TOPOLOGY OF SOME KÄHLER MANIFOLDS

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Goldberg and Bishop have shown that a homogeneous Kähler manifold of positive holomorphic curvature is isometric to the complex projective space with the usual metric. The aim of this note is to prove that such a Kähler manifold is isomorphic to the complex projective space.

We recall that a compact Kähler manifold $M$ of positive (resp. negative) holomorphic sectional curvature is always algebraic by a well-known theorem of Kodaira since its Ricci curvature is positive (resp. negative) [5]. The positively curved compact Kähler manifolds are simply-connected (cf p. 528, [3]) and their second Betti number $b_2$ is equal to one [2]. In §2, we prove that the first Betti number $b_1$ of a negatively curved compact Kähler surface is always zero.

In what follows, we assume that $M$ is homogeneous and its group of automorphisms acts effectively; recall that a homogeneous Kähler manifold is complete.

**Theorem.** A homogeneous Kähler $n$-manifold $M$ of positive holomorphic curvature is isomorphic to $\mathbb{P}C_n$.

**Proof.** It is well-known (p. 527, [3]) that a complete Kähler manifold $M$ of positive holomorphic curvature is compact and is simply-connected; moreover, its second Betti number is 1 [2] and its Euler-Poincaré characteristic $E$ is positive (Theorem 2, [9]). Thus we may assume that $M = K/L$ is the quotient of a compact semi-simple Lie group by a closed subgroup by a well-known theorem of Montgomery. It is well-known that $L$ is of maximal rank in $K$ and $K$ has trivial center. Moreover, $L$ is the centralizer of a 1-parameter subgroup of $K$ [9]. We first prove that $K$ is simple; in fact, let us assume that $K = K_1 \times \cdots \times K_m$ with $K_i$ compact, connected and simple. Since $L$ is of maximal rank, we have $L = L_1 \times \cdots \times L_m$, where $L_i \subset K_i$, $i = 1, 2, \cdots, m$. Thus $M = \prod (K_i/L_i)$ which is impossible in view of the fact $b_2(M) = 1$. Consider now the fibration of $K$ onto $K/L$ with fibre $L$; since $K$ is simple, the transgression defines an isomorphism of $H^i(L)$ onto $H^i(K/L)$ where the cohomology is taken with real coefficients. But $H^i(L)$ is isomorphic to the center of $L$; since $b_2(K/L) = 1$, we see that the center of $L$ is of dimension one. $K$ being effective, the isotropy representation of $L$ is faithful and hence the linear isotropy group is irreducible; consequently $K/L$ is irreducible hermitian symmetric (cf., p. 52, [4] and [8]). But the only irreducible...
compact hermitian symmetric space of positive holomorphic curvature in the list of E. Cartan is the complex projective space.

**Remark.** In fact we have shown above the following more general result: Let $M$ be a compact, simply-connected homogeneous complex manifold whose Euler-Poincaré characteristic is positive; if its second Betti number is one, then $M$ is isomorphic to an irreducible hermitian symmetric space (cf. Théorème 1, C.R.A.S. Paris 252, pp. 3377–3378 (1961), and [6]).

2. Let $D$ be an irreducible symmetric bounded domain of one of the following types: $I_{m,m'}$ ($m > m' > 6$), $II_m$ ($m > 7$), $III_m$ ($m > 7$) or IV. If $M$ is a compact quotient of $D$ by a properly discontinuous subgroup of automorphisms of $D$, it is well known that $b_1(M) = 0$ and $b_2(M) = 1$. In fact, we have the following result essentially due to Remmert-Van de Ven (cf. p. 456, [7]):

**Proposition 1.** Let $M$ be a compact Kähler manifold of dimension greater than one; if $b_2 = 1$, then its first Betti number is zero.

**Proof.** Suppose that $b_1 = 2q$, $q = h^{1,0}(M)$, is positive; let $A(M)$ denote the Albanese manifold of $M$ and let $\phi: M \rightarrow A(M)$ be the non-constant holomorphic onto projection. Since $b_2 = 1$, we have $h^{2,0}(M) = 0$ and hence $M$ is algebraic by Kodaira's theorem. Therefore $\dim M = \dim A(M)$ by Theorem 1.3 of [7]; let $\omega$ be a nonzero holomorphic 2-form on $A(M)$; then $\phi^*\omega$ is a nonzero holomorphic 2-form on $M$, a contradiction.

In fact, we can prove the following result for negatively curved Kähler surfaces which generalizes a result of [3]:

**Proposition 2.** Let $M$ be a compact Kähler surface of negative Ricci curvature; then its first Betti number is zero.

**Proof.** Since the Ricci curvature is negative, we have $H^p(M, \Omega^q(K)) = 0$ if $p + q = 1$ by a result of Akizuki-Nakano [1]; consequently, $H^1(M, \Omega^1(K)) = H^{0,1}(K) = 0$ by Dolbeaut's theorem. But $H^{0,1}(K) = H^{0,1}(M, K \otimes K^*) = H^{0,1}(M, 1)$ where 1 denote the trivial line bundle, by the duality theorem of Serre. Thus $h^{0,1} = \dim H^{0,1}(M, 1) = 0$ and hence $b_1 = 0$.

**Remark.** Note that the Euler-Poincaré characteristic of such a surface is positive (cf., [3]).
BIBLIOGRAPHY


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