GENERALIZED FRATTINI SUBGROUPS OF FINITE GROUPS

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The purpose of this paper is to generalize some of the fundamental properties of the Frattini subgroup of a finite group. For this purpose we call a proper normal subgroup H of G a generalized Frattini subgroup if and only if $G = N_G(P)$ for each normal subgroup L of G and each Sylow p-subgroup P, p is a prime, of L such that $G = HN_G(P)$. Here $N_G(P)$ is the normalizer of P in G. Among the generalized Frattini subgroups of a finite nonnilpotent group G are the center, the Frattini subgroup, and the intersection $L(G)$ of all self-normalizing maximal subgroups of G. The product of two generalized Frattini subgroups of a group G need not be a generalized Frattini subgroup, hence G may not have a unique maximal generalized Frattini subgroup.

Let H be a generalized Frattini subgroup of G and let K be normal in G. If K/H is nilpotent, then K is nilpotent. Similarly, if the hypercommutator of K is contained in H, then K is nilpotent. We consider the Fitting subgroup $F(G)$ of a nonnilpotent group G, and prove $F(G)$ is a generalized Frattini subgroup of G if and only if every solvable normal subgroup of G is nilpotent.

Now let H be a maximal generalized Frattini subgroup of a finite nonnilpotent group G. Following Bechtell we introduce the concept of an H-series for G and prove that if G possesses an H-series, then $H = L(G)$.

2. **Notation** The only groups considered here are finite.

If H is a subgroup of a group G, then H' is the commutator (derived) subgroup of H,

$H^{(k)}(k > 1)$ is the k-th commutator subgroup of H,

$H^x = x^{-1}Hx$ for each $x \in G$,

$Z(H)$ is the center of H,

$Z^*(H)$ is the hypercenter of H(i.e. the terminal member of the upper central series of H), $D(H)$ is the hypercommutator of H(i.e. the terminal member of the lower central series of H),

$\phi(H)$ is the Frattini subgroup of H,

$F(H)$ is the Fitting subgroup of H(i.e. the largest nilpotent normal subgroup of H),

$N_G(H)$ is the normalizer of H in G.

If H is a subset of a group G, then denote by $\langle H \rangle$ the subgroup of G generated by H.

In a group G, $L(G)$ is the intersection of the self-normalizing maximal subgroups of G and $R(G)$ is the intersection of the normal maximal subgroups of G; in each case one sets $L(G)$ or $R(G) = G$ if the respective maximal subgroups do not exist properly (see [1]).

3. **Generalized Frattini subgroups.** This section will be given to defining a generalized Frattini subgroup of a group and to the development of some properties of this type of subgroup.

DEFINITION 3.1. A proper normal subgroup H of a group G is called a *generalized Frattini subgroup* of G if and only if $G = N_G(P)$ for each normal subgroup L of G and each Sylow p-subgroup P, p is a prime, of L such that $G = HN_G(P)$.

We note that every proper normal subgroup of a nilpotent group G is a generalized Frattini subgroup of G. This is not the case if G is only supersolvable. For example, if S_3 is the symmetric group of three symbols, then the alternating subgroup A_3 is not a generalized Frattini subgroup.

THEOREM 3.1. *Let H be a generalized Frattini subgroup of a group G.*

   Then

   (a) *H is nilpotent,*

   (b) *A normal subgroup of G contained in H is a generalized Frattini subgroup of G,*

   (c) *$H\phi(G)$ is a generalized Frattini subgroup of G,*

   (d) *$HZ(G)$ is a generalized Frattini subgroup of G, whenever it is a proper subgroup.*

Proof. (a) Let P be a Sylow p-subgroup of H where p is a fixed prime. Because of Theorem 6.2.4 of [4], $G = HN_G(P)$, hence $G = N_G(P)$. Since all the Sylow subgroups of H are normal, H is nilpotent.

   (b) Let K be a normal subgroup of G contained in H, L a normal subgroup of G, and P a Sylow p-subgroup, p is a prime, of L such that $G = KN_G(P)$. Then $G = HN_G(P)$, hence $G = N_G(P)$.

   (c) This is an immediate consequence of Theorem 7.3.8 of [4].

   (d) Since $Z(G)$ is contained in the normalizer of every subgroup of G, $HZ(G)$ is a generalized Frattini subgroup of G.

We now note that the intersection of generalized Frattini subgroups of a group G is a generalized Frattini subgroup of G. However, this is not true in general when we consider products of subgroups (see Example 3.3).

As a consequence of Theorem 3.1 we have the following.

COROLLARY 3.1.1 *The Frattini subgroup of G is a generalized*

Frattini subgroup of G. *Moreover, if* G *is nonabelian, then* $Z(G)$ *is a generalized Frattini subgroup of* G.

The following result is a generalization of Theorem 7.4.8 of [4].

THEOREM 3.2. *Let* H *be a generalized Frattini subgroup of* G. *If* K *is a normal subgroup of* G *and* K/H *is nilpotent, then* K *is nilpotent.*

Proof. Let K be a normal subgroup of G such that K/H is nilpotent. Let P be a Sylow p-subgroup of K for a fixed prime p. Then HP/H is a Sylow p-subgroup of K/H, hence HP/H is a characteristic subgroup of K/H. Therefore, HP/H is normal in G/H, and so HP is normal in G. Since P is a Sylow p-subgroup of HP, $G = (HP)N_G(P)$ because of Theorem 6.2.4 of [4]. Hence $G = HN_G(P)$, which implies $G = N_G(P)$. Since all the Sylow subgroups of K are normal, K is nilpotent.

Let H be a generalized Frattini subgroup of G. Then by Theorem 3.1 $F(G)$ contains H. From Theorem 3.2 $F(G/H) = F(G)/H$, hence we obtain the following corollaries.

COROLLARY 3.2.1. *If* H *is a generalized Frattini subgroup of* G, *then* $F(G/H) = F(G)/H$.

COROLLARY 3.2.2. *Let* H *be a generalized Frattini subgroup of* G. *Then* G *is nilpotent if and only if* G/H *is nilpotent*

COROLLARY 3.2.3. *A group* G *is nilpotent if and only if its commutator subgroup* G' *is a generalized Frattini subgroup of* G.

The next result is similar to Theorem 2.3 of [1], however it generalizes Bechtell's result.

THEOREM 3.3. *Let* H *be a generalized Frattini subgroup of* G. *If* K *is a normal subgroup of* G *whose hypercommutator* $D(K)$ *is contained in* H, *then* $D(K) = 1$ *and* K *is nilpotent.*

Proof. From Theorem 3.1 it follows that $D(K)$ is a generalized Frattini subgroup of G. Since $K/D(K)$ is nilpotent, K is nilpotent by Theorem 3.2, hence $D(K) = 1$.

COROLLARY 3.3.1. *A proper normal subgroup* K *of a group* G *is nilpotent if and only if its commutator subgroup* K' *is a generalized Frattini subgroup of* G.

Proof. By Theorem 7.3.17 of [4], $\phi(K) \subseteq \phi(G)$. Hence the corollary follows from Theorem 7.3.5 of [4], Corollary 3.1.1 and Theorem 3.3.

Our next objective of this section is to show that $L(G)$ is a generalized Frattini subgroup of G whenever G is nonnilpotent. We begin with the following theorem.

THEOREM 3.4. *Let H be a generalized Frattini subgroup of G and let K be a proper normal subgroup of G containing H. Then K/H is a generalized Frattini subgroup of G/H if and only if K is a generalized Frattini subgroup of G.*

Proof. Assume that K is a generalized Frattini subgroup of G. Let L/H be a normal subgroup of G/H and let P be a Sylow p-subgroup, p is a prime, of L such that $G/H = (K/H)N_{G/H}(HP/H)$. Then $G = KN_G(HP)$. Let $g = kx$, where $k \in K$ and $x \in N_G(HP)$. Then $P^x \subseteq HP$. Since L is a normal subgroup of G, P^x and P are Sylow p-subgroups of $L \cap HP$. Therefore, there is an element y of $L \cap HP$ such that $P^{xy} = P$, hence xy is an element of $N_G(P)$. Therefore $gy = k(xy)$ is contained in $KN_G(P)$. Since $KN_G(P)$ contains HP, it follows that y is an element of $KN_G(P)$, and therefore $g \in KN_G(P)$. This shows that $G = KN_G(P)$, and hence $G = N_G(P)$ since K is a generalized Frattini subgroup of G. From this we conclude that HP/H is normal in G/H, and so K/H is a generalized Frattini subgroup of G/H.

Conversely, assume that K/H is a generalized Frattini subgroup of G/H. Let L be a normal subgroup of G and let P be a Sylow p-subgroup, p is a prime, of L such that $G = KN_G(P)$. Then $G/H = (K/H)N_{G/H}(HP/H)$, hence $N_{G/H}(HP/H) = G/H$ since HP/H is a Sylow p-subgroup of HL/H and HL/H is normal in G/H. Therefore, HP is a normal subgroup of G. Let P_1 be a Sylow p-subgroup of HP which contains P. By Theorem 6.2.4 of [4], $G = (HP)N_G(P_1) = HN_G(P_1)$, hence $G = N_G(P_1)$ since H is a generalized Frattini subgroup of G. From this it follows that the Fitting subgroup $F(G)$ of G contains P, hence KP is a nilpotent subgroup because of Theorems 3.1 and 3.2. Since $G = KN_G(P)$, it follows that KP is a normal nilpotent subgroup of G.

We now show $N_G(P)$ contains KP. For let P_2 be a Sylow p-subgroup of KP. Then P_2 is normal in G, hence $P \subseteq P_2$. Let P_3 be a Sylow p-subgroup of G containing P_2. Since L is normal in G, $L \cap P_3 = P$ and $N_G(P_3) \subseteq N_G(P)$. From this it follows that $P_2 \subseteq N_G(P)$. Hence $KP \subseteq N_G(P)$, since KP is nilpotent. This shows $G = N_G(P)$, and therefore K is a generalized Frattini subgroup of G.

Because of Corollary 3.1.1, Theorem 2.2 of [1] and Theorem 3.4,

we obtain the following theorem.

THEOREM 3.5.   *If $L(G)$ is a proper subgroup of G, then $L(G)$ is a generalized Frattini subgroup of G.*

We now give several examples that will help illustrate the theory of this section.

EXAMPLE 3.1.   Let $Q = \langle a, b | a^4 = 1, a^2 = b^2, ba = a^{-1}b \rangle$ and let G be the direct product of Q and S_3, the symmetric group on three symbols.   Then $\phi(G) = Z(G) = \langle a^2 \rangle$, $L(G) = Q$ and $F(G) = Q \times A_3$. $F(G)$ is not a generalized Frattini subgroup of G.   We note that $L(G)$ properly contains $\phi(G)$.

EXAMPLE 3.2.   Let $H = \langle h \rangle$ be a cyclic group of order 49 and let G be the direct product of H and S_5, where S_5 is the symmetric group on five symbols.   Then $\phi(G) = \langle h^7 \rangle$ and $F(G) = L(G) = Z(G) = H$. Hence $F(G)$ is a generalized Frattini subgroup of G which properly contains $\phi(G)$.

Our examples indicate that the Fitting subgroup of a group G need not be a generalized Frattini subgroup of G.   However, the next two theorems provide a necessary and sufficient condition for $F(G)$ to be a generalized Frattini subgroup of G.

THEOREM 3.6.   *If the Fitting subgroup $F(G)$ of G is a generalized Frattini subgroup of G, then every solvable normal subgroup of G is nilpotent.*

Proof.   Let H be a solvable normal subgroup of G and let k be the smallest positive integer such that $H^{(k+1)} = 1$.   Then $F(G)$ contains $H^{(k)}$, hence by Theorem 3.1 $H^{(k)}$ is a generalized Frattini subgroup. Since $H^{(k-1)}/H^{(k)}$ is abelian, $H^{(k-1)}$ is nilpotent by Theorem 3.2.   Hence $F(G)$ contains $H^{(k-1)}$.   Proceeding in this way we can prove $H' \subseteq F(G)$, hence H' is a generalized Frattini subgroup of G by Theorem 3.1. By applying Theorem 3.2, we see that H is nilpotent.

As a consequence of Theorem 3.6 we have the following.

COROLLARY 3.6.1.   *If $F(G)$ is a generalized Frattini subgroup of G, then G can not be solvable.*

DEFINITION 3.2.   For a group G denote by $S(G)$ the *radical* of G (i.e. the unique maximal solvable normal subgroup of G).

THEOREM 3.7.   *Let G be a nonnilpotent group.   If $S(G) = F(G)$,*

then $F(G)$ is a generalized Frattini subgroup of G.

Proof. Let H be a normal subgroup of G and let P be a Sylow p-subgroup of H, p is a fixed prime, such that $G = F(G)N_G(P)$. Then $F(G)P/F(G)$ is a solvable normal subgroup of $G/F(G)$, hence $F(G)P$, is a solvable normal subgroup of G. Since $F(G)$ is the radical of G, $F(G)$ contains P. Hence P is a Sylow p-subgroup of $H \cap F(G)$. Since $H \cap F(G)$ is a nilpotent normal subgroup, P is normal in G. Therefore, $F(G)$ is a generalized Frattini subgroup of G.

From Theorems 3.6 and 3.7 we have the following.

THEOREM 3.8. *Let G be a nonnilpotent group. The Fitting subgroup of G is a generalized Frattini subgroup of G if and only if it is the radical of G.*

From Theorem 3.8 and the fact that a solvable subnormal subgroup of a group G is contained in the radical of G we obtain the following result.

THEOREM 3.9. *If the Fitting subgroup of a group G is a generalized Frattini subgroup of G, then every solvable subnormal subgroup of G is nilpotent.*

A generalized Frattini subgroup of a group G is called *maximal* if it is not properly contained in any other generalized Frattini subgroup of G. We now consider maximal generalized Frattini subgroups of a group G.

Let H be a maximal generalized Frattini subgroup of G. Then H contains $\phi(G)$ by Theorem 3.1. Now suppose that $L(G)$ is a proper subgroup (i.e. G is nonnilpotent). By Theorem 2.2 of [1] $L(G)/\phi(G) = Z(G/\phi(G))$, hence H contains $L(G)$ by Theorems 3.1 and 3.4. We have proved the following.

THEOREM 3.10. *A maximal generalized Frattini subgroup of a nonnilpotent group G contains $L(G)$.*

COROLLARY 3.10.1 *A maximal generalized Frattini subgroup of a nonnilpotent group G contains the hypercenter of G.*

Proof. It is sufficient to apply Theorem 2.2 of [1] and Theorem 3.10.

We conclude this section with an example which illustrates several properties of generalized Frattini subgroups.

EXAMPLE 3.3. Let G be a group of order 84 with the following properties:

(a) G has 28 Sylow 3-subgroups,

(b) G has a normal Sylow 7-subgroup H,

(c) G has a normal Sylow 2-subgroup K which is isomorphic to the Klein four-group.

We note that such a group exists (see 9.2.14 of [4]).

Then H and K are generalized Frattini subgroups of G, however $F(G) = HK$ is not a generalized Frattini subgroup. We also note that both H and K are maximal generalized Frattini subgroups of G. Hence a maximal generalized Frattini subgroup need not be unique. Finally, $L(G) = \phi(G) = Z(G) = 1$, and therefore a maximal generalized Frattini subgroup may contain the intersection of the self-normalizing maximal subgroups properly.

4. Small subgroups. This section is devoted to the study of generalized Frattini subgroups which are small in a group G.

DEFINITION 4.1 A proper normal subgroup H of a group G is said to be *small* in G if and only if $G = K$ for each other normal subgroup K of G such that $G = HK$ (see [2]).

Let H be a small subgroup of G which is contained in $L(G)$. Suppose $R(G)$ does not contain H. Then there exists a normal maximal subgroup B such that $G = HB$, which implies $G = B$. Hence $R(G)$ contains H, and therefore $\phi(G)$ must contain H. We have established the following two results.

THEOREM 4.1. *Let H be a proper normal subgroup of G which is contained in $L(G)$. If H is small in G, then $\phi(G)$ contains H.*

THEOREM 4.2. *If $L(G)$ is small in G, then $L(G) = \phi(G)$.*

We note that Example 3.1 shows that the assumption that $L(G)$ is small in Theorem 4.2 is needed.

Since the center of a group G is contained in $L(G)$, we obtain the following result from Theorem 4.1.

THEOREM 4.3. *If the center $Z(G)$ is small in G, then $Z(G)$ is contained in $\phi(G)$.*

Let H be a generalized Frattini subgroup of G. Suppose that H is small in G and every proper normal subgroup of G/H is nilpotent. Let K be a proper normal subgroup of G. Then HK is also a proper normal subgroup of G. Hence HK/H is nilpotent, and so HK is

nilpotent by Theorem 3.2. Therefore K is nilpotent. We have proved the theorem which follows.

THEOREM 4.4. *Let H be a generalized Frattini subgroup of G which is small in G. If every proper normal subgroup of G/H is nilpotent, then every proper normal subgroup of G is nilpotent.*

Since an extention of a solvable group by a solvable group is solvable, we obtain the following result from Theorem 4.4 and Corollary 3.6.1.

COROLLARY 4.4.1. *Let G' be a proper subgroup of G and let H be a generalized Frattini subgroup of G. If H is small in G and every proper normal subgroup of G/H is nilpotent, then G is solvable and $F(G)$ is not a generalized Frattini subgroup of G.*

We note that in Corollary 4.4.1 it is necessary to assume G' is a proper subgroup of G. For we need only to consider the alternating group on five symbols.

5. *H-series.* Let H be a (fixed) maximal generalized Frattini subgroup of G. In this section we define an H-series for G and develop some of its elementary properties. We note that part of this section is closely related to Bechtell's results on L-series in [1].

DEFINITION 5.1. Let H be a maximal generalized Frattini subgroup. Then
   (a)   an *H-series* for G is a series

$$H = B_0 \supseteqq B_1 \supseteqq B_2 \supseteqq \cdots \supseteqq B_j \supseteqq \cdots$$

such that B_i is normal in G and $B_i/B_{i+1} \subseteq Z(G/B_{i+1})$ for $i = 0, 1, 2, \cdots$,
   (b)   the *upper H-series* is a series

$$H = H_0 \supseteqq H_1 \supseteqq \cdots \supseteqq H_j \supseteqq \cdots$$

in which $[H_{i-1}, G] = H_i$, for $i = 1, 2, \cdots$, and
   (c)   the *lower H-series* is a series

$$1 = Z_0 \subseteqq Z_1 \subseteqq \cdots Z_j \subseteqq \cdots$$

in which $Z_j/Z_{j-1} = Z(G/Z_{j-1})$, for $j = 1, 2, \cdots$, (i.e. the lower H-series for G is the upper central series in the sense of Scott [4]).

REMARK 5.1. If one replaces H by $L(G)$ in the above definition, then we obtain the concepts of L-series, upper L-series and lower L-series given by Betchell [1]. However, we mention that $L(G)$ need not be a maximal generalized Frattini subgroup.

Let H be a maximal generalized Frattini subgroup of G. Then we say that G *possesses* an H-series if there exists an H-series for G which terminates with the identity subgroup.

REMARK 5.2. A group G is said to possess an L-series if it has an L-series which terminates with the identity subgroup (see [1]).

From Theorem 6.4.1 of [4] we have the following.

THEOREM 5.1. *Let H be a maximal generalized Frattini subgroup of G. If G possesses an H-series $H = B_0 \supseteq B_1 \supseteq \cdots \supseteq B_k = 1$, then $B_j \supseteq H_j$, for $j = 0, 1, 2, \cdots, k$, and $H_{k-j} \subseteq B_{k-j} \subseteq Z_j$, for $j = 0, 1, 2, \cdots, k$.*

Now let H be a maximal generalized Frattini subgroup of non-nilpotent group G. By Theorem 3.10 and Theorem 2.2 of [1], it follows that $H \supseteq L(G) \supseteq Z^*(G)$. Hence, if G possesses an H-series, then $H = L(G) = Z^*(G)$ by Theorem 5.1. We have established the following.

THEOREM 5.2. *Let H be a maximal generalized Frattini subgroup of a nonnilpotent group G. If G possesses an H-series, then $H = L(G) = Z^*(G)$.*

The fact that G possesses an H-series in Theorem 5.2 cannot be omitted.

EXAMPLE 5.1. Let G be the group of order 84 presented in Example 3.3 and let N be a cyclic group of order 5. Let M be the direct product of G and N. Then $\phi(M) = 1$ and therefore $L(M) = Z(M) = N$. We also note that H and K generalized Frattini subgroups of M, however $HZ(M)$ and $KZ(M)$ are maximal generalized Frattini subgroups of M which properly contain $L(M) = Z(M)$. Now let $W = HZ(M)$. Then M does not possess a W-series and $W \neq L(M)$. However, since $L(M) = Z(M)$, G possesses an L-series by Corollary 3.1.1 of [1].

The converse of Theorem 5.2 is not true in general.

EXAMPLE 5.2. Let $G = \langle a, b | a^9 = b^2 = ba\,ba = 1 \rangle$. Then $F(G) = \langle a \rangle$, $L(G) = \phi(G) = \langle a^3 \rangle$, and $Z^*(G) = Z(G) = 1$. However, $L(G)$ is a maximal generalized Frattini subgroup of G.

We conclude this section with two corollaries to Theorem 5.2.

COROLLARY 5.2.1. *Let $F(G)$ be a generalized Frattini subgroup*

of G. If G possesses an F(G)-series, then F(G) = L(G) = Z*(G).

Proof. In this case $F(G)$ is a maximal generalized Frattini subgroup, hence the corollary follows from Theorem 5.2.

From Corollary 5.2.1 and Theorem 4.2 we have the following.

COROLLARY 5.2.2. *Let F(G) be a generalized Frattini subgroup of G. If F(G) is small in G and G possesses an F(G)-series, then F(G) = φ(G).*

6. **Remarks.** In [3] Huppert proved the following theorem: A finite group G is supersolvable if and only if $F/\phi(G)$ is supersolvable. Hence one might raise the following question: If H is a generalized Frattini subgroup of G and G/H is supersolvable, then is G supersolvable? The answer to this question is no in general. For let G be the group of order 84 given in Example 3.3 and let H be the Sylow 2-subgroup of G. Then H is a generalized Frattini subgroup and G/H is supersolvable. However, G is not supersolvable.

We mention that one can prove the following using results of Huppert [3]. If $L(G)$ is a proper subgroup of G and $G/L(G)$ is supersolvable, then G is supersolvable.

In a later paper the authors will study those groups for which Huppert's result is true whenever generalized Frattini subgroups are considered instead of the Frattini subgroup.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics
Vol. 23, No. 3   May, 1967