

Pacific Journal of Mathematics

DIOPHANTINE SYSTEMS

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We concern ourselves in this paper with integral solutions of three Diophantine systems, generalizations of

$$x + y + z = u + v + w, xyz = uvw$$

and of $xy + xz + yz = uv + uw + vw, xyz = uvw$. The solutions are given in terms of parameters and are integral for an integral choice of the parameters. Throughout the paper the integer n will be greater than 1.

Heron [3] in the first century B. C. considered the problem of finding two rectangles such that the area of the first is three times the area of the second and the perimeter of the second is three times the perimeter of the first. He also considered a second problem which results in the Diophantine system $x + y = u + v, xy = 4uv$. Planude [3] discussed the system $x + y = u + v, xy = buv$, and Cantor [3] gave general solutions to this problem. Tannery [3] generalized the two problems of Heron. Moessner [7] and [8] gave particular solutions, while Dickson [4] and Gloden [6] gave parametric solutions of the system

$$(1) \quad \begin{aligned} x + y + z &= u + v + w, \\ xyz &= uvw, \end{aligned}$$

Bini [1] considered a system equivalent to (1) and Buquet [2] extended this system to $2n$ unknowns.

All of the above systems are special cases of the system

$$(2) \quad \begin{aligned} A(x, y) &= 0, \\ cP(x) &= dP(y), \end{aligned}$$

where $A(\alpha, \beta) = \sum_{i=1}^n (a_i \alpha_i - b_i \beta_i)$, $P(\alpha) = \prod_{i=1}^n \alpha_i$, a_i, b_i are integers, and c and d are nonzero integers. We make the following definitions:

$$A_p(\alpha, \beta) = A(\alpha, \beta) - (a_p \alpha_p - b_p \beta_p),$$

$P_p(\alpha) = P(\alpha)/\alpha_p$, $\pi_i(\alpha, \beta) = cb_p P_p(\alpha) - da_p P_p(\beta)$, p is a fixed integer, $1 \leq p \leq n$, and the α 's and β 's are arbitrary integers.

We agree that solutions in which some unknown vanishes, or those for which $a_i x_i = b_i y_i$, ($i = 1, \dots, n$) are trivial solutions.

THEOREM. 1. *Any nontrivial integral solution of (2) is proportional to a solution given by*

$$(3) \quad \begin{aligned} x_i &= \pi_1(\alpha, \beta)\alpha_i, & i \neq p, \\ y_i &= \pi_1(\alpha, \beta)\beta_i, & i \neq p, \\ x_p &= dP_p(\beta)A_p(\alpha, \beta), \\ y_p &= cP_p(\alpha)A_p(\alpha, \beta). \end{aligned}$$

Proof. Since the solution is nontrivial there is an integer p , $1 \leq p \leq n$, such that $a_p x_p \neq b_p y_p$. If for $i \neq p$ we set

$$(4) \quad \begin{aligned} x_i &= \pi_1(\alpha, \beta)\alpha_i, \\ y_i &= \pi_1(\alpha, \beta)\beta_i, \end{aligned}$$

then (2) becomes

$$\begin{aligned} a_p x_p - b_p y_p &= -\pi_1(\alpha, \beta)A_p(\alpha, \beta), \\ cP_p(\alpha)\pi_1^{n-1}(\alpha, \beta)x_p - dP_p(\beta)\pi_1^{n-1}(\alpha, \beta)y_p &= 0. \end{aligned}$$

The solution of this system is

$$(5) \quad \begin{aligned} x_p &= dP_p(\beta)A_p(\alpha, \beta), \\ y_p &= cP_p(\alpha)A_p(\alpha, \beta). \end{aligned}$$

From (4) and (5) it follows that (3) is a solution of (2).

Suppose now that $x_i = \lambda_i$, $y_i = \mu_i$ is a non trivial integral solution of (2). Then $A(\lambda, \mu) = 0$, $cP(\lambda) = dP(\mu)$, and $a_p \lambda_p \neq b_p \mu_p$. If in (3) we choose $\alpha_i = \lambda_i$, $\beta_i = \mu_i$ we get

$$\begin{aligned} x_i &= \pi_1(\lambda, \mu)\lambda_i, & (i = 1, \dots, n), \\ y_i &= \pi_1(\lambda, \mu)\mu_i, & (i = 1, \dots, n), \end{aligned}$$

which is proportional to the solution $x_i = \lambda_i$, $y_i = \mu_i$, since $\pi_1(\lambda, \mu)$ is integral and $\pi_1(\lambda, \mu) = cP(\lambda)/\lambda_p \mu_p (b_p \mu_p - a_p \lambda_p) \neq 0$.

Dickson [5] has given solutions of the system

$$(6) \quad \begin{aligned} xy + xz + yz &= uv + uw + vw, \\ xyz &= uvw. \end{aligned}$$

He [4] also indicates that this system may be solved by the same method he used to solve (1). Our second theorem generalizes (6).

We wish to solve the system

$$(7) \quad \begin{aligned} \sum_{i=1}^n \left[\frac{a_i P(x)}{x_i} - \frac{b_i P(y)}{y_i} \right] &= 0, \\ cP(x) &= dP(y), \end{aligned}$$

where $a_i, b_i, c, d, P(\alpha), P_p(\alpha)$ are the same as in Theorem 1. We set

$$P_{pi}(\alpha) = \frac{P(\alpha)}{\alpha_p \alpha_i}, B(x) = \sum_{i=1}^n 'a_i P_{pi}(x), C(y) = \sum_{i=1}^n 'b_i P_{pi}(y),$$

where \sum' indicates that the p^{th} term is omitted from the summations, $\pi_2(\alpha, \beta) = cP_p(\alpha)C(\beta) - dP_p(\beta)B(\alpha)$, p is a fixed integer, $1 \leq p \leq n$, and the α 's and β 's are arbitrary integers.

We agree that solutions in which some unknown vanishes or any solution for which $a_iP(x)/x_i = b_iP(y)/y_i$, ($i = 1, \dots, n$), are trivial solutions.

THEOREM¹ 2. *Any nontrivial integral solution of (7) is proportional to a solution given by*

$$(8) \quad \begin{aligned} x_i &= \pi_2(\alpha, \beta)\alpha_i, & i \neq p, \\ y_i &= \pi_2(\alpha, \beta)\beta_i, & i \neq p, \\ x_p &= dP_p(\beta)(a_pP_p(\alpha) - b_pP_p(\beta)), \\ y_p &= cP_p(\alpha)(a_pP_p(\alpha) - b_pP_p(\beta)). \end{aligned}$$

Proof. Since the solution is nontrivial there is an integer p , $1 \leq p \leq n$, such that $a_pP(x)/x_p \neq b_pP(y)/y_p$. If for $i \neq p$ we set

$$(9) \quad \begin{aligned} x_i &= \pi_2(\alpha, \beta)\alpha_i, \\ y_i &= \pi_2(\alpha, \beta)\beta_i, \end{aligned}$$

then (7) becomes

$$\begin{aligned} \pi_2^{n-2}(\alpha, \beta)B(\alpha)x_p - \pi_2^{n-2}(\alpha, \beta)C(\beta)y_p &= \pi_2^{n-1}(\alpha, \beta)(b_pP_p(\beta) - a_pP_p(\alpha)), \\ c\pi_2^{n-1}(\alpha, \beta)P_p(\alpha)x_p - d\pi_2^{n-1}(\alpha, \beta)P_p(\beta)y_p &= 0. \end{aligned}$$

The solution of this system is

$$(10) \quad \begin{aligned} x_p &= dP_p(\beta)(a_pP_p(\alpha) - b_pP_p(\beta)), \\ y_p &= cP_p(\alpha)(a_pP_p(\alpha) - b_pP_p(\beta)). \end{aligned}$$

It follows from (9) and (10) that (8) is a solution of (7).

Suppose now that $x_i = \lambda_i, y_i = \mu_i$ is a nontrivial integral solution of (7). Then $\lambda_p\beta(\lambda) + a_pP_p(\lambda) = \mu_pC(\mu) + b_pP_p(\mu)$, $cP(\lambda) = dP(\mu)$, and $a_pP(\lambda)/\lambda_p \neq b_pP(\mu)/\mu_p$. If in (8) we choose $\alpha_i = \lambda_i, \beta_i = \mu_i$ we obtain

$$\begin{aligned} x_i &= \pi_2(\lambda, \mu)\lambda_i, & (i = 1, \dots, n), \\ y_i &= \pi_2(\lambda, \mu)\mu_i, & (i = 1, \dots, n), \end{aligned}$$

which is proportional to the solution $x_i = \lambda_i, y_i = \mu_i$ since $\pi_2(\lambda, \mu)$ is integral and

$$\pi_2(\lambda, \mu) = \frac{cP(\lambda)}{\lambda_p\mu_p} \left[\frac{a_pP(\lambda)}{\lambda_p} - \frac{b_pP(\mu)}{\mu_p} \right] \neq 0.$$

¹ This theorem also solves the problems of Heron and Planude.

The method of the two preceding theorems may be used to obtain solutions of the system

$$(11) \quad \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left[\frac{a_{ij}x_i}{x_j} - \frac{b_{ij}y_i}{y_j} \right] = 0 ,$$

$$cP(x) = dP(y) ,$$

where $c, d, P(\alpha)$ are defined above and the a_{ij}, b_{ij} are integers. We define

$$P_{ij}(\alpha) = \frac{P(\alpha)}{\alpha_i \alpha_j} ,$$

$$D(\alpha, \beta) = \sum_{j=2}^n a_{1j} P_{1j}(\alpha) P_1(\beta) ,$$

$$E(\alpha, \beta) = \sum_{j=2}^n b_{1j} P_1(\alpha) P_{1j}(\beta) ,$$

$$F(\alpha, \beta) = \sum_{i=2}^{n-1} \sum_{j=i+1}^n (a_{ij} \alpha_i P_{1j}(\alpha) P_1(\beta) - b_{ij} \beta_i P_1(\alpha) P_{1j}(\beta)) ,$$

$$\pi_3(\alpha, \beta) = cP_1(\alpha)E(\alpha, \beta) - dP_1(\beta)D(\alpha, \beta) ,$$

$$G(\alpha, \beta) = \sum_{i=1}^{n-1} a_{in} \alpha_i P_n(\alpha) P_n(\beta) ,$$

$$H(\alpha, \beta) = \sum_{i=1}^{n-1} b_{in} \beta_i P_n(\alpha) P_n(\beta) ,$$

$$I(\alpha, \beta) = \sum_{j=2}^{n-1} \sum_{i=1}^{j-1} (a_{ij} \alpha_i P_{nj}(\alpha) P_n(\beta) - b_{ij} \beta_i P_n(\alpha) P_{nj}(\beta)) ,$$

$$\pi_4(\alpha, \beta) = cdP_n(\alpha)P_n(\beta)I(\alpha, \beta) .$$

THEOREM 3. *Any nonzero integral solution of (11) which does not satisfy*

$$(12) \quad \sum_{j=2}^n \left[\frac{b_{1j}\mu_1}{\mu_j} - \frac{a_{1j}\lambda_1}{\lambda_j} \right] = 0 ,$$

is proportional to a solution given by

$$(13) \quad \begin{aligned} x_i &= \pi_3(\alpha, \beta)\alpha_i , & i \neq 1 , \\ y_i &= \pi_3(\alpha, \beta)\beta_i , & i \neq 1 , \\ x_1 &= dP_1(\beta)F(\alpha, \beta) , \\ y_1 &= cP_1(\alpha)F(\alpha, \beta) , \end{aligned}$$

and any non-zero integral solution which does not satisfy

$$(14) \quad \sum_{i=1}^{n-1} \left[\frac{b_{in}\mu_i}{\mu_n} - \frac{a_{in}\lambda_i}{\lambda_n} \right] = 0 ,$$

is proportional to a solution given by

$$\begin{aligned}
 (15) \quad & x_i = \pi_4(\alpha, \beta)\alpha_i, \quad i \neq n, \\
 & y_i = \pi_4(\alpha, \beta)\beta_i, \quad i \neq n, \\
 & x_n = dP_n(\beta) (dP_n(\beta)H(\alpha, \beta) - cP_n(\alpha)G(\alpha, \beta)), \\
 & y_n = cP_n(\alpha) (dP_n(\beta)H(\alpha, \beta) - cP_n(\alpha)G(\alpha, \beta)).
 \end{aligned}$$

Proof. If we multiply the first of the equations in (11) by $P_1(x)P_1(y)$ and for $i \neq 1$ set

$$\begin{aligned}
 (16) \quad & x_i = \pi_3(\alpha, \beta)\alpha_i, \\
 & y_i = \pi_3(\alpha, \beta)\beta_i,
 \end{aligned}$$

the system becomes

$$\begin{aligned}
 & \pi_3^{2n-3}(\alpha, \beta)D(\alpha, \beta)x_1 - \pi_3^{2n-3}(\alpha, \beta)E(\alpha, \beta)y_1 = -\pi_3^{2n-2}(\alpha, \beta)F(\alpha, \beta), \\
 & c\pi_3^{n-1}(\alpha, \beta)P_1(\alpha)x_1 - d\pi_3^{n-1}(\alpha, \beta)P_1(\beta)y_1 = 0,
 \end{aligned}$$

which has as solution

$$\begin{aligned}
 (17) \quad & x_1 = dP_1(\beta)F(\alpha, \beta), \\
 & y_1 = cP_1(\alpha)F(\alpha, \beta).
 \end{aligned}$$

From (16) and (17) it follows that (13) is a solution of (11).

Suppose now that $x_i = \lambda_i, y_i = \mu_i$ is any nonzero integral solution of (11) which does not satisfy (12). If we choose $\alpha_i = \lambda_i, \beta_i = \mu_i$ then (13) becomes

$$\begin{aligned}
 & x_i = \pi_3(\lambda, \mu)\lambda_i, \quad (i = 1, \dots, n), \\
 & y_i = \pi_3(\lambda, \mu)\mu_i, \quad (i = 1, \dots, n),
 \end{aligned}$$

which is proportional to the solution $x_i = \lambda_i, y_i = \mu_i$ since $\pi_3(\lambda, \mu)$ is integral and

$$\pi_3(\lambda, \mu) = \frac{cP(\lambda)P_1(\lambda)P_1(\mu)}{\lambda_1\mu_1} \sum_{j=2}^n \left[\frac{b_{1j}\mu_1}{\mu_j} - \frac{a_{1j}\lambda_1}{\lambda_j} \right] \neq 0.$$

We may now write (11) as

$$\begin{aligned}
 (18) \quad & \sum_{j=2}^n \sum_{i=1}^{j-1} \left[\frac{a_{ij}x_i}{x_j} - \frac{b_{ij}y_i}{y_j} \right] = 0, \\
 & cP(x) = dP(y).
 \end{aligned}$$

If we multiply the first equation in (18) by $P_n(x)P_n(y)$ and for $i \neq n$ set

$$\begin{aligned}
 (19) \quad & x_i = \pi_4(\alpha, \beta)\alpha_i, \\
 & y_i = \pi_4(\alpha, \beta)\beta_i,
 \end{aligned}$$

the system becomes

$$\frac{1}{x_n} \pi_4^{2n-1}(\alpha, \beta) G(\alpha, \beta) - \frac{1}{y_n} \pi_4^{2n-1}(\alpha, \beta) H(\alpha, \beta) = - \pi_4^{2n-2}(\alpha, \beta) I(\alpha, \beta),$$

$$\frac{1}{x_n} \pi_4^{n-1}(\alpha, \beta) dP_n(\beta) - \frac{1}{y_n} \pi_4^{n-1}(\alpha, \beta) cP_n(\alpha) = 0$$

which has solution

$$(20) \quad \begin{aligned} x_n &= dP_n(\beta) (dP_n(\beta)H(\alpha, \beta) - cP_n(\alpha)G(\alpha, \beta)), \\ y_n &= cP_n(\alpha) (dP_n(\beta)H(\alpha, \beta) - cP_n(\alpha)G(\alpha, \beta)). \end{aligned}$$

It follows from (19) and (20) that (15) is a solution of (18) and hence of (11).

Suppose now that $x_i = \lambda_i, y_i = \mu_i$ is any nonzero integral solution of (18) which does not satisfy (14). If in (15) we choose $\alpha_i = \lambda_i, \beta_i = \mu_i$ we get

$$\begin{aligned} x_i &= \pi_4(\lambda, \mu)\lambda_i, & (i = 1, \dots, n), \\ y_i &= \pi_4(\lambda, \mu)\mu_i, & (i = 1, \dots, n), \end{aligned}$$

which is proportional to the solution $x_i = \lambda_i, y_i = \mu_i$ since $\pi_4(\lambda, \mu)$ is integral and

$$\pi_4(\lambda, \mu) = cdP_n^2(\lambda)P_n^2(\mu) \sum_{i=1}^{n-1} \left[\frac{b_{in}\mu_i}{\mu_n} - \frac{a_{in}\lambda_i}{\lambda_n} \right] \neq 0.$$

The following example shows that not all systems of type (11) can be solved by the method of this paper. The system

$$\sum_{i=1}^3 \sum_{j=i+1}^4 \left[\frac{a_{ij}x_i}{x_j} - \frac{b_{ij}y_i}{y_j} \right] = 0,$$

$$c \prod_{i=1}^4 x_i = d \prod_{i=1}^4 y_i,$$

where $a_{12} = 3, a_{13} = -2, a_{14} = 3, a_{23} = 4, a_{24} = 4, a_{34} = 3, b_{12} = 6, b_{13} = -2, b_{14} = 3, b_{23} = 8, b_{24} = 2, b_{34} = 3, c = 2, d = 1$, has the solution

$$x_1 = 2, x_2 = 3, x_3 = -4, x_4 = -2, y_1 = 4, y_2 = -3, y_3 = -4, y_4 = 2,$$

which also satisfies

$$\sum_{j=2}^4 \left[\frac{a_{1j}x_1}{x_j} - \frac{b_{1j}y_1}{y_j} \right] = 0$$

and

$$\sum_{i=1}^3 \left[\frac{a_{i4}x_i}{x_4} - \frac{b_{i4}y_i}{y_4} \right] = 0.$$

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May, 1967

A. A. Aucoin, <i>Diophantine systems</i>	419
Charles Ballantine, <i>Products of positive definite matrices. I</i>	427
David Wilmot Barnette, <i>A necessary condition for d-polyhedrality</i>	435
James Clark Beidleman and Tae Kun Seo, <i>Generalized Frattini subgroups of finite groups</i>	441
Carlos Jorge Do Rego Borges, <i>A study of multivalued functions</i>	451
William Edwin Clark, <i>Algebras of global dimension one with a finite ideal lattice</i>	463
Richard Brian Darst, <i>On a theorem of Nikodym with applications to weak convergence and von Neumann algebras</i>	473
George Wesley Day, <i>Superatomic Boolean algebras</i>	479
Lawrence Fearnley, <i>Characterization of the continuous images of all pseudocircles</i>	491
Neil Robert Gray, <i>Unstable points in the hyperspace of connected subsets</i>	515
Franklin Haimo, <i>Polynomials in central endomorphisms</i>	521
John Sollion Hsia, <i>Integral equivalence of vectors over local modular lattices</i>	527
Jim Humphreys, <i>Existence of Levi factors in certain algebraic groups</i>	543
E. Christopher Lance, <i>Automorphisms of postliminal C^*-algebras</i>	547
Sibe Mardesic, <i>Images of ordered compacta are locally peripherally metric</i>	557
Albert W. Marshall, David William Walkup and Roger Jean-Baptiste Robert Wets, <i>Order-preserving functions: Applications to majorization and order statistics</i>	569
Wellington Ham Ow, <i>An extremal length criterion for the parabolicity of Riemannian spaces</i>	585
Wellington Ham Ow, <i>Criteria for zero capacity of ideal boundary components of Riemannian spaces</i>	591
J. H. Reed, <i>Inverse limits of indecomposable continua</i>	597
Joseph Gail Stampfli, <i>Minimal range theorems for operators with thin spectra</i>	601
Roy Westwick, <i>Transformations on tensor spaces</i>	613
Howard Henry Wicke, <i>The regular open continuous images of complete metric spaces</i>	621
Abraham Zaks, <i>A note on semi-primary hereditary rings</i>	627
Thomas William Hungerford, <i>Correction to: "A description of $\text{Mult}_i(A^1, \dots, A^n)$ by generators and relations"</i>	629
Uppuluri V. Ramamohana Rao, <i>Correction to: "On a stronger version of Wallis' formula"</i>	629
Takesi Isiwata, <i>Correction: "Mappings and spaces"</i>	630
Henry B. Mann, Josephine Mitchell and Lowell Schoenfeld, <i>Correction to: "Properties of differential forms in n real variables"</i>	631
James Calvert, <i>Correction to: "An integral inequality with applications to the Dirichlet problem"</i>	631
K. Srinivasacharyulu, <i>Correction to: "Topology of some Kähler manifolds"</i>	632