A GENERALIZATION OF THE BORSUK-WHITEHEAD-HANNER THEOREM

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Let $A$ and $B$ be metric spaces and let $f: A \to B$ be a map. Suppose that $X$ and $Y$ are ANR's containing $A$ and $B$, respectively, as closed subsets, and consider $f$ to be a map from $A$ into $Y$. One of the results of this paper is that the question as to whether or not the adjunction space $X \cup_f Y$ is an absolute neighborhood extensor for metric pairs (or ANR if $X \cup_f Y$ is metrizable) depends only on $f$ and not on $X$ and $Y$; that is, if $X \cup_f Y$ is an ANE (metric) and if $X$ and $Y$ are replaced by ANR's $X'$ and $Y'$, respectively, then $X' \cup_f Y'$ is an ANE (metric). This result is a consequence of the main theorem:

Let $B$ be a strong neighborhood deformation retract of a space $Y$ and suppose that both $B$ and $Y - B$ are ANE (metric). If $Y - B$ has a certain type of covering, then $Y$ is an ANE (metric). This generalizes the known result that if $Y$ is metrizable, then $Y$ is an ANR.

By a pair $(X, A)$ we shall mean a space $X$ together with a closed subset $A$. If a space $Y$ has the property that for every metric pair $(X, A)$, each map $f: A \to Y$ has a neighborhood extension, then $Y$ is called an absolute neighborhood extensor for metric pairs (abbreviated ANE). In particular, a space is an ANR if and only if it is a metrizable ANE [2].

Let $(X, A)$ be a pair, and let $f: A \to Y$ be a map. It is well known [4, p. 178] that if $X$, $A$ and $Y$ are ANR's, then the adjunction space $X \cup_f Y$ is an ANR provided that it is metrizable. This result was essentially proved in successive stages by Borsuk [1], Whitehead [7], and Hanner [3]. Our purpose is to generalize this theorem.

The author wishes to thank Professors James Dugundji and A. H. Kruse for many valuable suggestions.

2. The main theorem. Let $(Y, B)$ be a pair. Generalizing the notion of a canonical cover [2], we say that a collection \{\(V_\alpha\)\} of open subsets of $Y$ is a semi-canonical cover of $(Y, B)$ if (1) $\bigcup_\alpha V_\alpha = Y - B$ and (2) for each $b \in B$ and each neighborhood $U$ of $b$ there is a neighborhood $W$ of $b$ such that $V_\alpha \subseteq U$ whenever $V_\alpha$ meets $W$.

A semi-canonical cover differs from a canonical cover only in that a semi-canonical cover is not required to be locally finite.
canonical covers.

**Lemma 2.1.** Suppose that \( \{V_a\} \) is a semi-canonical cover for a pair \((Y, B)\). Let \( \{x_u\} \) and \( \{y_v\} \) be two nets in \( Y - B \), and suppose that for each \( v, x_v \) and \( y_v \) lie in a common element \( V_v \) of \( \{V_a\} \). Then \( \{x_v\} \) converges to a point \( b \in B \) if and only if \( \{y_u\} \) converges to \( b \).

**Proof.** Suppose that \( \{x_v\} \) converges to \( b \). Let \( U \) be any neighborhood of \( b \), and let \( W \) be a neighborhood of \( b \) such that \( V_a \subset U \) whenever \( V_a \cap W \neq \emptyset \). Since \( \{x_v\} \) is eventually in \( W \), the sets \( \{V_v\} \) eventually lie in \( U \), and since \( y_v \in V_v \), it follows that \( \{y_u\} \) converges to \( b \). The converse is proved similarly.

**Remark.** If \( \{V_a\} \) is a semi-canonical cover of \((Y, B)\) and if for each \( y \in Y - B \) an element—call it \( V_y \)—of \( \{V_a\} \) containing \( y \) is chosen, then the collection \( \{V_y\}, y \in Y - B \), is a semi-canonical cover of \((Y, B)\).

A closed subset \( B \subset Y \) is called a strong neighborhood deformation retract of \( Y \) if there exists a neighborhood \( W \) of \( B \) and a homotopy \( h: W \times I \to Y \) such that \( h_0 \) is the inclusion, \( h_1 \) is a retraction of \( W \) onto \( B \), and \( h(b, t) = b \) for all \( b \in B, t \in I \). \( h \) is called a strong deformation retraction of \( W \) onto \( B \).

We now establish the main theorem.

**Theorem 2.2.** Let \((Y, B)\) be a semi-canonical pair such that \( B \) is a strong neighborhood deformation retract of \( Y \). If both \( B \) and \( Y - B \) are ANE, then \( Y \) is an ANE.

**Proof.** By hypothesis, there exists a strong deformation retraction \( h: W \times I \to Y \) onto \( B \). Let \( \{V_y\}, y \in Y - B \), be a semi-canonical cover for \((Y, B)\) as in the remark above.

To prove that \( Y \) is an ANE it is sufficient to show that for any metric pair \((X, A)\), each map \( f: A \to W \) has a neighborhood extension \( F: U \to Y \). For from this it follows first that \( F| F^{-1}(W): F^{-1}(W) \to W \) is a neighborhood extension of \( f \), so that \( W \) is an ANE; and then \( Y \), being the union of the open ANE subspaces \( W \) and \( Y - B \), is itself an ANE \([4, \text{p. 44}]\). Given \((X, A)\) and \( f: A \to W \), we proceed to construct \( F \).

Let \( A_0 = f^{-1}(B), A_1 = A - A_0 \) and \( X_1 = X - A_0 \). Then \( f(A_1) \subset Y - B \), and since \( Y - B \) is an ANE, there is a neighborhood \( G_1 \) of \( A_1 \) in \( X_1 \) and a map \( \phi_1: G_1 \to Y - B \) such that \( \phi_1| A_1 = f| A_1 \). Let \( d \) be a metric on \( X \). For each \( a \in A_1 \), let \( G_a \) be the set of points \( x \) in \( G_1 \) such that

1. \( d(x, A_0) > 1/2d(a, A_0) \),
2. \( d(x, a) < d(a, A_0) \),
(3) \( x \in \phi_{1}^{-1}(V_{\phi_{1}(a)}) \), and
(4) \( x \in \phi_{2}^{-1}(W) \).

Let \( G_{2} = \bigcup \{ G_{a} \mid a \in A_{0} \} \). \( G_{2} \) is open in \( X \) and contains \( A \). Let \( G \) be a neighborhood of \( A \) in \( X \) such that its closure \( K \) (in \( X \)) is contained in \( G_{2} \), and let \( \lambda : X \rightarrow [0,1] \) be a map such that \( \lambda(A) = 0 \) and \( \lambda(X - G) = 1 \).

Define \( \phi_{2} : K \cup A_{0} \rightarrow Y \) by
\[
\phi_{2}(x) = \begin{cases} h(\phi_{1}(x), \lambda(x)) & \text{if } x \in K, \\ f(x) & \text{if } x \in A_{0}. \end{cases}
\]

\( \phi_{2} \) is well-defined and extends \( f \). Furthermore, \( \phi_{2} \) is clearly continuous except possibly at those points of \( A_{0} \) which are limit points of \( K - A \). To prove its continuity at these points also, we suppose \( a \in A_{0} \) is the limit of a sequence \( \{ x_{n} \} \) in \( K - A \) and show that \( \{ \phi_{2}(x_{n}) \} \) converges to \( \phi_{2}(a) \). For each \( n \), choose \( a_{n} \in A_{1} \) such that \( x_{n} \in G_{a_{n}} \). Since \( \{ x_{n} \} \) converges to \( a \in A_{0} \), it follows from (1) that \( \{ d(a_{n}, A_{1}) \} \rightarrow 0 \), and from (2) that \( d((x_{n}, a_{n})) \rightarrow 0 \). Therefore \( \{ a_{n} \} \) converges to \( a \). Since \( \{ \phi_{2}(a_{n}) \} = \{ f(a_{n}) \} \) converges to \( f(a) \), we find by (3) and 2.1 that \( \{ \phi_{2}(x_{n}) \} \) converges to \( f(a) \). Given a neighborhood \( V \) of \( f(a) \) in \( Y \), there is a neighborhood \( V_{1} \) of \( f(a) \) such that \( h(V_{1} \times I) \subset V \). Since \( \{ \phi_{2}(x_{n}) \} \) converges to \( f(a) \), \( \{ \phi_{2}(x_{n}) \} \) is eventually in \( V_{1} \), and by the definition of \( \phi_{2} \), \( \{ \phi_{2}(x_{n}) \} \) is eventually in \( V \). Therefore \( \phi_{2} \) is continuous at \( a \), and hence is continuous on \( K \cup A_{0} \).

Since \( \lambda = 1 \) on the boundary (in \( X \)) of \( G \), and since \( h \) maps \( W \times 1 \) onto \( B \), it follows that \( \phi_{2} \) maps the boundary (in \( X \)) of \( K \cup A_{0} \) into \( B \). Since \( B \) is an ANE, it follows that \( \phi_{2} \) has an extension \( F : U \rightarrow Y \) for some open set \( U \) in \( X \), and the proof is complete.

3. Applications. In order to apply Theorem 2.2, it is necessary to have on hand some semi-canonical pairs. For this purpose we establish.

**Lemma 3.1.** Every metric pair \((Y, B)\) is semi-canonical.

**Proof.** As in [2], for each \( y \in Y - B \) let \( V_{y} \) be the open \( \varepsilon/2 \) ball centered at \( y \), where \( \varepsilon \) is the distance from \( y \) to \( B \) under some fixed metric for \( Y \). The collection \( \{ V_{y} \} \) is a semi-canonical cover for \((Y, B)\).

Combining 3.1 and 2.2, we obtain the following result, which was first proved in [5]:

**Theorem 3.2. (Kruse-Liebnitz).** Let \((Y, B)\) be a metric pair such that \( B \) is a strong neighborhood deformation retract of \( Y \). If \( B \) and \( Y - B \) are ANR's, then \( Y \) is an ANR.

Given a metric space \( A \), let \( \text{ANR}(A) \) denote the class of all ANR's
that contain \( A \) as a closed subset. Let \( f \) be a map from \( A \) into an ANR \( Y \). Our next result (3.5) states that either the adjunction space \( X \cup_f Y \) is an ANE for every \( X \in \text{ANR}(A) \) or for no \( X \in \text{ANR}(A) \). Therefore, given an \( X \in \text{ANR}(A) \), the question of whether or not \( X \cup_f Y \) is an ANE depends only on the map \( f \), and not on the choice of \( X \).

To obtain this result from 2.2, some additional information concerning semi-canonical covers and strong neighborhood deformation retractions will be needed. The necessary facts are supplied by the following lemmas.

For any pair \((X, A)\) and map \( f: A \rightarrow Y \), let \( X + Y \) denote the disjoint union of \( X \) and \( Y \), and let \( p: X + Y \rightarrow X \cup_f Y \) be the natural projection.

**Lemma 3.3.** Let \((X, A)\) be a pair and let \( f: A \rightarrow Y \) be a map. If \( \{V_a\} \) is a semi-canonical cover for \((X + Y, A + Y)\), then \( \{p(V_a)\} \) is a semi-canonical cover for \((X \cup_f Y, p(Y))\).

**Proof.** Since \( p \) maps \( X - A \) homeomorphically onto \( X \cup_f Y - p(Y) \), it follows that each \( p(V_a) \) is open and \( \bigcup_a p(V_a) = X \cup_f Y - p(Y) \). Let \( y \in p(Y) \) and let \( U \) be a neighborhood of \( y \). Since \( \{V_a\} \) is semi-canonical, for each \( x \in p^{-1}(U \cap p(Y)) \) there is a neighborhood \( W_x \subset p^{-1}(U) \) such that \( V_a \subset p^{-1}(U) \) whenever \( V_a \cap W_x \neq \emptyset \). Let \( W = \bigcup \{W_x \mid x \in p^{-1}(U \cap p(Y))\} \).

From our construction it is clear that \( y \in p(W) \) and that \( p(V_a) \subset U \) whenever \( p(V_a) \cap p(W) \neq \emptyset \). It remains to show that \( p(W) \) is open. Since \( p \) is an identification, it is sufficient to show that \( W \) is saturated, that is, \( W = p^{-1}(S) \) for some \( S \subset X \cup_f Y \). From our construction we have \( W \cap p^{-1}(p(Y)) = p^{-1}(U) \cap p^{-1}(p(Y)) = p^{-1}(U \cap p(Y)) \). Moreover, since \( p \) is one-to-one on \( (X + Y) - p^{-1}(p(Y)) \) it follows that \( W - p^{-1}(p(Y)) \) is saturated. Since \( W \) is the union of the saturated sets \( W \cap p^{-1}(p(Y)) \) and \( W - p^{-1}(p(Y)) \), \( W \) itself is saturated, and the lemma is proved.

**Lemma 3.4.** Let \( X \) and \( Y \) be ANR’s, and let \( f: A \rightarrow Y \) be a map, where \( A \) is a closed subset of \( X \). Then \( X \cup_f Y \) is an ANE if and only if \( p(Y) \) is a strong neighborhood deformation retract of \( X \cup_f Y \).

**Proof.** Suppose that \( X \cup_f Y \) is an ANE. Since \( Y \) is an ANR, \( f \) has an extension \( F: \bar{U} \rightarrow Y \), where \( U \) is some neighborhood of \( A \) in \( X \). Define a map \( g: X \times \{0\} \cup A \times I \cup \bar{U} \times \{1\} \rightarrow X \cup_f Y \) by
Since $X \cup_f Y$ is an ANE, $g$ has an extension $G: V \to X \cup_f Y$, for some open subset $V$ of $X \times I$. Let $W$ be a neighborhood of $A$ in $X$ such that $W \times I \subseteq V$. The map $h: p(W + Y) \times I \to X \cup_f Y$ defined by

$$h(z, t) = G((p|X)^{-1}(z), t) \quad \text{if} \quad z \in p(W), \quad 0 \leq t \leq 1,$$

$$= z \quad \text{if} \quad z \in p(Y), \quad 0 \leq t \leq 1,$$

is the desired deformation.

The converse is an immediate consequence of 3.3 and 2.2.

We now obtain the main result of this section.

**Theorem 3.5.** Let $f$ be a map from an arbitrary metric space $A$ into an ANR $Y$. If $X_0 \cup_f Y$ is an ANE for some $X_0 \in \text{ANR}(A)$, then $X \cup_f Y$ is an ANE for every $X \in \text{ANR}(A)$.

*Proof.* Given $X \in \text{ANR}(A)$, let $p: X + Y \to X \cup_f Y$ and $q: X_0 + Y \to X_0 \cup_f Y$ be the natural projections. To prove that $X \cup_f Y$ is an ANE it is sufficient, by 3.4, to show that $p(Y)$ is a strong neighborhood deformation retract of $X \cup_f Y$.

Since $X$ is an ANR, there exists a neighborhood $G$ of $A$ in $X_0$ and a map $\phi: G \to X$ such that $\phi|A$ is the identity map. By 3.4, there is a neighborhood $W$ of $q(Y)$ in $X_0 \cup_f Y$ and a strong deformation retraction $h$ of $W$ onto $q(Y)$ over $q(G + Y)$. Since $q^{-1}(W) \cap X_0$ is open in $X_0$, $q^{-1}(W) \cap X_0$ is an ANR; therefore there exists a neighborhood $U$ of $A$ in $X$ and a map $\psi: U \to q^{-1}(W) \cap X_0$ such that $\psi|A$ is the identity map. Since $U$ is open in $X$, $U$ is an ANR; and it follows that there exists a neighborhood $V$ of $A$ in $U$ and a deformation $j: V \times I \to U$ such that $j(a, t) = a$, for all $a \in A$, $0 \leq t \leq 1$, and such that $j_t = \phi \psi|V$. Letting $\phi + 1_Y: G + Y \to X + Y$ be the map defined by $\phi$ and the identity on $Y$, define a map $k: p(V + Y) \times I \to X \cup_f Y$ by

$$k_t(z) = p j_{2t}(p|X)^{-1}(z) \quad \text{if} \quad z \in p(V), \quad 0 \leq t \leq 1/2,$$

$$= p(\phi + 1_Y) q^{-1} h_{2t-1} q \psi(p|X)^{-1}(z) \quad \text{if} \quad z \in p(V) , \quad 1/2 \leq t \leq 1,$$

$$= z \quad \text{if} \quad z \in p(Y), \quad 0 \leq t \leq 1.$$

It is easily verified that $k$ is a strong deformation retraction of $p(V + Y)$ onto $p(Y)$, and the proof is complete.

An application of 3.5 gives a direct generalization of the BWH theorem:
Corollary 3.6. Let \((X, A)\) be a pair, and let \(f: A \to Y\) be a map. If \(X, A\) and \(Y\) are ANR's, then \(X \cup_f Y\) is an ANE.

Proof. This result can be obtained as a consequence of 3.3 and 2.2, but it also follows quite simply from 3.5: Taking \(X_0 = A\), we see that \(X_0 \cup_f Y\) is an ANR, since it is homeomorphic to \(Y\). Therefore by 3.5, \(X \cup_f Y\) is an ANE.

If we take \(Y\) in 3.5 to be a single point, we obtain

Corollary 3.7. If \(A\) is a metric space, then either \(X/A\) is an ANE for every \(X \in \text{ANR}(A)\) or for no \(X \in \text{ANR}(A)\).

If \(A\) is a compact subset of a metric space \(X\), then \(X/A\) is metrizable [6]. Therefore we have from 3.7

Corollary 3.8. If \(A\) is a compact metric space, then either \(X/A\) is an ANR for every \(X \in \text{ANR}(A)\) or for no \(X \in \text{ANR}(A)\).

We have seen that for a map \(f: A \to Y\), the question of whether or not \(X \cup_f Y\) is an ANE is independent of the choice of \(X \in \text{ANR}(A)\). Our final result, which slightly generalizes 3.5, shows that this question is also independent of \(Y\). Precisely, we have

Theorem 3.9. Let \(A\) and \(B\) be metric spaces and let \(f: A \to B\) be a map. Either \(X \cup_f Y\) is an ANE for every \(X \in \text{ANR}(A)\) and \(Y \in \text{ANR}(B)\) or for no \(X \in \text{ANR}(A)\) and \(Y \in \text{ANR}(B)\).

Remark. For \(Y \in \text{ANR}(B)\), we consider \(f\) to be not only a map from \(A\) into \(B\) but also from \(A\) into \(Y\). This justifies the symbol \(X \cup_f Y\).

Proof of Theorem. Suppose that \(X \cup_f Y_0\) is an ANE for some \(X \in \text{ANR}(A)\) and some \(Y_0 \in \text{ANR}(B)\). In view of 3.5, we need only to show that if \(Y \in \text{ANR}(B)\) then \(X \cup_f Y\) is an ANE.

Since \(Y\) is an ANR, there is a neighborhood \(U\) of \(B\) in \(Y_0\) and a map \(\phi: U \to Y\) such that \(\phi(b) = b\) for all \(b \in B\).

Letting \(p: X + Y \to X \cup_f Y\) and \(q: X + U \to X \cup_f U\) be the natural projections, define a map \(\psi: X \cup_f U \to X \cup_f Y\) by

\[
\psi(z) = p(q \mid X)^{-1}(z) \quad \text{if} \quad z \in q(X),
\]

\[
= p\phi(q \mid U)^{-1}(z) \quad \text{if} \quad z \in q(U).
\]

\(X \cup_f U\) is open in \(X \cup_f Y_0\), and therefore \(X \cup_f U\) is an ANE. By 3.4 there is a strong deformation retraction \(h\) of an open set \(W\) onto \(q(U)\) in \(X \cup_f U\). Define a homotopy \(k_t: \psi(W) \cup p(Y) \to X \cup_f Y\) by
$k_t(z) = \psi h_t \psi^{-1}(z)$ if $z \in \psi(W)$, 
$= z$ if $z \in p(Y)$.

It follows from the equation $\psi(W) \cup p(Y) = p((q | X)^{-1}(W) + Y)$ that $\psi(W) \cup p(Y)$ is an open subset of $X \cup Y$, and it is easily verified that $k$ is a strong deformation retraction of $\psi(W) \cup p(Y)$ onto $p(Y)$. The result now follows from 3.4.

4. Results for AR’s. In this section we establish results for AR’s and AE’s analogous to Theorems 2.2 and 3.9. A space $Y$ is called an absolute extensor for metric pairs (abbreviated AE) if for every metric pair $(X, A)$ each map $f: A \to Y$ has an extension $F: X \to Y$. A link between AE’s and ANE’s is provided by the following

**Lemma 4.1.** If $Y$ is an ANE and if $Y$ can be deformed into an AE subspace, then $Y$ is an AE.

**Proof.** Let $B \subset Y$ be an AE and let $h: Y \times I \to Y$ be a deformation such that $h_t(Y) \subset B$. Suppose that $(X, A)$ is a metric pair and let $f: A \to Y$ be a map. Since $Y$ is an ANE, there is a neighborhood $U$ of $A$ in $X$ and an extension $F: \bar{U} \to Y$ of $f$. Let $g: X \to [0, 1]$ be a map such that $g(A) = 0$ and $g(X - U) = 1$. Since $B$ is an AE, there is a map $G: X - U \to B$ such that $G|\text{bdry } U = h_tF|\text{bdry } U$. Define a map $\phi: X \to Y$ by

$$
\phi(x) = h(F(x), g(x)) \quad \text{if } x \in \bar{U},
$$
$$
= G(x) \quad \text{if } x \in X - U.
$$

$\phi$ extends $f$, and the lemma is proved.

We now establish the analog of 2.2.

**Theorem 4.2.** Let $(Y, B)$ be a semi-canonical pair such that $B$ is a strong deformation retract of $Y$. If $B$ is an AE and if $Y - B$ is an ANE, then $Y$ is an AE.

**Proof.** By 2.2, $Y$ is an ANE. Since by hypothesis $Y$ is deformable into $B$, $Y$ is an AE by 4.1.

In order to obtain the analog of 3.9, we will need the analog of 3.4.

**Lemma 4.3.** Let $X$ and $Y$ be AR’s, and let $f: A \to Y$ be a map, where $A$ is a closed subset of $X$. Then $X \cup_f Y$ is an AE if and only if $p(Y)$ is a strong deformation retract of $X \cup_f Y$. 


Proof. Suppose that $X \cup_f Y$ is an AE. Since $Y$ is an AR, $f$ has an extension $F: X \to Y$. Since $X \cup_f Y$ is an AE, the map 

$$g: X \times [0] \cup A \times I \cup X \times \{1\} \to X \cup_f Y$$

defined by

$$g(x, 0) = p(x) \quad \text{if } x \in X,$$

$$g(a, t) = p(a) \quad \text{if } a \in A, \; 0 \leq t \leq 1,$$

$$g(x, 1) = pF(x) \quad \text{if } x \in X,$$

has an extension $G: X \times I \to X \cup_f Y$. The map $h: X \cup_f Y \times I \to X \cup_f Y$ defined by

$$h(z, t) = G((p \mid X)^{-1}(z), t) \quad \text{if } z \in p(X), \; 0 \leq t \leq 1$$

$$= z \quad \text{if } z \in p(Y), \; 0 \leq t \leq 1$$

is the desired deformation.

Conversely, if $p(Y)$ is a strong deformation retract of $X \cup_f Y$, then $X \cup_f Y$ is an ANE by 3.4 and an AE by 4.1.

We now establish the analog of 3.9.

**Theorem 4.4.** Let $A$ and $B$ be metric spaces and let $f: A \to B$ be a map. Either $X \cup_f Y$ is an AE for every $X \in AR(A)$ and $Y \in AR(B)$ or for no $X \in AR(A)$ and $Y \in AR(B)$.

Proof. Suppose $X_0 \cup_f Y_0$ is an AE for some $X_0 \in AR(A)$ and $Y_0 \in AR(B)$, and suppose $X \in AR(A)$ and $Y \in AR(B)$. Let $p: X + Y \to X \cup_f Y$ and $q: X_0 + Y_0 \to X_0 \cup_f Y_0$ be the natural projections.

By 3.9, $X \cup_f Y$ is an ANE; to prove that it is an AE it is sufficient, by 4.3, to show that $X \cup_f Y$ can be deformed into $p(Y)$.

Since $X$ and $X_0$ are AR's, there are maps $\phi: X \to X_0$ and $\phi_0: X_0 \to X$, each extending the identity on $A$, and a deformation $j_t$ on $X$ leaving $A$ pointwise fixed and such that $j_1 = \phi \circ \phi_0$. Similarly, there are maps $\psi: Y \to Y_0$ and $\psi_0: Y_0 \to Y$, each extending the identity on $B$, and a deformation $k_t$ on $Y$ leaving $B$ pointwise fixed and such that $k_1 = \psi \circ \psi_0$.

By 4.3, there is a strong deformation retraction $h_t$ of $X_0 \cup_f Y_0$ onto $q(Y_0)$. Define a deformation $g_t$ on $X \cup_f Y$ by

$$g_t(z) = pj_{2t}(p \mid X)^{-1}(z) \quad \text{if } z \in p(X), \; 0 \leq t \leq 1/2,$$

$$= pk_{2t}(p \mid Y)^{-1}(z) \quad \text{if } z \in p(Y), \; 0 \leq t \leq 1/2,$$

$$= p(\phi_0 + \psi_0)q^{-1}h_{2t-1}q\phi(p \mid X)^{-1}(z) \quad \text{if } z \in p(X), \; 1/2 \leq t \leq 1,$$

$$= p(\phi_0 + \psi_0)q^{-1}h_{2t-1}q\psi(p \mid Y)^{-1}(z) \quad \text{if } z \in p(Y), \; 1/2 \leq t \leq 1,$$

where $\phi_0 + \psi_0: X_0 + Y_0 \to X + Y$ is the map defined by $\phi_0$ and $\psi_0$. $g$
deforms $X \cup Y$ into $p(Y)$, and the proof is complete.

By taking $B$ to be a single point, we obtain

**Corollary 4.5.** If $A$ is a metric space, then either $X/A$ is an $AE$ for every $X \in AR(A)$ or for no $X \in AR(A)$.

**Corollary 4.6.** If $A$ is a compact metric space, then either $X/A$ is an $AR$ for every $X \in AR(A)$ or for no $X \in AR(A)$.

**References**


Received December 12, 1966.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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