

Pacific Journal of Mathematics

TOPOLOGY OF SOME KÄHLER MANIFOLDS

K. SRINIVASACHARYULU

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Goldberg and Bishop have shown that a homogeneous Kähler manifold of positive holomorphic curvature is isometric to the complex projective space with the usual metric. The aim of this note is to prove that such a Kähler manifold is isomorphic to the complex projective space.

We recall that a compact Kähler manifold M of positive (resp. negative) holomorphic sectional curvature is always algebraic by a well-known theorem of Kodaira since its Ricci curvature is positive (resp. negative) [5]. The positively curved compact Kähler manifolds are simply-connected (cf p. 528, [3]) and their second Betti number b_2 is equal to one [2]. In §2, we prove that the first Betti number b_1 of a negatively curved compact Kähler surface is always zero.

In what follows, we assume that M is homogeneous and its group of automorphisms acts *effectively*; recall that a homogeneous Kähler manifold is complete.

THEOREM. *A homogeneous Kähler n -manifold M of positive holomorphic curvature is isomorphic to PC_n .*

Proof. It is well-known (p. 527, [3]) that a complete Kähler manifold M of positive holomorphic curvature is compact and is simply-connected; moreover, its second Betti number is 1 [2] and its Euler-Poincaré characteristic E is positive (Theorem 2, [9]). Thus we may assume that $M = K/L$ is the quotient of a compact semi-simple Lie group by a closed subgroup by a well-known theorem of Montgomery. It is well-known that L is of maximal rank in K and K has trivial center. Moreover, L is the centralizer of a 1-parameter subgroup of K [9]. We first prove that K is *simple*; in fact, let us assume that $K = K_1 \times \cdots \times K_m$ with K_i compact, connected and simple. Since L is of maximal rank, we have $L = L_1 \times \cdots \times L_m$, where $L_i \subset K_i$, $i = 1, 2, \dots, m$. Thus $M = \prod_1^m (K_i/L_i)$ which is impossible in view of the fact $b_2(M) = 1$. Consider now the fibration of K onto K/L with fibre L ; since K is simple, the transgression defines an isomorphism of $H^1(L)$ onto $H^2(K/L)$ where the cohomology is taken with real coefficients. But $H^1(L)$ is isomorphic to the center of L ; since $b_2(K/L) = 1$, we see that the center of L is of dimension one. K being effective, the isotropy representation of L is faithful and hence the linear isotropy group is irreducible; consequently K/L is irreducible hermitian symmetric (cf., p. 52, [4] and [8]). But the only irreducible

compact hermitian symmetric space of positive holomorphic curvature in the list of \tilde{E} . Cartan is the complex projective space.

REMARK. In fact we have shown above the following more general result: Let M be a compact, simply-connected homogeneous complex manifold whose Euler-Poincaré characteristic is positive; if its second Betti number is one, then M is isomorphic to an irreducible hermitian symmetric space (cf. Théorème 1, C.R.A.S. Paris 252, pp. 3377-3378 (1961), and [6]).

2. Let D be an irreducible symmetric bounded domain of one of the following types: $I_{m,m'}$ ($m > m' > 6$), II_m ($m > 7$), III_m ($m > 7$) or IV. If M is a compact quotient of D by a properly discontinuous subgroup of automorphisms of D , it is well known that $b_1(M) = 0$ and $b_2(M) = 1$. In fact, we have the following result essentially due to Remmert-Van de Ven (cf. p. 456, [7]):

PROPOSITION 1. Let M be a compact Kähler manifold of dimension greater than one; if $b_2 = 1$, then its first Betti number is zero.

Proof. Suppose that $b_1 = 2q$, $q = h^{1,0}(M)$, is positive; let $A(M)$ denote the Albanese manifold of M and let $\phi: M \rightarrow A(M)$ be the non-constant holomorphic onto projection. Since $b_2 = 1$, we have $h^{2,0}(M) = 0$ and hence M is algebraic by Kodaira's theorem. Therefore $\dim M = \dim A(M)$ by Theorem 1.3 of [7]; let ω be a nonzero holomorphic 2-form on $A(M)$; then $\phi^*\omega$ is a nonzero holomorphic 2-form on M , a contradiction.

In fact, we can prove the following result for negatively curved Kähler surfaces which generalizes a result of [3]:

PROPOSITION 2. Let M be a compact Kähler surface of negative Ricci curvature; then its first Betti number is zero.

Proof. Since the Ricci curvature is negative, we have $H^q(M, \Omega^p(K)) = 0$ if $p + q = 1$ by a result of Akizuki-Nakano [1]; consequently, $H^1(M, \Omega^0(K)) = H^{0,1}(K) = 0$ by Dolbeault's theorem. But $H^{0,1}(K) = H^{0,1}(M, K \otimes K^*) = H^{0,1}(M, 1)$ where 1 denote the trivial line bundle, by the duality theorem of Serre. Thus $h^{0,1} = \dim H^{0,1}(M, 1) = 0$ and hence $b_1 = 0$.

REMARK. Note that the Euler-Poincaré characteristic of such a surface is positive (cf., [3]).

BIBLIOGRAPHY

1. Y. Akizuki and S. Nakano, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. of the Jap. Acad. **30** (1954), 266-272.
2. R. L. Bishop and S. I. Goldberg, *On the second cohomology group of a Kähler manifold of positive curvature*, Proc. Amer. Math. Soc. **16** (1965), 119-222.
3. ———, *Some implications of the generalized Gauss-Bonnet theorem*, Trans. Amer. Math. Soc. **112** (1964), 508-535.
4. É. Cartan, *La théorie des groupes finis et continus et l'analyse situs*, Mémorial Sci. Math., Paris, 1930.
5. T. Frankel, *Manifolds with positive curvature*, Pacific J. Math. **11** (1961), 165-174.
6. R. Remmert and A. Borel, *Über kompakte homogene kählersche Mannigfaltigkeiten*, Math. Annalen **145** (1962), 429-439.
7. R. Remmert and T. Van de Ven, *Über holomorphe Abbildungen projektiv-algebraischen Mannigfaltigkeiten auf komplexe Räume* Math. Ann. **142** (1961), 453-486.
8. J. de Siebenthal et A. Borel, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, commentari Math. Helv. **23** (1949), 202-221.
9. H. C. Wang, *Some geometrical aspects of coset spaces of Lie groups*, Proc. Inter. Congress of Math. 1958 (1960), 500-509.

Received February 29, 1966.

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Pacific Journal of Mathematics

Vol. 23, No. 1

March, 1967

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