TWO SOLVABILITY THEOREMS

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In this paper we prove two theorems which have certain similarities.

**Theorem I.** Let $G$ be a group with a cyclic $S_p$ subgroup $P$ such that every $p'$-subgroup of $G$ is abelian. Then either $G$ has a normal $p$-complement or else $P 	riangle G$.

**Theorem II.** Let $G$ be a group and let $p \neq 2$ and $q$ be primes dividing $|G|$. Suppose for every $H < G$ which is not a $p'$-group or a $q'$-group that $p \mid |H|$. If $q^a$ is the $q$-part of $|G|$ and $p > q^a - 1$ or if $p = q^a - 1$ and an $S_p$ of $G$ is abelian then no primes but $p$ and $q$ divide $|G|$.

Both theorems are proved by studying minimal counter-examples and in both cases contradictions are obtained for $p > 3$ without the use of character theory. When $p = 3$ both minimal counterexamples satisfy the hypotheses of the same character theoretic proposition which is actually a special case of Theorem II, and this yields the desired contradictions.

Both theorems imply that the respective groups in question are solvable. In the first case the Schur-Zassenhaus Theorem (see 9.3.6 of [5]) is used and in the second case Burnside’s $p^aq^b$ theorem (see 12.3.3 of [5]) yields the solvability.

1. In this section we prove the character theoretic proposition which is a special case of Theorem II and which is used to prove both of our main results. We begin by giving a lemma which is a restatement of some of the results of §II of [1].

**Lemma 1.** (Brauer-Fowler) Let $G$ be a group of even order which has only one class of involutions $K_0$ with $m = |K_0|$. Let $K_i, 1 \leq i \leq r$ be the remaining nonidentity real classes in $G$. Then

$$m^2 = um + \sum_{i=1}^{r} v_i |K_i|$$

where $u$ is the number of involutions in the centralizer of an involution and $v_i$ is the number of involutions which transform $x$ to $x^{-1}$ when $x \in K_i$.

**Proposition.** Let $G$ be a group with an abelian $S_3$ subgroup $P$ with the properties

(1) $|\mathcal{N}_o(P)| = 4 |P|$, $|\mathcal{C}_o(P)| = 2 |P|$,
(2) $G(P)$ is a T.I. set and
(3) if $H < G$ has even order then $|H| |(4|P|)$. Then $G$ is not simple.

Proof. Suppose $G$ is simple. It is clear that the order of an $S_2$ of $G$ is 4 and thus by Burnside's theorem it must be elementary and all of its involutions are conjugate in its normalizer. Put

$$S = G(P) = P \times \langle s \rangle \quad \text{and} \quad N = G(P) = S \langle t \rangle,$$

where $s$ and $t$ are commuting involutions. Since $G$ is simple and $P$ is abelian, we have $P \cap G(P) = 1$ by 13.5.5 of [5] and thus $G_P(t) = 1$ and $t$ acts on $P$ with no nontrivial fixed points. Therefore $t$ transforms every element of $P$ and thus also of $S$ into its inverse. Clearly $S \triangle N$ and $P \triangle G(P)$ and thus $N = G(P(S))$. If two elements of $S$ are conjugate in $G$ they are conjugate in $N$ since $G$ is a T. I. set and if they are distinct they are inverses. Since the only elements of $S$ equal to their inverses are $s$ and 1, the remaining $2|P| - 2$ elements of $S$ span $|P| - 1$ classes of $G$.

If $y \neq 1$ is a real element of $G$ which is not an involution then $G(P(y)) < G$ has even order and thus $y$ has order divisible by 3 and centralizes some element of order 3. By taking conjugates we may suppose that this element is in $P$ and therefore $y \in N$. Since no element of $N - S$ centralizes any element $\neq 1$ in $P$, we conclude that $y \in S$. Therefore the $|P| - 1$ classes spanned be the nonself-inverse elements of $S$ are the classes $K_i$ of the lemma and $r = |P| - 1$.

Since $G(s) \supseteq N$ and $|G(s)| |(4|P|)$ we must have $G(s) = N$. Every element of $N - S$ is an involution and therefore in the lemma we have $\nu = 2|P| + 1$. Since $G(s) = N$, $m = [G:N] = |G||4|P|$. If $x \in S$ and $x \neq 1$, $s$ then $G(x) = S$ and $|K_i| = [G:S] = 2m$. Finally, the only involutions transforming $x$ to $x^{-1}$ are the elements of $N - S$ and hence each $v_i = 2|P|$ and the lemma yields

$$m^2 = (2|P| + 1)m + (|P| - 1)(2|P|)(2m)$$

and therefore $m = 4|P|^2 - 2|P| + 1$ and $|G| = 4|P|m$.

Now $G$ has $|P| + 1$ real classes and thus by Theorem 12.4 of [4] it has $|P|$ irreducible, nonprincipal real valued characters, $\chi_i$, $1 \leq i \leq |P|$. Since $G$ has $m$ involutions,

$$m = \sum_{i=1}^{|P|} \chi_i(1) \epsilon_i$$

where $\epsilon_i = \pm 1$ by Theorem 3.6 of [4]. Therefore $m \leq \sum_{i=1}^{|P|} \chi_i(1)$ and we have
$m^2 \leq \left[ \sum_{i=1}^{[P]} \chi_i(1) \right]^2 \leq |P| \left[ \sum_{i=1}^{[P]} \chi_i(1)^2 = |P| (|G| - \sum \psi_j(1)^2 - 1) \right]$  

where the $\psi_j$ are the irreducible nonreal valued characters. Thus  

$|P| \sum \psi_j(1)^2 \leq |P| (|G| - 1) - m^2 \leq m(4 |P|^2 - m)$  

since $|G| = 4 |P| m$. Since $4 |P|^2 - m = 2 |P| - 1 < 2 |P|$, we have $\sum \psi_j(1)^2 < 2m$. Because $G$ contains elements of order prime to 6, not every class of $G$ is real and thus some $\psi$ exists with $\psi \neq \overline{\psi}$ and hence $\psi(1)^2 < m$.

Now $[N : S] = 2$ and $S$ is abelian and thus all nonlinear irreducible characters of $N$ have degree 2. Since $t$ acts without fixed points on $P$, it is clear that $N' = P$ and $N$ has exactly 4 linear characters and thus has $|P| - 1$ distinct irreducible characters of degree 2, say $\lambda_1, \cdots, \lambda_{|P|-1}$. Since $[N : S] = 2$ and $\chi|S$ is reducible, it follows that $\lambda_i$ vanishes on $N - S$ and we may apply Theorem 38.16 of [3] since $S$ is a $T. I$. set. Therefore $G$ has irreducible characters

$$\zeta_1, \zeta_2, \cdots, \zeta_{|P|-1}$$

and there is $\varepsilon = \pm 1$ with $\lambda_i^\zeta - \lambda_j^\zeta = \varepsilon(\zeta_i - \zeta_j)$. Since each $\lambda_i^\zeta$ is real valued, the same is true of the $\zeta_i$ and thus we have the inner product $[\psi, (\lambda_i^\zeta - \lambda_j^\zeta)] = 0$. Therefore

$$[\psi, \lambda_i^\zeta] = [\psi, \lambda_j^\zeta]$$

and by Frobenius Reciprocity, $[\psi | N, \lambda_i] = [\psi | N, \lambda_j]$. We conclude that the multiplicities of each $\lambda_i$ in $\psi | N$ are equal. Since $\psi$ is faithful and $N$ is nonabelian, $\psi | N$ has some nonlinear constituent and thus this common multiplicity is $\geq 1$ and therefore $\psi(1) \geq 2(|P| - 1)$. Since $\psi(1)^2 < m < 4 |P|^2$, we have $\psi(1) < 2 |P|$ and thus

$$\psi(1) = 2 |P| - 2 \text{ or } 2 |P| - 1.$$  

Let $q$ be the largest prime divisor of $\psi(1)$. If $q = 2$ then since $\psi(1) | |G|$ we must have $\psi(1) = 4 |P| - 2$ and $|P| = 3$. In this situation $m = 31$ and $|G| = 12 \cdot 31$ and since no simple group can have this order, we have a contradiction. Thus $q \neq 2$ and since $3 | |P|$, $q > 3$. Since $q | |G|$ we must have $q | m$ and $4 |P|^2 - 2 |P| + 1 \equiv 0 \mod q$. Since $2 |P| \equiv 1$ or $2 \mod q$, we have $4 |P|^2 - 2 |P| + 1 \equiv 1$ or $3 \mod q$. Since $q > 3$ this is our final contradiction.

2. In this section we prove the first of our main results. We begin with a lemma.

**Lemma 2.** Let $H$ be an abelian group with a collection of proper subgroups $\{K_i\}$ such that $H = \bigcup K_i$ and $K_i \cap K_j = 1$ if $i \neq j$. Then
$H$ is an elementary abelian $p$-group for some prime $p$.

Proof. If $x, y \in H^*$ have different orders $m$ and $n$ respectively, with $m > n$, choose $K_i$ with $x \in K_i$. Then $1 \neq (xy)^n = x^n \in K_i$. If $xy \in K_j$ then $(xy)^n \in K_i \cap K_j$ and therefore $i = j$ and $xy \in K_i$. Thus $y \in K_i$. If $z \in H^*$ is arbitrary then the order of $z$ is different from at least one of $m$ and $n$ and thus $z \in K_i$. Thus $K_i = H$ and this contradiction shows that all elements of $H^*$ have equal orders and the result follows.

Theorem I. Let $G$ be a group with a cyclic $S_p$ subgroup $P$ such that every $p'$-subgroup of $G$ is abelian. Then $G$ has a normal $p$-complement or else $P \triangleleft G$.

Proof. Suppose the theorem is false and let $G$ be a minimal counterexample. Let $N = \mathfrak{R}_G(P)$ and let $K$ be an $S_p$, ($p$-complement) of $N$ whose existence is guaranteed by the Schur-Zassenhaus Theorem (9.3.6 of [5]). If any element $x \in K$ centralizes a nonidentity element of $P$, then because $P$ is cyclic, $x$ centralizes all of $P$. (See for instance 20.1 of [4]).

Every proper subgroup of $G$ satisfies the hypotheses and thus has either a normal $S_p$ or $S_p'$. If $L \triangleleft G$ and $p \nmid |L|$ then $G/L$ satisfies the hypotheses and does not have a normal $S_p'$ and therefore if $L > 1$, $PL \triangleleft G$. By Burnside's theorem, $K \triangleleft N$ and thus $NL$ does not have a normal $S_p'$ and if $NL < G$, $L$ normalizes $P$ and $P$ is characteristic in $PL$ and thus is normal in $G$. This contradiction shows that $NL = G$. Now put $M = \bigcap_{e \in G} N^e \triangleleft G$. Since $x = uv$ for some $u \in N$ and $v \in L$ we have $N^u = N^v = N^v \supseteq K^e$. However $KL$ is a $p'$-subgroup and thus is abelian and $K^e = K$. Since $x$ was arbitrary, $M \supseteq K$ and thus $M \supseteq K^e$ for all $u \in N$. Since $K$ is an $S_p'$ of the solvable group $M$ we may conclude that $K^e$ is conjugate to $K$ in $M$ by P. Hall's theorem (9.3.10 of [5]) and therefore there exists $w \in M$ with $uw^{-1} \in \mathfrak{R}_N(K)$. If $\mathfrak{R}_N(K) > K$ then $\mathfrak{R}_p(K) > 1$. This group is normalized and thus centralizes by $K$ and thus all of $P$ is also. This contradiction shows that $\mathfrak{R}_N(K) = K$, $uw^{-1} \in K$, and thus $N = MK$. Since $p \nmid |K|$, $P \subseteq M$ and we have $M = N$ and thus all $N^e$ are equal and $N \triangleleft G$. Thus $P \triangleleft G$ and we have a contradiction. Our assumption on the existence of $L$ is therefore invalid and $\mathfrak{D}_p(G) = 1$.

If $P \triangleleft G$ is a $p$-group, put $C = \mathfrak{C}_G(P) \triangleleft G$. If $C = G$ then $K$ centralizes $P$ and therefore $K$ centralizes all of $P$ and we have a contradiction. Thus $C < G$ and since $P \subseteq C$, $C$ does not have a normal $S_p$. Therefore $C$ is not a $p$-group and has a normal $S_p$ and this contradicts $\mathfrak{D}_p(G) = 1$ and we conclude that $\mathfrak{D}_p(G) = 1$. If $L \neq 1$
is any proper normal subgroup of $G$ then either an $S_p$ or an $S_p^*$ of $L$ is normal in $G$ and is $>1$ and this contradiction shows that $G$ is simple.

If $P$ and $P^*$ are two $S_p$ subgroups of $G$ and $P_0 = P \cap P^* > 1$, then since $P$ is cyclic, $U = \mathcal{R}_d(P_0) \subseteq N$ and $U < G$. Since $N$ fails to have a normal $S_p^*$, the same is true of $U$ and thus the $S_p$ $P$ of $U$ is normal and $P = P^*$. Therefore $P$ is a T. I. set. Now let

$$S = \mathcal{C}_g(P) \subseteq N.$$ 

If $P^*$ is another $S_p$ of $G$ and $S^* = \mathcal{C}(P^*)$, suppose that $S_0 = S \cap S^* > 1$. Now $S_0$ is not a $p$-group for otherwise $S_0 \subseteq P \cap P^* = 1$, and thus there is some $x \neq 1$ in $S_0$ which is a $p'$-element. Since

$$P, P^* \subseteq \mathcal{C}_g(x) < G,$$

$\mathcal{C}_g(x)$ has a normal $S_p^*$ $L$. Since $x$ is a $p'$-element of $N$ we may suppose that $x \in K$ and hence $K \subseteq \mathcal{C}(x)$ because $K$ is abelian. Thus $K \subseteq L$ and $K = \mathcal{R}_L(P)$. Since $P$ normalizes $L$, it also normalizes $K$ and this is a contradiction. Therefore $S_0 = 1$ and $S$ is a T. I. set.

Now let $A$ be any maximal $p'$-subgroup of $G$ and $B$ a $p'$-subgroup with $A \cap B \neq 1$. If $V = \mathcal{C}_g(A \cap B) < G$ then $A, B \subseteq V$. If $V$ has a normal $S_p^*$ $L$ then $A \subseteq L$ and by maximality $A = L$ and $B \subseteq A$. If $V$ has a normal $S_p$ $P$ then $V$ has a possibly not normal $S_p^*$ $L$ and since $V$ is solvable, we may suppose that $A \subseteq L$ by P. Hall's theorem. Thus $A = L$ and some conjugate of $B$ is contained in $A$. In this situation, since $A$ normalizes $P_0$ and $P$ is a T. I. set we may conclude that $A$ normalizes some $S_p$ of $G$.

If $q$ is a prime, $q \mid |A|$, let $Q$ be an $S_q$ of $G$ with $Q \cap A \neq 1$. Then some conjugate of $Q$ is $\subseteq A$ and thus $A$ is a Hall subgroup of $G$. If $A^*$ is another maximal $p'$-subgroup of $G$ with $q \mid |A^*|$ then $A^*$ meets some conjugate of $A$ and we may conclude that $A^*$ is conjugate to $A$ and $|A| = |A^*|$. If $A$ does not normalize an $S_p$ of $G$ then $A$ is disjoint from all other maximal $p'$-subgroups of $G$ and $A$ is a T. I. set. In this situation let $Q \subseteq A$ be an $S_q$ of $G$. Since $A$ is abelian, $Q \triangle \mathcal{R}_d(A)$ and since $A$ is a T. I. set, $\mathcal{R}_d(Q) = \mathcal{R}_d(A)$ and thus by Burnside's theorem, $\mathcal{R}_d(A) > A$. By the maximality of $A$ it follows that $p \mid |\mathcal{R}(A)|$ and some element of order $p$ normalizes $A$.

Continuing with the situation where $A$ does not normalize an $S_p$ of $G$, suppose some element $y$ of order $p$ centralizes some $a \neq 1$ in $A$. We may suppose $y \in P$ and since $y \in P^*$ also, we conclude that $P = P^*$ and we may suppose $a \in K$. Then $K \cap A \neq 1$ and therefore $K \subseteq A$. Since $A$ is a T. I. set, $y$ normalizes $A$ and $K = \mathcal{R}_d(\langle y \rangle)$ and thus $y$ normalizes and hence centralizes $K$ and therefore $K$ centralizes all of $P$ and we have a contradiction. Thus no $a \in A$ different from
1 commutes with any element of order $p$ and since $A$ is normalized by such an element we have $|A| = 1 \mod p$.

Let $A_0, A_1, \cdots, A_s$ be a collection of maximal $p'$-subgroups of $G$ with all $|A_i|$ distinct and including all possibilities and with $K \subseteq A_t$. If $q \mid |G|$ and $q \neq p$ then some $A_i$ contains an $S_p$ of $G$ and if $q \mid |A_j|$ also, then $A_j$ meets some conjugate of $A_i$ and as we have seen this implies that $|A_j| = |A_i|$ and thus $j = i$. Therefore

$$|G| = |P| \prod_{i=0}^s |A_i|.$$  

Since $K \subseteq A_0$, no $A_i$ for $i > 0$ can normalize an $S_p$ of $G$ and if $A_0 > K$, the same is true of $A_i$. In this situation no $p$-element commutes with a $p'$-element nontrivially and thus $\mathfrak{S}_d(P) = P$ and $K$ is isomorphic with a subgroup of the automorphisms of $P$ and since $P$ is cyclic and $p \mid |K|$, $|K| \leq p - 1$. Continuing with the assumption that $A_0 > K$ we see that all $|A_i| = 1 \mod p$ and thus $|G|/|P| = 1 \mod p$. By Sylow’s theorem, $|G|/|P| = 1 \mod p$ and therefore $1 = |G|/|P| = |K| \mod p$. Since $|K| < p$ we must have $|K| = 1$ and this is a contradiction by Burnside’s theorem. Therefore $A_0 = K$ and $K$ is a maximal $p'$-subgroup.

Let $Z = \mathfrak{S}_K(P) < K$ and let $Q$ be an $S_q$ of $K$. Clearly, $K \subseteq \mathfrak{R}_d(Q)$ and thus by Burnside’s theorem, $K < \mathfrak{R}_d(Q)$ and hence $p \mid |\mathfrak{R}(Q)|$. Since $Z < K$ we may choose $q$ with $Q \nsubseteq Z$. If $\mathfrak{R}(Q)$ has a normal $S_p$, $P_0$ then $Q$ centralizes $P_0$ and therefore $Q$ centralizes all of some $S_p$ subgroup of $G$. It follows that $Q$ is contained in some conjugate of $Z$ and thus $Q^* \subseteq Z$. However $Q^*$ is therefore an $S_p$ of the abelian $K$ and $Q^* = Q$. This contradicts $Q \nsubseteq Z$ and thus $\mathfrak{R}(Q)$ fails to have a normal $S_p$ and hence has a normal $S_p$, $L$ and $L \supseteq K$. By the maximality of $K$, $K = L$ and $K$ is normalized by an element $x$ of order $p$. If $x \in P^*$, an $S_p$ of $G$, suppose $K \subseteq \mathfrak{R}(P^*)$. Then $K \subseteq \mathfrak{R}(\langle x \rangle)$ and thus $x$ centralizes $K$ and therefore $K$ centralizes all of $P^*$. Since $KP^* = N_q(P^*)$ we have a contradiction and no $S_p$ containing $x$ is normalized by $K$. In particular, $x \notin P$. We conclude that each of $P, P^*, \cdots, P^{s^{p-1}}$ is normalized by $K$ and they are all distinct. Now $\mathfrak{S}_x(P^{z^i}) = Z^{z^i}$ and since $\mathfrak{S}_d(P)$ is a $T$. I. set $Z^{z^i} \cap Z^{z^j} = 1$ unless $i = j$.

Put $|Z| = c$. Since the direct product $Z \times Z^c \subseteq K$ we have $c^2 \mid |K|$ and we set $|K| = c^2t$. We have $|K - \bigcup Z^{z^i}| = c^2t - p(c - 1) - 1$. Now $K/Z$ is a $p'$-group isomorphic with a subgroup of the automorphisms of $P$ and thus is cyclic of order dividing $p - 1$. Since $[K:Z] = ct$, we have $ct \mid (p - 1)$.

If $x$ centralizes any $a \neq 1$ in $K$ then $a$ normalizes and thus centralizes a full $S_p$ $P^*$ of $G$ with $x \in P^*$. If $b \in K$ then $a^b = a$ centralizes
(P*)^b and thus P* = (P*)^b because G_\phi(P*) is a T. I. set and thus K normalizes P*. We have seen that this is impossible and thus x acts without nontrivial fixed points on K and p | (c^t - 1).

We have then, p | (p - 1 + c^t) and since ct | (p - 1),

\[ p \mid \left[ \frac{p - 1}{ct} + c \right]. \]

Since both p - 1/ct and c divide p - 1, we have (p - 1)/ct + c < 2p and thus (p - 1)/ct + c = p. This implies that c | ((p - 1)/ct - 1) and p - 1/ct \mid (c - 1). It follows that either p - 1/ct = 1 or c = 1. If c = 1 then t = 1 and thus \( |K| = 1 \) and this is a contradiction and therefore p - 1/ct = 1. This yields t = 1 and c = p - 1 and thus \( |K| = (p - 1)^2 \).

We have then \( |K - \bigcup Z^{et}| = c^t - p(c - 1) - 1 = 0 \) and thus \( K = \bigcup Z^{et} \). We may therefore apply Lemma 2 to K and conclude that K is an elementary abelian p-group for some prime q. Since K/Z is cyclic of order ct = p - 1, we conclude that p - 1 = q and thus p = 3 and q = 2. Therefore \( |R_\phi(P)| = |P| |K| = 4 |P| \) and

\[ |G_\phi(P)| = |P| |Z| = 2 |P|. \]

If H < G has even order then so does an S_p of H and thus a maximal p'-subgroup containing it has even order and this order must equal \( |A_p| = |K| = 4 \) and therefore \( |H| |(4 |P|)\). Since G_\phi(P) is a T. I. set, the proposition applies and G is not simple. This contradiction proves the theorem.

We note here that an alternate method of completing the proof is to use the theorem of Brauer, Suzuki and Wall [2] instead of the proposition given here in § 1. While there are some similarities in the proofs of these two results, the Brauer-Suzuki-Wall theorem is considerably deeper.

3. Here we prove our second theorem.

**Theorem II.** Let G be a group and let p \neq 2 and q be primes dividing \( |G| \). Suppose for every H < G which is not a q-group or a q'-group that p \mid |H|. If q^a is the q-part of \( |G| \) and p > q^a - 1 or if p = q^a - 1 and an S_p of G is abelian then no primes but p and q divide \( |G| \).

**Proof.** If the theorem is false, let G be a minimal counter-example. Every H < G which is neither a q-group nor a q'-group satisfies the hypotheses and thus none has order divisible by any prime different from p and q. Suppose N \triangle G with 1 < N < G. If q \mid |N| then no other prime but p can also divide it and thus some prime
r \neq p$, $q$ divides $[G : N]$. If $Q$ is an $S_q$ of $N$ then $\mathcal{R}_d(Q)N = G$ and since $r \nmid |N|$, $r \mid |\mathcal{R}_d(Q)|$ and thus $G$ has a subgroup of order $r \mid |Q|$. This contradiction shows that $q \nmid |N|$. If any $r \neq p$ divides $|N|$, let $R$ be an $S_r$ of $N$. Then $\mathcal{R}_d(R)N = G$ and since $q \nmid |N|$, $q \mid |\mathcal{R}_d(R)|$ and $G$ has a subgroup of order $q \mid |R|$. This contradiction shows that $N$ must be a $p$-group.

If $Q$ is any $q$-subgroup of $G$ then $\mathcal{R}_d(Q) < G$ and thus is not divisible by any prime different from $p$ or $q$. If for every $Q > 1$, $\mathcal{R}_d(Q)/\mathcal{C}_d(Q)$ is a $q$-group then by Frobenius' theorem (see for instance 21.8 of [4]) $G$ has a normal $S_r$, which must be a $p$-group and this is a contradiction. Thus for some $Q$, an $S_p$ of $\mathcal{R}_d(Q)$ fails to centralize $Q$ and in particular is not normal. Thus an $S_p$ of $G$ is not normal and $Q$ is normalized by an element $x$ of order $p^d$ which does not centralize it. Some orbit of the elements of $Q$ thus has size $\geq p$ and $q^a \geq |Q| \geq p + 1 \geq q^a$. We have equality and thus $p + 1 = q^a$ and $Q$ is a full $S_q$ of $G$, all of whose nonidentity elements are conjugate under $x$. Thus since $p \neq 2$, $q = 2$ and all 2-elements of $G$ are involutions and in one class. Furthermore, by hypothesis, an $S_p$ subgroup $P$ of $G$ is abelian.

If $G$ has the proper normal subgroup $N$ then we have seen that $N$ is a $p$-group but since $G$ does not have a normal $S_p$, $p \mid [G : N]$. If $N \triangleleft H < G$ and $q \mid [H : N]$ then the only other prime which can divide $[H : N]$ is $p$ and thus $G/N$ satisfies the hypothesis and if $N > 1$ we have a contradiction. This shows that $G$ is simple.

If $H < G$ has even order and an $S_2$ of $H$ is not normal then $H$ does not have a normal $p$-complement. If $P_0$ is an $S_p$ of $H$ then by Burnside's theorem, $P_0$ is properly contained in its normalizer in $H$. Therefore $[H : \mathcal{R}_d(P_0)] < [H : P_0] \leq 2^a = p + 1$. By Sylow's theorem then, $P_0 \triangle H$.

Suppose $x \neq 1$ is a real element of $G$. Then $\mathcal{R}_d(x) < G$ has even order and since the only 2-elements are involutions, the order of $x^2$ is a power of $p$ and $x^2$ is a real element. If $G$ has no nonidentity real $p$-elements then for every real $x \in G$, $x^2 = 1$. Since the product of two involutions is real, the set $\{x \mid x^2 = 1\}$ is a normal subgroup of $G$. Therefore there exists $y \neq 1$, a real $p$-element. Since $y$ is transformed into its inverse by an element of $\mathcal{R}_d(y)$, $y$ is not central in that group and thus $\mathcal{R}_d(y)$ does not have a normal $S_p$. It therefore has a normal $S_p$ which is a full $S_p$ subgroup, $P$ of $G$ and thus $\mathcal{R}_d(P)$ has even order. It follows that $\mathcal{R}(P) = PS$ where $S$ is contained in an $S_t$ of $G$ and $P$ is the unique $S_p$ of $G$ containing $y$.

If no involution centralizes any nonidentity $p$-element then $S$ acts in a Frobenius manner on $P$ and being abelian, it must be cyclic and thus have order 2. If $t \in T$ is an involution then $\mathcal{C}_d(t) = T$ and in
the terminology of Lemma 1, \( m = |G|/2^α \) and \( u = 2^α - 1 \). If \( 1 \neq s \in S \) then \( s \) inverts every element of \( P \). Therefore each nonidentity element of \( P \) is real and thus is contained in a unique \( S_p \) and hence \( P \) is a \( T.I \) set. Thus if any two elements of \( P \) are conjugate in \( G \) they are conjugate in \( \mathcal{N}_G(P) \) and thus are inverses and the nonidentity elements of \( P \) span \((|P| - 1)/2\) classes of \( G \). These are the only real classes other than \{1\} and the class of involutions and thus in Lemma 1, \( r = (|P| - 1)/2 \). If \( x \neq 1, x \in P \) then \( \mathcal{C}_G(x) = \mathcal{C}_{P_1}(x) = P \) and the set of involutions transforming \( x \) to \( x^{-1} \) is the coset \( P_1 \). Therefore in Lemma 1, \( v_i = |P| \) and \( |K_i| = [G : P] \) for each \( i \). The lemma yields

\[
m^2 = m(2^α - 1) + \frac{|P| - 1}{2} |P| [G : P].
\]

Since \( |P| \) divides \( m \) and \( 2^α - 1 = p \), \( p | P \) divides the left side and the first term on the right side but not the remainder of the right side of the above equation and thus we have a contradiction. Therefore an involution centralizes some element of order \( p \).

Now let \( C = \mathcal{C}_G(T) \) and suppose \( C > T \). Then \( C = T \times P_1 \) where \( P_1 > 1 \) is a \( p \)-subgroup of \( G \). Set \( A = \mathcal{C}_G(P_1) \supseteq C \). Either \( T \triangle A \) or an \( S_p \) subgroup \( P^* \) of \( A \) (which is a full \( S_p \) of \( G \)) is normal. If \( P^* \triangle A \) then since \( |A| = |P| |T| \geq |\mathcal{N}_G(P)|, A = \mathcal{N}_G(P^*) \) and

\[
1 \neq P_1 \subseteq P^* \cap \mathcal{N}_G(P^*)
\]

and this is impossible in a simple group by 13.5.5 of [5]. Thus \( T \triangle A \). Let \( s \in S, s \neq 1 \) and let \( B = \mathcal{C}_G(s) \). If \( P_2 \) is an \( S_p \) of \( B \) then \( s \in \mathcal{R}_A(P_2) \) and thus \( [B : \mathcal{R}_A(P_2)] < p + 1 \) and \( P_2 \triangle B \). Since \( P_1 \subseteq B \) we have \( P_1 \subseteq P_2 \) and thus \( P_2 \subseteq A \) and thus \( P_2 \) normalizes \( T \). Since \( T \subseteq B, T \) normalizes \( P_2 \) and thus \( P_2 \) centralizes \( T \) and \( P_2 \subseteq P_1 \). Now

\[
\mathcal{C}_G(s) = P \cap B = P \cap P_2 \subseteq P \cap P_1 \subseteq P \cap \mathcal{N}_G(P) = 1
\]

and therefore \( S \) acts without nontrivial fixed points on \( P \) and every \( p \)-element of \( G \) is real. In particular \( x \in P, x \neq 1 \) is real. However, we have \( \mathcal{R}_A(\langle x \rangle) \supseteq A \) and since \( |A| = |P| |T| \), we have equality and \( x \) is central in \( \mathcal{R}_A(\langle x \rangle) \) and this is a contradiction. We have shown that \( C = \mathcal{C}_G(T) = T \).

If \( x \neq 1 \) is a \( p \)-element centralized by an involution then \( \mathcal{C}_G(x) \) has even order but does not contain a full \( S_p \) of \( G \) and thus has a normal \( S_p \) which is a full \( S_p \) of \( G \). Hence \( x \) is contained in a unique \( S_p \) of \( G \) which is normalized by an involution centralizing \( x \). By taking conjugates we may suppose that \( x \in P \) is centralized by \( s \in S \). Put \( E = \mathcal{C}_G(s) > 1 \). Now \( \mathcal{C}_G(s) \) has the normal \( S_p, P_1 \supseteq E \) and since \( E \) can meet no \( S_p \) of \( G \) other than \( P \) we see that \( P_1 \subseteq P \) and thus
If $P^* \neq P$ is an $S_p$ of $G$ then $P \cap P^* = 1$ and thus $C_{P^*}(s) = 1$.

Choose $t \in S$, $t \neq 1$. Since all involutions of $T$ are conjugate in $R(T)$, choose $u \in R(T)$ with $s = t^u$. If $P^* \neq P$, then $1 = C_{P^*}(s) = C_{P^*}(t^u) = C_P(t)^u$ and thus $C_P(t) = 1$. Otherwise, $P^* = P$ and $u \in R(P) = PS$ so that $u = r y$ for some $r \in S$ and $y \in P$. Now $S^*$ normalizes $P$ and $S^* \subseteq T$ and thus $S^* \subseteq R_{T}(P) = S$ and therefore $S = S^* = S^y$ and $y \in R_P(S)$. This group is normalized and thus centralized by $S$ and $y \in P \cap \overline{\mathfrak{Z}}(R_{\delta}(P))$ which as we have seen is trivial. Thus $y = 1$ and $u = r$ and hence $s = t$. We have therefore shown that $s$ is the only involution in $S$ which centralizes any nonidentity element of $P$.

If $|S| = 2$ then $1 \neq C_P(s) \subseteq P \cap \overline{\mathfrak{Z}}(R_{\delta}(P))$ and this is a contradiction. Thus $|S| \geq 4$ and we may find two involutions $t$ and $t'$ in $S$, both different from $s$. Then both $t$ and $t'$ invert every element of $P$. Therefore $tt'$ centralizes $P$ and hence $tt' = s$ and $\langle s \rangle$ has index 2 in $S$. We have now $|R_{\delta}(P)| = |S||P| = 4|P|$ and $|C_{\delta}(P)| = |\langle P, s \rangle| = 2|P|$. Since we have seen that a nontrivial $p$-element which is centralized by an involution is in only one $S_p$, $P$ is a $T. I.$ set. If $P^* \neq P$ is an $S_p$ of $G$ then if $C(P) \cap C(P^*) > 1$ it is not a $p$-group and thus contains an involution. Because $P \triangle C_{\delta}(s)$ this is impossible and $C_{\delta}(P)$ is a $T. I.$ set. Furthermore, since $T \subseteq C(s)$, $T$ normalizes $P$ and $T = S$. Therefore $|T| = 4 = p + 1$ and $p = 3$. If $H < G$ has even order then $|H| \mid (|T||P|)$ and the hypotheses of the proposition are satisfied. Since $G$ is simple, we have a contradiction and the theorem is proved.

We note that for $p = 2$ we can get a counterexample to the theorem by taking $G = A_5$ and $q = 3$.

**References**


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