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TOPOLOGY OF SOME KÄHLER MANIFOLDS

K. SRINIVASACHARYULU

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Goldberg and Bishop have shown that a homogeneous Kähler manifold of positive holomorphic curvature is isometric to the complex projective space with the usual metric. The aim of this note is to prove that such a Kähler manifold is isomorphic to the complex projective space.

We recall that a compact Kähler manifold M of positive (resp. negative) holomorphic sectional curvature is always algebraic by a well-known theorem of Kodaira since its Ricci curvature is positive (resp. negative) [5]. The positively curved compact Kähler manifolds are simply-connected (cf p. 528, [3]) and their second Betti number b_2 is equal to one [2]. In § 2, we prove that the first Betti number b_1 of a negatively curved compact Kähler surface is always zero.

In what follows, we assume that M is homogeneous and its group of automorphisms acts *effectively*; recall that a homogeneous Kähler manifold is complete.

THEOREM. *A homogeneous Kähler n -manifold M of positive holomorphic curvature is isomorphic to PC_n .*

Proof. It is well-known (p. 527, [3]) that a complete Kähler manifold M of positive holomorphic curvature is compact and is simply-connected; moreover, its second Betti number is 1 [2] and its Euler-Poincaré characteristic E is positive (Theorem 2, [9]). Thus we may assume that $M = K/L$ is the quotient of a compact semi-simple Lie group by a closed subgroup by a well-known theorem of Montgomery. It is well-known that L is of maximal rank in K and K has trivial center. Moreover, L is the centralizer of a 1-parameter subgroup of K [9]. We first prove that K is *simple*; in fact, let us assume that $K = K_1 \times \cdots \times K_m$ with K_i compact, connected and simple. Since L is of maximal rank, we have $L = L_1 \times \cdots \times L_m$, where $L_i \subset K_i$, $i = 1, 2, \dots, m$. Thus $M = \prod_1^m (K_i/L_i)$ which is impossible in view of the fact $b_2(M) = 1$. Consider now the fibration of K onto K/L with fibre L ; since K is simple, the transgression defines an isomorphism of $H^1(L)$ onto $H^2(K/L)$ where the cohomology is taken with real coefficients. But $H^1(L)$ is isomorphic to the center of L ; since $b_2(K/L) = 1$, we see that the center of L is of dimension one. K being effective, the isotropy representation of L is faithful and hence the linear isotropy group is irreducible; consequently K/L is irreducible hermitian symmetric (cf., p. 52, [4] and [8]). But the only irreducible

compact hermitian symmetric space of positive holomorphic curvature in the list of \tilde{E} . Cartan is the complex projective space.

REMARK. In fact we have shown above the following more general result: Let M be a compact, simply-connected homogeneous complex manifold whose Euler-Poincaré characteristic is positive; if its second Betti number is one, then M is isomorphic to an irreducible hermitian symmetric space (cf. Théorème 1, C.R.A.S. Paris 252, pp. 3377-3378 (1961), and [6]).

2. Let D be an irreducible symmetric bounded domain of one of the following types: $I_{m,m'}$ ($m > m' > 6$), II_m ($m > 7$), III_m ($m > 7$) or IV. If M is a compact quotient of D by a properly discontinuous subgroup of automorphisms of D , it is well known that $b_1(M) = 0$ and $b_2(M) = 1$. In fact, we have the following result essentially due to Remmert-Van de Ven (cf. p. 456, [7]):

PROPOSITION 1. Let M be a compact Kähler manifold of dimension greater than one; if $b_2 = 1$, then its first Betti number is zero.

Proof. Suppose that $b_1 = 2q$, $q = h^{1,0}(M)$, is positive; let $A(M)$ denote the Albanese manifold of M and let $\phi: M \rightarrow A(M)$ be the non-constant holomorphic onto projection. Since $b_2 = 1$, we have $h^{2,0}(M) = 0$ and hence M is algebraic by Kodaira's theorem. Therefore $\dim M = \dim A(M)$ by Theorem 1.3 of [7]; let ω be a nonzero holomorphic 2-form on $A(M)$; then $\phi^*\omega$ is a nonzero holomorphic 2-form on M , a contradiction.

In fact, we can prove the following result for negatively curved Kähler surfaces which generalizes a result of [3]:

PROPOSITION 2. Let M be a compact Kähler surface of negative Ricci curvature; then its first Betti number is zero.

Proof. Since the Ricci curvature is negative, we have $H^q(M, \Omega^p(K)) = 0$ if $p + q = 1$ by a result of Akizuki-Nakano [1]; consequently, $H^1(M, \Omega^0(K)) = H^{0,1}(K) = 0$ by Dolbeault's theorem. But $H^{0,1}(K) = H^{0,1}(M, K \otimes K^*) = H^{0,1}(M, 1)$ where 1 denote the trivial line bundle, by the duality theorem of Serre. Thus $h^{0,1} = \dim H^{0,1}(M, 1) = 0$ and hence $b_1 = 0$.

REMARK. Note that the Euler-Poincaré characteristic of such a surface is positive (cf., [3]).

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TRW SYSTEMS
NAVAL ORDNANCE TEST STATION

M. J. C. Baker, <i>A spherical Helly-type theorem</i>	1
Robert Morgan Brooks, <i>On locally m-convex $*$-algebras</i>	5
Lindsay Nathan Childs and Frank Rimi DeMeyer, <i>On automorphisms of separable algebras</i>	25
Charles L. Fefferman, <i>A Radon-Nikodym theorem for finitely additive set functions</i>	35
Magnus Giertz, <i>On generalized elements with respect to linear operators</i>	47
Mary Gray, <i>Abelian objects</i>	69
Mary Gray, <i>Radical subcategories</i>	79
John A. Hildebrandt, <i>On uniquely divisible semigroups on the two-cell</i>	91
Barry E. Johnson, <i>AW^*-algebras are QW^*-algebras</i>	97
Carl W. Kohls, <i>Decomposition spectra of rings of continuous functions</i>	101
Calvin T. Long, <i>Addition theorems for sets of integers</i>	107
Ralph David McWilliams, <i>On w^*-sequential convergence and quasi-reflexivity</i>	113
Alfred Richard Mitchell and Roger W. Mitchell, <i>Disjoint basic subgroups</i>	119
John Emanuel de Pillis, <i>Linear transformations which preserve hermitian and positive semidefinite operators</i>	129
Qazi Ibadur Rahman and Q. G. Mohammad, <i>Remarks on Schwarz's lemma</i>	139
Neal Jules Rothman, <i>An L^1 algebra for certain locally compact topological semigroups</i>	143
F. Dennis Sentiilles, <i>Kernel representations of operators and their adjoints</i>	153
D. R. Smart, <i>Fixed points in a class of sets</i>	163
K. Srinivasacharyulu, <i>Topology of some Kähler manifolds</i>	167
Francis C.Y. Tang, <i>On uniqueness of generalized direct decompositions</i>	171
Albert Chapman Vosburg, <i>On the relationship between Hausdorff dimension and metric dimension</i>	183
James Victor Whittaker, <i>Multiply transitive groups of transformations</i>	189