A NECESSARY CONDITION FOR $d$-POLYHEDRALITY

DAVID WILMOT BARNETTE
A NECESSARY CONDITION FOR $d$-POLYHEDRALITY

DAVID BARNETTE

A graph is $d$-polyhedral provided it is isomorphic to the graph of a $d$-dimensional convex polytope. One of the unsolved problems in the field of convex polytopes is to characterize the $d$-polyhedral graphs for $d > 3$. There are, however, several necessary conditions known for a graph to be $d$-polyhedral. In this paper we present a new necessary condition which is not implied by the other conditions but which has two of them as corollaries. We also show how this new condition may be useful in solving problems dealing with ambiguity of $d$-polyhedral graphs.

2. Preliminary remarks. The 3-polyhedral graphs have been shown by Steinitz [8] to be those that are planar and 3-connected. The known necessary conditions for $d$-polyhedrality, $d > 3$, are:

(i) A $d$-polyhedral graph is $d$-connected [1].

(ii) A $d$-polyhedral graph contains a subgraph homeomorphic to the complete graph of $d + 1$ vertices [3, 4, 6].

(iii) The maximum number of components into which a $d$-polyhedral graph may be separated by removing $n > d$ vertices is equal to the maximum number $\sigma(d, n)$ of facets of a $d$-dimensional polytope with $n$ vertices [7].

Let $G$ be a graph homeomorphic to $C_d$ (the complete graph of $d$ vertices). If $\varphi$ is the homeomorphism then the images under $\varphi$ of the vertices of $C_d$ will be called the principal vertices of $G$.

3. The main result.

THEOREM 1. Given a vertex $v$ of a $d$-polyhedral graph $G$, let $V$ be the set of vertices joined to $v$ by edges of $G$. Then $V$ is contained in a $(d - 1)$-polyhedral subgraph $G'$ of $G \sim \{v\}$, and $G'$ contains a homeomorphic image of $C_d$ whose principal vertices lie in the set $V$.

Proof. We shall make use of the pulling process which is discussed in detail in [2]. We say that a $d$-polytope $P'$ is obtained from a $d$-polytope $P$ by pulling vertex $v$ of $P$, when we replace the vertex $v$ by a point $v'$ such that the segment $(v, v')$ does not intersect any $(d - 1)$-flat determined by vertices of $P \sim \{v\}$, and then take as $P'$ the convex hull of $v'$ and the remaining vertices of $P$. It follows from the results in [2] that when $v$ has been pulled to $v'$ all $(d - 1)$-
dimensional faces of $P'$ incident to $v'$ are pyramids over certain sub-
facets of $P$.

We define the linked complex of a vertex $v$ in $P$ to be the inter-
section of the star of $v$ with the antistar of $v$ in the boundary
complex of $P$. It also follows from the results in [2] that if $v$ is
pulled to $v'$ then the linked complex remains unchanged.

Let $P$ be a $d$-polytope ($d > 2$) and let a vertex $v$ be specified.
We pull $v$ to a new vertex $v'$ and then intersect this new polytope
$P'$ by a hyperplane, $H$, which separates $v'$ from the remaining ver-
tices of $P'$. This intersection is a $(d - 1)$-polytope $P^*$ and we shall
show that $P^*$ is combinatorially equivalent to the linked complex of
$v$. Let $F$ be any facet (i.e. $(d - 1)$-face) of $P^*$. The facet $F$ is the
intersection of $H$ with some pyramidal facet $F'$ of $P'$. Let $\varphi$ be a
function from the collection of facets of $P^*$ to the collection of
$(d - 1)$-faces of the linked complex of $v$, which takes $F$ into the facet
of $\mathcal{F}$ which misses $v'$. Since every face of the linked complex of $v$
is a face of some pyramidal face of $P'$ which meets $v'$, the function
$\varphi$ is one-to-one and preserves incidences and is therefore a combina-
torial equivalence. For $d \geq 2$ the graph of a $d$-polytope is a subset
of the boundary complex and thus the graph consisting of the vertices
and edges of the linked complex of $v$ is the required $(d - 1)$-polyhedral
subgraph of $G \sim \{v\}$.

We now prove the second part of our theorem by induction on
$d$. The theorem is obvious for $d = 3$. For $d > 3$ we intersect $P$ with
a hyperplane $H$ which separates $V$ from the remaining vertices of
$P$. Let $Q = H \cap P$. The projection from $v$ of $P$ onto the linked
complex of $v$ provides a homeomorphism of the graph of $Q$ onto a
subgraph of the linked complex of $v$. Let $v'$ be a vertex of $Q$; then
by the induction hypothesis the set of vertices $V'$ connected to $v'$
contains a set of principal vertices of a homeomorphic image of $C_{d-1}$.
Thus $\{v'\} \cup V'$ contains a set of principal vertices of a homeomorphic
image of $C_d$ whence the linked complex contains a graph homeomorphic
to $C_d$. The principal vertices of this graph in the linked complex are
joined to $v$ and the theorem is proved.

**COROLLARY 1.** A $d$-polyhedral graph contains a subgraph homeo-
morphic to $C_{d+1}$.

**COROLLARY 2.** A $d$-polyhedral graph is $d$-connected.

**Proof.** The result is obvious for $d \leq 2$. Proceeding by induction,
we assume the theorem to be true for all dimensions less than $d$.
Suppose $G$ is a $d$-polyhedral graph which can be separated by $d - 1$
or fewer vertices. Let $V$ be a minimal set of vertices separating $G$. 
Let $v \in V$, then there are vertices $v_1$ and $v_2$ each joined to $v$ by an edge and lying in different components of the separation. By the above theorem, $v_1$ and $v_2$ lie in a $(d-1)$-polyhedral subgraph $Q$ of $G \sim \{v\}$ and thus $Q$ can be separated by removing $d-2$ or fewer vertices which contradicts the induction hypothesis.

The following example shows that (i), (ii) and (iii) do not imply the conditions of Theorem 1.

Let $G$ be the graph obtained from $C_7$ by removing a circuit of length 3 (Figure I). This graph satisfies (i), (ii) and (iii) for $d=4$, however, the vertices connected to $v_0$ do not belong to any 3-polyhedral subgraph of $G \sim \{v_0\}$. To see this, suppose $G'$ were such a subgraph. Then $G'$ must contain each edge incident to $v_1$, $v_2$ and $v_3$, but these are the edges of the complete bipartite graph $K_{3,3}$ which is not planar and thus $G'$ is not 3-polyhedral.

![Figure I](image1.png)

![Figure II](image2.png)

A more complicated example is the graph in Figure II which satisfies (i), (ii), (iii) and the first part of our theorem for $d=4$ but there is no 3-polyhedral subgraph containing the set $V$ of vertices attached to $v_0$ which contains a homeomorphic image of $C_4$ whose principal vertices lie in $V$.

A graph may satisfy the conditions of Theorem 1 and (iii) and still not be polyhedral as is shown by the following theorem and example.

The *complete bipartite graph* $K_{a,b}$ is defined to be the graph whose vertices consist of two nonempty sets, $A$ and $B$ with $a$ and $b$ vertices respectively, and two vertices are joined by an edge if and only if one is in $A$ and the other is in $B$. 
Theorem 2. The complete bipartite graph $K_{a,b}$ is $d$-polyhedral only if $d = 2$, $a = 2$, and $b = 2$; or $d = 1$ $a = 1$, and $b = 1$.

Proof. The theorem is obvious for $d \leq 2$. Suppose $K_{a,b}$ is 3-polyhedral. The sets $A$ and $B$ must each contain three vertices for otherwise $K_{a,b}$ would not be 3-connected. But if $a$ and $b$ are greater than or equal to three then $K_{a,b}$ contains $K_{3,3}$ and is therefore not planar.

Suppose $K_{a,b}$ is $d$-polyhedral for $d > 3$. Let $G$ be the graph of some 3-dimensional face $F$ of a $d$-polytope whose graph is $K_{a,b}$. If two vertices of $G$ are joined by an edge in $K_{a,b}$ then they are joined by an edge in $G$ because $F$ is convex. Therefore, $G$ is a complete bipartite graph which has been shown to be impossible.

It is easy to check (using the results mentioned in Remark 2 in §4 of this paper) that $K_{19,19}$ satisfies (iii) for $d = 4$. The graph in Figure III has 37 vertices, is bipartite and is 3-polyhedral. The vertex set with 19 vertices contains the principal vertices of a homeomorphic image of $C_4$ as is indicated in Figure III. This shows that $K_{19,19}$ satisfies the conditions of Theorem 1, but Theorem 2 shows that it is not 4-polyhedral.

4. Unambiguous graphs. A polyhedral graph $G$ is said to be completely unambiguous provided

(i) $G$ is the graph of only one $d$-polytope $P$ for some $d$
(ii) $G$ is not the graph of any $e$-polytope for $e \neq d$

(iii) $G$ can be realized as the graph of $P$ in only one way. That is, if $P_1$ and $P_2$ are two combinatorially equivalent realizations of $G$ and if a set of vertices of $G$ determine a face of $P_1$ then it determines a face of $P_2$.

One of the unsolved problems on polyhedral graphs is whether there exist completely unambiguous graphs of $d$-polytopes for $d \geq 5$. The main difficulty is in finding graphs which are $d$-polyhedral for $d \geq 5$ but are not $e$-polyhedral for any $e < d$. The only examples of such graphs have been constructed by Klee [7] using (iii) and by Grünbaum [5] using related ideas, but these graphs are ambiguous. An affirmative answer to the following question would provide a completely unambiguous 5-polyhedral graph:

**Does there exist a 4-valent 4-polyhedral graph $G$ which is completely unambiguous and which does not contain a 3-polyhedral subgraph using all vertices of $G$?**

If $P$ were a realization of such a 4-polyhedral graph $G$ then a pyramid over $P$ would provide a 5-polytope whose graph $G'$ could not be 4-polyhedral by our theorem. It could not be $e$-polyhedral for $e > 5$ because it contains 5-valent vertices and is thus not $e$-connected. There must be at least one 4-dimensional face of any realization of $G'$, which misses the vertex $v$ of $G' \sim G$. Since $G$ is 4-valent it has no 4-polyhedral subgraphs and thus only one 4-dimensional face misses $v$, and any realization of $G'$ must be a pyramid over a realization of $G$. Since $G$ has only one realization, $G'$ is completely unambiguous.

5. Remarks.
1. It is not known whether all graphs satisfying Theorem 1 also satisfy (iii).
2. The function $\sigma(d, n)$ has been determined for $d \leq 8$ in which case we have

$$\sigma(d, n) = \binom{n - \left\lfloor \frac{d + 1}{2} \right\rfloor}{n - d} + \binom{n - \left\lfloor \frac{d + 2}{2} \right\rfloor}{n - d}.$$ 

This equality also holds if $n \leq d + 3$ or $n > \lfloor d/2 \rfloor$, see [3].

3. The graph consisting of $C_7$ minus a circuit of length three was first used by Grünbaum and Motzkin [5], as an example of a graph which satisfies (i) and (ii) for $d = 4$ but is not 4-polyhedral.

4. It would be of interest to determine the maximum number $C(d)$, such that given any $C(d)$ vertices of a $d$-polyhedral graph there is a subgraph homeomorphic to $C_{d+1}$ whose principal vertices include these given vertices. Clearly $C(2) = 3$ and it may easily be shown that $C(d) \leq d$ for $d > 2$. Theorem 1 implies $C(d) \geq 1$. 


REFERENCES


Received February 21, 1967. Research supported by a National Science Foundation graduate fellowship.

UNIVERSITY OF WASHINGTON

SEATTLE, WASHINGTON
A. A. Aucoin, *Diophantine systems* ........................................ 419
Charles Ballantine, *Products of positive definite matrices. I* ............ 427
David Wilmot Barnette, *A necessary condition for d-polyhedrality* .......... 435
James Clark Beidleman and Tae Kun Seo, *Generalized Frattini subgroups of finite groups* ................................................................. 441
Carlos Jorge Do Rego Borges, *A study of multivalued functions* .......... 451
William Edwin Clark, *Algebras of global dimension one with a finite ideal lattice* ................................................................. 463
Richard Brian Darst, *On a theorem of Nikodym with applications to weak convergence and von Neumann algebras* ................................. 473
George Wesley Day, *Superatomic Boolean algebras* .......................... 479
Lawrence Fearnley, *Characterization of the continuous images of all pseudocircles* ................................................................. 491
Neil Robert Gray, *Unstable points in the hyperspace of connected subsets* ..... 515
Franklin Haimo, *Polynomials in central endomorphisms* ..................... 521
John Sollion Hsia, *Integral equivalence of vectors over local modular lattices* ................................................................. 527
Jim Humphreys, *Existence of Levi factors in certain algebraic groups* .... 543
E. Christopher Lance, *Automorphisms of postliminal C*-algebras* ........ 547
Sibe Mardesic, *Images of ordered compacta are locally peripherally metric* ... 557
Albert W. Marshall, David William Walkup and Roger Jean-Baptiste Robert Wets, *Order-preserving functions: Applications to majorization and order statistics* ................................................................. 569
Wellington Ham Ow, *An extremal length criterion for the parabolicity of Riemannian spaces* ................................................................. 585
Wellington Ham Ow, *Criteria for zero capacity of ideal boundary components of Riemannian spaces* ................................................................. 591
J. H. Reed, *Inverse limits of indecomposable continua* ....................... 597
Joseph Gail Stampfli, *Minimal range theorems for operators with thin spectra* ................................................................. 601
Roy Westwick, *Transformations on tensor spaces* ................................ 613
Howard Henry Wicke, *The regular open continuous images of complete metric spaces* ................................................................. 621
Abraham Zaks, *A note on semi-primary hereditary rings* ....................... 627
Thomas William Hungerford, *Correction to: “A description of Mult(D, A^1, \ldots, A^n) by generators and relations”* ................................................................. 629
Uppuluri V. Ramamohana Rao, *Correction to: “On a stronger version of Wallis’ formula”* ................................................................. 629
Takeshi Isiwata, *Correction: “Mappings and spaces”* ....................... 630
Henry B. Mann, Josephine Mitchell and Lowell Schoenfeld, *Correction to: “Properties of differential forms in n real variables”* ....................... 631
James Calvert, *Correction to: “An integral inequality with applications to the Dirichlet problem”* ................................................................. 631
K. Srinivasacharyulu, *Correction to: “Topology of some Kähler manifolds”* .. 632