INTEGRAL KERNEL FOR ONE-PART FUNCTION SPACES

HERBERT STANLEY BEAR, JR. AND BERTRAM JOHN WALSH
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H. S. Bear and Bertram Walsh

Let $X$ be a separable compact Hausdorff space, and let $B$ be a linear space of continuous real functions on $X$, where $1 \in B$ and $B$ separates the points of $X$. Let $\Gamma$ denote the Silov boundary of $B$ in $X$, and assume that $\Delta = X \sim \Gamma \neq 0$. Further assumptions on $B$ are made which are in the nature of axioms for an abstract potential theory. These assumptions are more global than is usual, and in particular a sheaf axiom is not assumed, nor is the existence of a base of regular neighborhoods. Instead the assumptions are concerned with equicontinuity properties of $B$ on $\Delta$, and the consequences of $\Delta$ being a single Gleason part of $X$. With suitable hypotheses on $B$ and $\Delta$ there is an integral kernel representation of the following sort:

$$u(x) = \int_{\Gamma} u(\theta)Q(x, \theta)d\mu(\theta),$$

where $Q$ is a jointly measurable function on $\Delta \times \Gamma$ which is "in $B$" (i.e., abstractly harmonic) as a function of $x$ for each fixed $\theta \in \Gamma$.

2. Topologies on $\Delta$. Let $\mathcal{F}$ denote the given compact topology of $X$, usually considered as relativized to $\Delta$. Since $X$ is compact, $\mathcal{F}$ is the weak topology induced by $B$. Let $\|x\|_*$ be the norm of $B^*$ transferred to points of $\Delta$ by considering them as evaluation functionals. Let $\mathcal{F}_*$ be the metric topology on $\Delta$ obtained from the norm $\|\|_*$ of $B^*$. Clearly $\mathcal{F}_* \supseteq \mathcal{F}$. We will later introduce other topologies on $\Delta$ which are germane in the presence of additional assumptions on $B$.

Let ball $B = \{u \in B : \|u\| \leq 1\}$, and let

$$B^+(z) = \{u : \Delta : u \in B, u > 0, u(z) = 1\}$$

be the section of $B^+$ normalized at some $z \in \Delta$. We will be concerned with conditions implying the equicontinuity of ball $B$ and $B^+(z)$. We remark that Loeb and Walsh [7] have recently shown that equicontinuity of $B^+(z)$ can be taken as the convergence axiom of Brelot's axiomatic potential theory.

**Theorem 1.** If $B^+(z)$ is equicontinuous on $\Delta$, then ball $B$ is equicontinuous on $\Delta$. Ball $B$ is equicontinuous on $\Delta$ if and only if $\mathcal{F} = \mathcal{F}_*$ on $\Delta$.

**Proof.** Suppose that $B^+(z)$ is equicontinuous on $\Delta$ (with respect to $\mathcal{F}$) and that $\|u\| \leq 1$. Then $v = (u + 2)/(u(z) + 2) \in B^+(z)$. Given $\varepsilon > 0$ and $x \in \Delta$ there is a neighborhood $U$ of $x$ such that
\( |w(y) - w(x)| < \varepsilon \) for all \( w \in B^+(z) \) and all \( y \in U \). In particular

\[
\frac{|u(y) + 2|}{u(z) + 2} - 1 < \varepsilon
\]

for all \( y \in U \), and consequently \( |u(y) - u(x)| < 3\varepsilon \) if \( y \in U \) and \( ||u|| \leq 1 \). Hence ball \( B \) is equicontinuous.

We have already observed that \( \mathcal{F} \subset \mathcal{F}_* \). If \( x_n \to x \) implies that \( u(x_n) \to u(x) \) uniformly for \( ||u|| \leq 1 \) (equicontinuity of ball \( B \)), then certainly \( ||x_n - x||_e \to 0 \). That is, equicontinuity of ball \( B \) implies \( \mathcal{F} = \mathcal{F}_* \). The converse is clear.

We recall (see [1]) that \( X \) is decomposed into parts by the equivalence relation \( x \sim y \) if and only if \( 1/\alpha \leq u(x)/u(y) \leq \alpha \) for some \( \alpha \geq 1 \) and all positive \( u \in B \). If \( x \sim y \), let \( R(x, y) \) be the infimum of the numbers \( \alpha \) which satisfy the inequality. Then \( \log R(x, y) = d(x, y) \) is a metric on each part. We call \( d \) the “part metric”, and let \( d \mathcal{F}_d \) be the part metric topology. It will simplify the exposition without any real loss of generality to assume that \( \Delta \) is a single part. Otherwise the statements below would hold for individual parts within \( \Delta \).

**Theorem 2.** If \( \Delta \) is a part, then \( \mathcal{F}_d \supset \mathcal{F}_* \supset \mathcal{F} \), and \( B^+(z) \) is equicontinuous if and only if \( \mathcal{F} = \mathcal{F}_d \).

**Proof.** Suppose \( x_n, x \in \Delta \) and \( d(x_n, x) \to 0 \); i.e., \( R(x_n, x) \to 1 \). Given \( \varepsilon > 0 \) there is \( N \) such that

\[
\frac{|u(x_n)|}{u(x)} - 1 = \frac{|u(x_n) - u(x)|}{u(x)} < \varepsilon
\]

for all \( u > 0 \), if \( u \geq N \). If \( ||v|| \leq 1 \), and \( u = v + 2 \), then \( 1 \leq u \leq 3 \), \( u(x_n) - u(x) = v(x_n) - v(x) \), and

\[
\frac{|v(x_n) - v(x)|}{v(x) + 2} < \varepsilon
\]

if \( n \geq N \). Therefore \( |v(x_n) - v(x)| < 3\varepsilon \) if \( n \geq N \) and \( ||v|| \leq 1 \), and \( ||x_n - x||_e \to 0 \) if \( d(x_n, x) \to 0 \).

It is shown in [2, Th. 1] that \( d(x_n, x) \to 0 \) if and only if \( u(x_n) \to u(x) \) uniformly for all \( u \in B^+(z) \). If \( B^+(z) \) is equicontinuous on \( \Delta \), then by definition we have such convergence uniformly over \( B^+(z) \) whenever \( x_n \to x \) (in \( \mathcal{F} \)). Hence \( \mathcal{F} \supset \mathcal{F}_d \) if \( B^+(z) \) is equicontinuous.

We will say that \( B \) is a \((U)\)-space if for each \( x \in \Delta \) the evaluation functional \( e_x \in B^* \) has a unique maximal (in the sense of [9, §§4, 6]) representing probability measure \( \mu_x \) on \( \Gamma \); recall that this measure is in an appropriate sense supported by the Choquet boundary \( bX \) of \( X \) with respect to \( B \). Clearly \( B \) is a \((U)\)-space whenever the base
\{F; F \in B^*, \|F\| = 1 = F(1)\} of the positive cone in \(B^*\) is a simplex [9, §9], since that means that every positive linear functional on \(B\) has a unique maximal representing measure. It is known (see [6, p. 63, (14b)]) that this occurs if \(B\) has the Riesz decomposition property and if and only if its uniform closure does, so \(B\) is a \((U)\)-space whenever it is a Dirichlet space [2, p. 294]. If \(B\) is a \((U)\)-space and \(\Delta\) is a part, then the maximal representing measures for the point of \(\Delta\) are all mutually absolutely continuous with bounded derivatives both ways; for in the argument in [4] in which representing measures are constructed, there would be no loss in generality in taking the measures \(\alpha\) and \(\beta\) to be maximal, whence (since the maximal measures form a cone [9, p. 65]) \(\mu_z\) and \(\mu_y\) as constructed there would also be maximal—but uniqueness guarantees that those are our \(\mu_z\) and \(\mu_y\).

Let \(\mu = \mu_z\) represent the point \(z \in \Delta\), and write \(d\mu = g_x \, d\mu\) for \(x \in \Delta\).

We then have \(\Delta\) identified with a subset \(\{g_x : x \in \Delta\}\) of \(L^\mu(\mu)\) so that \(u(x) = \int g_x \, d\mu\) for all \(u \in B\) and all \(x \in \Delta\). Let \(\|\cdot\|_\infty\) be the \(L^\mu(\mu)\) norm, and write \(\|x - y\|_\infty = \|g_x - g_y\|_\infty\) to transfer this norm-metric to \(\Delta\). Let \(\mathcal{F}_\infty\) be the resulting topology on \(\Delta\).

**Theorem 3.** If \(\Delta\) is a part and \(B\) is a \((U)\)-space, then \(\mathcal{F}_{\infty} = \mathcal{F}_d \supset \mathcal{F}_s \supset \mathcal{F}\). If in addition \(B^+(z)\) is equicontinuous on \(\Delta\), then \(\mathcal{F} = \mathcal{F}_d = \mathcal{F}_s = \mathcal{F}_\infty = \mathcal{F}_{\infty}^*\).

**Proof.** If \(u \in B^+(z)\), then

\[
|u(x_n) - u(x)| = \left|\int u(x_n - x) \, d\mu\right| 
\leq \|x_n - x\|_\infty \int |u| \, d\mu 
= \|x_n - x\|_\infty u(z).
\]

Since \(u(z) = 1\) for \(u \in B^+(z)\), \(u(x_n) \to u(x)\) uniformly for \(u \in B^+(z)\) if \(\|x_n - x\|_\infty \to 0\). Hence \(d(x_n, x) \to 0\) if \(\|x_n - x\|_\infty \to 0\) by Theorem 1 of [2].

Now we show that \(d\)-convergence implies \(L_\infty\) convergence. Since \(B\) is a \((U)\)-space and \(R(x, y)\) is the infimum of the constants \(c\) usable in the proof of [4, Th. 1], any two Radon-Nikodým derivatives \(g_x, g_y\) must satisfy

\[
\frac{1}{R(x, y)} \leq \frac{g_x}{g_y} \leq R(x, y)
\]

almost everywhere with respect to \(\mu\). We also have, comparing \(g_x\) and \(g_y \equiv 1\) in the inequality above, that \(0 \leq g_x \leq R(x, z) = \exp d(x, z)\) holds a.e. \(\mu\). For \(x, y \in \Delta\), we have
\[ |g_x - g_y| \leq g_x[R(x, y) - 1] \leq R(y, z)[R(x, y) - 1] \]

holding a.e. \( \mu \). Since \( d(x, y) \to 0 \) is equivalent to \( R(x, y) \to 1 \), and \( R(y, z) \) is fixed, we have that \( ||g_x - g_y||_\infty \to 0 \) if \( x \to y \) in \( \mathcal{F}_d \). Hence \( \mathcal{F}_d = \mathcal{F}_\infty \) on \( \Delta \).

The final statement of the theorem follows from the equivalence of \( \mathcal{F} = \mathcal{F}_d \) with equicontinuity of \( B^+(z) \).

3. An integral kernel for \( B \). The second half of the proof of Theorem 3 is a modification of that used by Nakai \[8\] in the case that \( B \) consists of all harmonic functions on a Riemann surface with an ideal boundary which makes \( B \) a Dirichlet space. The results below include those obtained by Nakai, and the proof of Theorem 4 is essentially a modification of Nakai’s technique to our general situation.

**Theorem 4.** If \( \Delta \) is a part, \( B^+(z) \) is equicontinuous, and \( B \) is a \((U)\)-space, then there is a positive measure \( \mu = \mu_z \) and a jointly measurable function \( Q(x, \theta) \) on \( \Delta \times \Gamma \) such that \( Q(\cdot, \theta) \) is continuous on \( \Delta \) for each \( \theta \in \Gamma \), \( 0 \leq Q(x, \theta) \leq R(x, z) \) for all \( (x, \theta) \in \Delta \times \Gamma \), and

\[
\int_{\Gamma} Q(x, \theta) d\mu(\theta)
\]

for all \( u \in B \), all \( x \in \Delta \).

**Proof.** Let \( \mu \) represent \( z \) and let \( D \) be a countable dense subset of \( \Delta \) containing \( z \). For each fixed \( x \in D \) pick a measurable function \( Q(x, \cdot) \) on \( \Gamma \) such that \( Q(\cdot, \theta) \) represents \( x \). Then the inequalities

\[
|Q(x, \cdot) - Q(y, \cdot)| \leq R(y, z)[R(x, y) - 1]
\]

and

\[
0 \leq Q(x, \cdot) \leq R(x, z)
\]

hold a.e. \( \mu \) for all \( x, y \in D \). Let \( E \) be the union of the countably many \( \mu \)-null subsets of \( \Gamma \) where the inequalities above fail. Then \( \mu(E) = 0 \) and

\[
|Q(x, \theta) - Q(y, \theta)| \leq R(y, z)[R(x, y) - 1],
\]

\[
0 \leq Q(x, \theta) \leq R(x, z),
\]

hold for all \( x, y \in D \) and all \( \theta \in \Gamma \sim E \). If \( \{x_n\}, \{x'_n\} \) are two sequences in \( D \) both approaching \( x \in \Delta \), then \( |Q(x_n, \cdot) - Q(x'_n, \cdot)| \) converges uniformly to zero on \( \Gamma \sim E \). For any \( x \in \Delta \), pick any sequence \( x_n \in D \) with \( x_n \to x \), and define \( Q(x, \theta) = \lim Q(x_n, \theta) \) for \( \theta \in E \), and \( Q(x, \theta) = 1 \) for \( \theta \in E \). The function \( Q \) is well defined on \( \Delta \times \Gamma \) and satisfies the
desired inequalities. Moreover, $Q$ is measurable in $\theta$ and continuous in $x$ by its definition. Therefore (see [5, p. 285]) $Q$ is jointly measurable. By the bounded convergence theorem, if $u \in B$ then

$$u(x) = \lim u(x_n) = \lim \int_{\Gamma} u(\theta)Q(x_n, \theta)d\mu(\theta) = \int_{\Gamma} u(\theta)Q(x, \theta)d\mu(\theta),$$

and hence $Q(x, \cdot)d\mu(\cdot)$ represents $x$.

The kernel obtained by Nakai [8] by the sort of argument above is harmonic in $x$ for each fixed $\theta$. Walsh and Loeb [10] have a generalization of this result in the setting of the abstract potential theory of Brelot. Nakai's result can also be obtained by specializing the results of [3]. We show below that our kernel can be taken to be "in $B$" as a function of $x$ with no local hypotheses whatsoever.

Let $\hat{B}$ denote the closure, in the topology of uniform convergence on compact subsets of $\Delta$, of $B \backslash \Delta$. This space $\hat{B}$ is our abstract replacement for the space of all harmonic functions on the open set $\Delta$.

**Lemma 5.** If $\Delta$ is a part, $B^+(z)$ is equicontinuous, and $B$ is a $(U)$-space, then the mapping $T: B \backslash \Gamma \rightarrow B \backslash \Delta$ given by

$$T(u)(x) = \int_{\Gamma} u(\theta)Q(x, \theta)d\mu(\theta)$$

extends to a mapping $T: L_\lambda(\mu) \rightarrow \hat{B}$ which is continuous with respect to the $L_1$ norm and the u.c.c. topology of $\hat{B}$.

**Proof.** $Q(\cdot, \theta)$ is uniformly bounded on $K \times \Gamma'$ for each compact $K \subseteq \Delta$. The uniqueness of the maximal representing measure $\mu = \mu$ implies that $B \backslash \Gamma$ is dense in $L_1(\mu)$, for if $g \in L_\infty(\mu)$ has the property that $g \cdot \mu$ annihilates $B \backslash \Gamma$, then (assuming without loss of generality that $\|g\|_\infty < 1$) the measure $(1 + g) \cdot \mu$ is also maximal (since by [9, p. 65] the cone of maximal measures is hereditary) and also represents $z$, so that $g = 0$. Thus the mapping $T$ can be extended by denseness and continuity to all of $L_1(\mu)$, and the images will remain in $\hat{B}$.

**Lemma 6.** If $\Delta$ is a part and $B^+(z)$ is equicontinuous, then $\Delta$ is $\sigma$-compact.

**Proof.** Since $\Delta$ is open in $X$ and $X$ is separable, $\Delta$ is also separable. Since $\mathcal{F} = \mathcal{F}_\Delta$ with our hypotheses, $\Delta$ is a metric space.
Let \( \{y_k\} \) be a countable dense subset of \( \Delta \), and let
\[
R_k = \sup \{r : S(y_k, r) \cap \Gamma = \emptyset\}
\]
where \( S(y_k, r) \) is the \( r \)-sphere about \( y_k \). If some \( R_k = \infty \), then the
sets \( S(y_k, r) \) are compact subsets of \( \Delta \) whose union is all of \( \Delta \), and
we are done. Otherwise each \( R_k < \infty \) and the spheres \( S(y_k, r) \), where \( r \) runs through all rationals \( < R_k \), exhaust \( \Delta \). To see this, notice
that for any \( x \in \Delta \) there is a rational \( \rho > 0 \) such that \( S(x, \rho) \subset \Delta \).
If \( y_k \in S(x, \rho/2) \), then \( x \in S(y_k, \rho/2) \subset \Delta \) and \( \rho/2 < R_k \).

**Theorem 7.** If \( \Delta \) is a part, \( B^+(z) \) is equicontinuous, and \( B \) is
a \((U)\)-space, then there is a function \( Q(x, \theta) \) as in Theorem 4 such
that \( Q(\cdot, \theta) \in B \) for each \( \theta \in \Gamma \).

**Proof.** We give \( C(\Delta) \) the locally convex topology of uniform
convergence on compact sets. Since \( \Delta \) is \( \sigma \)-compact, \( C(\Delta) \) is metrizable.
If \( \Delta = \bigcup K_n \) where each \( K_n \) is a compact (and metric) subset of \( \Delta \),
then \( C(K_n) \) is separable in the uniform topology, and hence \( C(\Delta) \) is separable in the u.c.c. topology. Since \( C(\Delta) \) has a countable base of
convex open sets, the open set \( C(\Delta) \sim \hat{B} \) can be written as a countable
union of open convex sets, and we can take each such set \( U \) to have
its closure disjoint from \( \hat{B} \).

If \( E = \{\theta \in \Gamma : Q(\cdot, \theta) \in \hat{B}\} \) has zero \( \mu \)-measure, then we can redefine
\( Q \) to be one on \( \Delta \times E \) and the resulting function will still satisfy
Theorem 4 and will be in \( \hat{B} \) as a function of \( x \) for each \( \theta \in \Gamma \).
Assume on the contrary that \( \mu(E) > 0 \). By the countable additivity
of \( \mu \), there is some \( U \) such that \( E_U = \{\theta \in \Gamma : Q(\cdot, \theta) \in U\} \) has positive
\( \mu \)-measure, provided these sets are \( \mu \)-measurable subsets of \( \Gamma \). To
show the measurability of \( E_U \), it suffices to consider \( E_U \) for a basic
open set \( U = \{g \in C(\Delta) : |g(x) - v(x)| < \epsilon \text{ for } x \in K\} \) where \( \epsilon > 0 \) and
\( K \) is compact. If \( \{x_n\} \) is a dense sequence in \( K \), and \( \theta \) is a fixed
point of \( \Gamma \), then \( |Q(x, \theta) - v(x)| \leq \epsilon' \) for all \( x \in K \) if and only if
\( |Q(x_n, \theta) - v(x_n)| \leq \epsilon' \) for all \( n \), since \( Q \) is continuous in \( x \). The set
\( \{\theta : |Q(x_n, \theta) - v(x_n)| \leq \epsilon'\} \) is measurable since \( Q \) is measurable in \( \theta \),
and hence the intersection \( \{\theta : |Q(x, \theta) - v(x)| \leq \epsilon' \text{ all } x \in K\} \) is measurable.
Finally, \( \{\theta : |Q(x, \theta) - v(x)| < \epsilon\} \) is a countable union of sets
corresponding to values of \( \epsilon' < \epsilon \).

By the Hahn-Banach theorem we can separate \( U \) from the closed
subspace \( \hat{B} \), and there is a functional \( F \in C(\Delta)^* \) such that \( F = 0 \) on
\( \hat{B} \) and \( F(\omega) > 0 \) for \( \omega \in U \). In particular, \( F(Q(\cdot, \theta)) > 0 \) for \( \theta \in E_U \).
For some \( \epsilon > 0 \), the set \( S = \{\theta : F(Q(\cdot, \theta)) \geq \epsilon\} \) must have positive
\( \mu \)-measure. The dual space of \( C(\Delta) \) can be represented by the space
of regular Borel measures with compact support in \( \Delta \), and we let \( \lambda \)
be the measure corresponding to $F$. Define $v$ on $\Delta$ by

$$v(x) = \int_r \chi_s(\theta)Q(x, \theta)d\mu(\theta).$$

By Lemma 5, $v \in \hat{B}$ and hence $F(v) = 0$:

$$0 = F(v) = \int_a v(x)d\lambda(x)$$
$$= \int_a \int_r \chi_s(\theta)Q(x, \theta)d\mu(\theta)d\lambda(x)$$
$$= \int_r \int_a Q(x, \theta)d\lambda(x)\chi_s(\theta)d\mu(\theta)$$
$$= \int_r F(Q(\cdot, \theta))\chi_s(\theta)d\mu(\theta)$$
$$\geq \varepsilon\mu(S) > 0.$$

The interchange of integrals is justified because $Q$ is jointly measurable and bounded for $x$ in the compact support $\lambda$. The contradiction completes the proof.

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