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Let μ be a signed measure, and denote the total measure of its positive and negative parts by P and N. Since the total variation of such a measure is $V = P + |N|$, and the maximum of the absolute value of the measure is $M = \max(P, |N|)$, we have the inequality $M \leqq V \leqq 2M$. We consider the following question.

What should replace the constant 2 in this inequality when we pass to higher-dimensional vecstor-valued measures?

This question has been answered by Kaufman and Rickert [1]. In addition to their own result, they describe a geometric proof of the two-dimensional case, due to Kakutani, and state that they know of no geometric proof for $n \geqq 3$. In this note we extend Kakutani's proof to all dimensions. The crucial step in his proof is the observation that the total variation of a 2-dimensional vector measure is proportional to the circumference of the convex hull of its range. The "obvious" attempt to generalize to higher dimensions by replacing "circumference" by "surface area" must fail, as a simple dimension analysis shows. The generalization succeeds, however, if we first replace "circumference" in Kakutani's observation by "average width".

For easy reference, we recall the definition of total variation:

The total variation of a vector measure μ is the supremum over all partitions $\{E_1, \cdots, E_n\}$ of the measure space of $\sum_{i=1}^{n} \| \mu(E_i) \|$, where $\| \cdot \|$ is Euclidean length.

THEOREM. *The total variation of an n-vector measure equals c_n times the average width of the range of the measure where $c_{2n} = 4^{-n} n (2n)! \, \pi / (n!)^2$ and $c_{2n+1} = 4^n (n!)^2 / (2n)!$.*

Proof. If the measure space consists of a single atom, the range of the measure consists of two vectors: the vector $\boldsymbol{a}$ assigned to the whole space, and the zero vector. The total variation is in this case just $\| \boldsymbol{a} \|$, the length of $\boldsymbol{a}$, and the average width of the two-point set is seen to be $\| \boldsymbol{a} \| / c_n$ by a simple calculation; therefore the theorem holds for a trivial measure space. Since direct sum formation of measure spaces gives rise to (group theoretic) addition of the ranges, their support functions undergo addition as well (see [4]), and so do their average widths. Clearly, total variation is additive as well, and the theorem follows for measure spaces with finitely many elements. By continuity, of widths and variations, it follows for arbitrary n-vector valued measures.

COROLLARY 1. (*Kaufman and Rickert*). *There exists a set in the measure space, the length of whose measure is at least* $(2c_n)^{-1}$ *times the total variation* V.

Proof. The maximal width M of the range is at most twice the length L of the longest vector in the range, yet it cannot be less than the average width A. Therefore $Vc_n^{-1} = A \leqq M \leqq 2L$.

For $n \geqq 2$ we also have

COROLLARY 2. *If and only if there is no set the length of whose measure exceeds* $(2c_n)^{-1}V$, *the range of the measure is a ball centered at the origin.*

Proof. The "only if" is trivial. For the "if": When the maximal width does not exceed the average width, the width must be constant, and the closed convex hull of range is a ball. When the longest vector does not exceed half the maximal width, the ball must be centered at zero.

If the measure space is nonatomic, the range is closed and convex by a well known theorem of Liapounoff [2], and is therefore itself a ball centered at zero.

If, on the other hand, there is an atom in the measure space, the convex closure of the range is the group theoretic sum of an interval and a closed convex set. Hence its boundary contains an interval, and in 2 or more dimensions it cannot be a ball.

REMARK 1. After applying a vector form of the Radon-Nikodym theorem (see the preceding paper [5] in this journal), these results can be translated from vector measures to probabilities and yield the following: If U is an n-dimensional unit vector valued random variable on a probability space, there is at least one event B such that

$$E(U \mid B)P(B) \geqq (2c_n)^{-1} .$$

REMARK 2. The numbers c_n occur in a bound, calculated by A. E. Mayer [3], for the diameter of a polyhedron with a one-dimensional skeleton of given total length.

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