distinguish three cases.

*Case 1.* Suppose  $Tp_i \in B_i - \{p_i\}$  for some  $i$ . Then we will show that  $T$  has a fixed point in  $B_i$ .

Define

$$S: B_i \rightarrow B_i \text{ by}$$

$$Sx = Tx \text{ if } Tx \in B_i$$

$$Sx = p_i \text{ if } Tx \notin B_i.$$

Then  $S$  is continuous by Lemma 2 below.

Since  $B_i$  has the fixed point property,  $Sx = x$  for some  $x$  in  $B_i$ . Now  $x \neq p_i$  (for  $x = p_i$  would give  $Sp_i = p_i$ , impossible since  $Sp_i = Tp_i \in B_i - \{p_i\}$ ). Thus  $Sx \neq p_i$  so that  $Tx = Sx = x$ .

*Case 2.* Suppose  $Tp_i = p_i$  for some  $i$ . Then  $p_i$  is a fixed point.

*Case 3.* Suppose  $Tp_i \in X - B_i$  for all  $i$ . Then we will show that  $T$  has a fixed point in  $A$ . Define  $R: A \rightarrow A$  by

$$Rx = Tx \text{ if } Tx \in A.$$

$$Rx = p_i \text{ if } Tx \in B_i .$$

Then  $R$  is continuous by Lemma 2. Since  $A$  has the fixed point property,  $R$  has a fixed point in  $A$ . The fixed point  $\xi$  cannot be a point  $p_i$  since  $Rx = p_i$  only if  $Tx \in B_i$ ; and  $Tp_i \notin B_i$ . Since the fixed point is not  $p_i$ ,  $T\xi \notin B_i$ . Thus  $T\xi \in A$  so that  $T\xi = R\xi = \xi$ .

Thus in each case  $T$  has a fixed point, which proves the theorem. The above proof depends on two lemmas.

LEMMA 1. *If  $z(t)$  is a continuous function on  $[0, 1]$  to a metric space and either*

(i)  *$w(t)$  is a constant, or*

(ii)  *$w(t) \equiv z(t)$  except on a non-overlapping sequence of intervals  $[t_{2n-1}, t_{2n}]$  ( $n \geq 1$ ) such that*

$$t_1 = 0 \text{ and } w(t) \equiv z(t_2) \text{ on } [t_1, t_2]$$

$$t_4 = 1 \text{ and } w(t) \equiv z(t_3) \text{ on } [t_3, t_4]$$

*and for  $n > 2$ ,  $z(t_{2n-1}) = z(t_{2n})$  and  $w(t) \equiv z(t_{2n})$  on  $[t_{2n-1}, t_{2n}]$ . Then  $w(t)$  is continuous on  $[0, 1]$ .*

*Proof.* Obvious. (One proof is: if  $z_n$  is the function obtained from  $z$  by changing its value to that of  $w$  on the first  $n$  intervals, then  $z_n$  is continuous. Also  $z_n \rightarrow w$  uniformly on  $[0, 1]$  since the length of  $[t_{2n-1}, t_{2n}]$  must tend to 0.

LEMMA 2. *Let  $Y$  be a closed simplex contained in a metric space  $X$ . Suppose that  $X - Y$  is the union of disjoint sets  $Z_i$ , that  $Z_i \cap Y$  is a one-point set  $\{q_i\}$ , and that any path from a point in a  $Z_i$  to a point in  $X - Z_i$  must pass through  $q_i$ . Let  $U$  be continuous on  $Y$ . Define  $T$  by*

$$Ty = Uy \text{ if } Uy \in Y$$

$$Ty = q_i \text{ if } Uy \in Z_i .$$

*Then  $T$  is continuous.*

*Proof.* If  $y_n \rightarrow y$  in  $Y$  we must show that  $Ty_n \rightarrow Ty$ . Consider a path  $g(t)$  in  $Y$  ( $0 \leq t \leq 1$ ) such that  $g(0) = y$  and  $g(1/n) = y_n$ . Writing  $Ug(t) = z(t)$  and  $Tg(t) = w(t)$  the conditions of Lemma 1 are satisfied. For if  $w(t)$  differs from  $z(t)$  the possibilities are:  $z(t)$  could be in some  $Z_i$  for all  $t$ , in which case  $w(t)$  is a constant; otherwise, there is an initial interval  $[0, t_2]$  where  $z(t)$  is in some  $Z_i$ , and/or some intermediate intervals  $[t_{2n-1}, t_{2n}]$  where  $z(t)$  is in some  $Z_{i(n)}$  and/or a final interval  $[t_3, 1]$  where  $z(t)$  is in some  $Z_j$ . By Lemma 1,  $w(t)$  is continuous. Thus  $Tg(1/n) \rightarrow Tg(0)$  as required.

The theorem can be used to establish some pathological examples. (It seems that all of these are already known.)

I. There exists a noncompact set having the fixed point property.

Take

$$A = \{(x, y) : 0 \leq x \leq 1, y = 0\}$$

$$B_n = \left\{ (x, y) : x = \frac{1}{n}, 0 \leq y \leq 1 \right\}.$$

(In this case  $\bar{X}$  also has the fixed point property.)

II. There exists an unbounded set having the fixed point property.

Take  $A$  as above.

$$B_n = \left\{ (x, y) : x = \frac{1}{n}, 0 \leq y \leq n \right\}.$$

III. There exists a set with the fixed point property whose closure lacks this property. Take  $X$  as in II.

IV. There exists a precompact set with the fixed point property, whose closure lacks this property.

Take

$$A = \left\{ e^{i\theta} : \frac{\pi}{2} \leq \theta \leq 2\pi \right\}$$

$$B_n = \left\{ \left( 1 + \frac{\theta}{n} \right) e^{i\theta} : 0 \leq \theta \leq \frac{\pi}{2} \right\}.$$

Several sets which have some interest in other contexts have the fixed point property in consequence of our theorem:-

V. The set

$$A \cup \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{n=1}^{\infty} C_n$$

where  $A$  is the unit interval,  $B_n$  is a unit line segment sloping up from  $(0,0)$  with slope  $1/n$ , and  $C_n$  is a unit line segment sloping up to  $(0,1)$  with slope  $1/n$ . (This is a non contractible set.)

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