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The primary purpose of this study is to determine which topological properties of a space are preserved by multivalued functions. Among other results, the following are proved:

(A) Let $F: X \to Y$ be a perfect map from X onto Y, with $F(x) \neq \emptyset$ for each $x \in X$, where X and Y are T_1-spaces whose diagonals are G_δ-sets. Then X is metrizable (stratifiable) if and only if Y is metrizable (stratifiable)-see Theorem 3.2.

(B) If $F: X \to Y$ is a multivalued Y-compact quotient map from a separable metrizable space X onto a regular first countable space Y with a G_δ-diagonal, then Y is separable metrizable (see Theorem 4.5).

(C) Every (usc-) lsc-function F from a closed subset of a stratifiable space X to a topological space Y admits a (usc-) lsc-extension to all of X (see Theorem 5.2).

Multivalued functions have been extensively studied by Kruse [6], Michael [7; 8], Ponomarev [12; 13; 14], Smithson [15] and Strother [17; 18]. Choquet [2] and Hahn [3] have also considered multivalued functions.

2. **Preliminary definitions and results.** Because there are many conflicting terminologies in the theory of multivalued functions, we find it necessary to attempt a terminology of our own, which is a direct extension of the most natural and simple terminology of Michael [10] and includes some of Ponomarev's terminology:

DEFINITION 2.1. For any sets X and Y, $F: X \to Y$ is a multivalued function provided that, for each $x \in X$, $F(x)$ is a subset of Y ($F(x)$ need not be a closed or nonempty set as required by Ponomarev and others).

Clearly, single-valued functions are just special cases of multivalued functions and indeed a multivalued function from X to Y can obviously be thought of as a single-valued function from X to $\mathscr{P}(Y)$—the family of all subsets of Y (including the empty set).

DEFINITION 2.2. Let $F: X \to Y$ be a multivalued function. Then

(a) $F(A) = \bigcup \{F(x) \mid x \in A\}$ for each $A \subset X$,

(b) $F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$ for each $B \subset Y$ (clearly F^{-1} is a multivalued function from Y to X).

It is quite easy to construct a multivalued function F from a

topological space X to a topological space Y such that $F^{-1}(U)$ is open for each open $U \subset Y$ and yet $F^{-1}(B)$ is not closed for each closed $B \subset Y$ and vice versa. Hence on immediately realizes that the various equivalent definitions of continuity of (single-valued) functions have to be considered separately for multivalued functions:

DEFINITION 2.3. Let $F: X \to Y$ be a multivalued function. Then we say that (the terminology of (e)—(g) was first developed by Ponomarev [13])

(a) F is a usc-function (i.e., upper semi-continuous function) provided that $F^{-1}(B)$ is closed for each closed $B \subset Y$,

(b) F is a lsc-function (i.e., lower semi-continuous function) provided that $F^{-1}(V)$ is open for each open $V \subset Y$,

(c) F is a continuous function provided that F is a usc-function and a lsc-function,

(d) F is a closed (open) function provided that $F(B)$ is closed (open) for each closed (open) $B \subset X$,

(e) F is Y-compact (Y-separable) (X-compact: X-separable) provided that $F(x)$ is compact (separable) for each $x \in X(F^{-1}(y)$ is compact for each $y \in Y$; $F^{-1}(y)$ is separable for each $y \in Y$),

(f) F is Y-perfect (X-perfect) provided that F is a closed, Y-compact, usc-function (F is a closed, X-compact, usc-function),

(g) F is perfect provided that F is X-perfect and Y-perfect.

Our terminology compares with others as follows (where "$\equiv$" means "the same as"):

1. usc $\equiv$ strong upper semi-continuity (Choquet [2]) $\equiv$ upper semi-continuous (Michael [7]) $\equiv$ continuous (Ponomarev [12]) $\equiv$ lower continuous (Hahn [3]).

2. lsc $\equiv$ strong lower semi-continuity (Choquet [2]) $\equiv$ lower semi-continuous (Michael [7]) $\equiv$ skew-continuous (Ponomarev [12]) $\equiv$ lower continuous (Hahn [3]).

DEFINITION 2.4. Let $F: X \to Y$ be a multivalued function. Then

(a) The graph of F is

$$grF = \{(x, y) \in X \times Y \mid y \in F(x) \text{ and } x \in X\} .$$

(b) The functions $p_X: grF \to X$ and $p_Y: grF \to Y$ are defined by

$$p_X(x, y) = x \text{ and } p_Y(x, y) = y$$

for each $(x, y) \in grF$.

We now state, without proof, some straightforward results of crucial importance which were first observed by Ponomarev and Smirnov.

LEMMA 2.5. *Let $F: X \to Y$ be a multivalued function from X onto Y. Then*

(a) $(F^{-1})^{-1} = F$,

(b) $F(x) = p_Y p_X^{-1}(x)$ *for each* $x \in X$,

(c) $F^{-1}(y) = p_X p_Y^{-1}(y)$ *for each* $y \in Y$,

(d) *the maps p_X and p_Y are always continuous (since they are restrictions of the projection maps $\pi_X: X \times Y \to X$ and $\pi_Y: X \times Y \to Y$).*

We now prove two very useful results which determine the relationship of a multivalued map $F: X \to Y$ and the maps p_X and p_Y. The proof of the first result is essentially due to Smirnov.

THEOREM 2.6. *Let X and Y be topological spaces and $F: X \to Y$ be a multivalued function. Then*

(a) *F is a* usc*-function and Y-compact if and only if the map p_X is perfect,*

(b) *F is closed and X-compact if and only if the map p_Y is perfect,*

(c) *F is perfect if and only if p_X and p_Y are perfect.*

Proof. The proof of the "only if" part of (a) appears in the footnote of page 123 of Ponomarev [13]. The proof of the "if" part of (a) follows easily from Lemma 2.5 and Definition 2.4(a). The proof of (b) is the same as the proof of (a) if one considers F^{-1} instead of F. Part (c) is an immediate consequence of parts (a) and (b).

REMARK 2.7. The appealing conjecture that $p_X(p_Y)$ is closed even though F is not Y-compact (X-compact) is false: Let R be the real line with the usual topology, I be the closed unit interval and $F: R \to R$ be defined by $F(x) = I$ for each $x \in R$. Then F is not X-compact ($F^{-1}(0) = R$) and p_Y is not a closed map ($p_Y(\{(x, y) \mid y = e^x$ and $x \leqq 0\}) = [0, 1]$). Furthermore, F is Y-compact, continuous and closed.

THEOREM 2.8. *Let $F: X \to Y$ be a multivalued function. Then*

(a) *if F is a* lsc*-function then p_X is an open map,*

(b) *if F is open then p_Y is open.*

Proof. It suffices to prove (a), since (b) is an immediate consequence of (a), due to Definition 2.4(b) and Lemma 2.5(a): Therefore, let us assume p_X is not open. Then there exists a closed $A \subset grF$ such that the set $B = \{x \in X \mid p_X^{-1}(x) \subset A\} = X - p_X(grF - A)$ is not a closed subset of X. Let $w \in B^- - B$. Then $p_X^{-1}(w) \not\subset A$. Let $(w, z) \in p_X^{-1}(w) - A$. Since A is a closed subset of grF there exists an

open neighborhood $N = U \times X$ of (w, z) in $X \times Y$ such that $N \cap A = \emptyset$. Letting

$$C = X - F^{-1}(V) = \{x \in X \mid F(x) \cap V = \emptyset\}$$

we get that C is not a closed subset of X, since $w \in U - C(z \in F(w) \cap V)$, $w \in U \cap B^-$ and $\emptyset \neq U \cap B \subset C$ (Suppose there exists $x \in (U \cap B) - C$. Then $V \subset F(x) \neq \emptyset$ and thus

$$\emptyset \neq (U \times V) \cap (\{x\} \times F(x)) = (U \times V) \cap p_X^{-1}(x) \subset N \cap A = \emptyset ,$$

a contradiction). Hence F^{-1} is clearly not an open function (i.e., F is not a lsc-function), a contradiction which completes the proof.

Finally we prove a rather interesting result (even though we cannot find it in the literature, it appears to be known).

THEOREM 2.9. *The relation between spaces X and Y expressed by "there is a perfect multivalued function F from X onto Y such that $F(x) \neq \emptyset$ for each $x \in X$" is an equivalence relation.*

Proof. The only difficult part of the proof involves showing that the composition of a Y-compact usc-function $F: X \to Y$ and a Z-compact usc-function $G: Y \to Z$ is Z-compact; but this is an immediate consequence of Corollary 9.6 of Michael [10] (see also 7.7 (p. 60) of Kruse [6], Th. 1 (p. 104) of Ponomarev [12] and 21.3.4 of Hahn [3]).

3. Preservation of topological properties. First we will prove necessary and sufficient conditions for the preservation of metrizability and stratifiability[1] by multivalued functions, for which we need the following lemma:

LEMMA 3.1. *Let X and Y be topological spaces with G_δ-diagonals.*[2] *Then $X \times X$ has a G_δ-diagonal.*

Proof. Suppose that U_n and V_n are open in $X \times X$ and $Y \times Y$ respectively, and $\bigcap_{n=1}^{\infty} U_n$ and $\bigcup_{n=1}^{\infty} V_n$ are the diagonals of $X \times X$ and $Y \times Y$ respectively. Let

$$W_n = \{((x_1, y_1), (x_2, y_2)) \mid (x_1, x_2) \in U_n \text{ and } (x_2, y_2) \in V_n\} .$$

Then W_n is open in $(X \times Y) \times (X \times Y)$ and $\bigcup_{n=1}^{\infty} W_n$ is the diagonal of $(X \times Y) \times (X \times Y)$.

[1] A summary of the properties of stratifiable spaces appears in the introduction of [1]. The most relevant results about these spaces which should be mentioned here are (a) Every CW-complex of Whitehead is stratifiable, (b) metrizability implies stratifiability which implies paracompactness and perfect normality.

[2] A topological space X has a G_δ-diagonal if $\{(x, x) \mid x \in X\}$ is a G_δ-subset of $X \times X$.

THEOREM 3.2. *Let $F: X \to Y$ be a perfect multivalued function, with $F(x) \neq \emptyset$ for each $x \in X$ (i.e., $F^{-1}(Y) = X$), where X and Y are T_1-spaces with G-diagonals. Then X is metrizable (stratifiable) if and only if Y is metrizable (stratifiable).*

Proof. We will first prove the "only if" part: Let X be metrizable (stratifiable) and Y have a G_δ-diagonal. Then grF has a G_δ-diagonal (a subspace of a space with a G_δ-diagonal clearly has a G_δ-diagonal) and hence grF is metrizable (stratifiable) because of Theorem 8.1 (Theorem 8.4) of [1].

Since p_Y is perfect, due to Theorem 2.6(c), we get that Y is metrizable (stratifiable) due to Theorem 1 of Stone [16] (Th. 3.1 of [1]). The "if" part follows immediately from Theorem 2.9.

We will now turn our attention to the preservation of various other topological properties by multivalued maps, for which we will need the following definition:

DEFINITION 3.3. Let $F: X \to Y$ be a multivalued map. Then F is said to be Y-monotone (monotone) provided that each $F(x)$ is connected (each $F^{-1}(y)$ is connected).

THEOREM 3.4. *Let $F: X \to Y$ be an onto (i.e., $F(x) = Y$) multivalued function such that $F(x) \neq \emptyset$ for each $x \in X$. Then*

(a) *if F is a* lsc*-function, $F(x)$ is separable for each $x \in X$ and X is separable then Y is separable,*

(b) *if F is a Y-monotone* lsc*-function and X is connected then Y is connected.*

Proof. (a) By Lemma 2.8(a), p_X is an open function and hence grF is easily seen to be separable. Hence Y is separable (p_Y is continuous).

(b) Clearly, it suffices to prove that grF is connected: Assume not. Then grF is the union of two disjoint nonempty open subsets U and V. Since P_X is clearly monotone we get that $p_X^{-1}(x) \cap U \neq \emptyset$ if and only if $p_X^{-1}(x) \subset U$. Hence $\{p_X(U), p_X(V)\}$ is an open disjoint cover of X, because of Lemma 2.8(a), a contradiction.

As pointed out by the referee, both parts of the preceding result become false if the hypothesis that $F(x) \neq \emptyset$, for each $x \in X$, is removed (Example (a). Let Z be a countably infinite discrete space, $Y = (\beta Z) - Z$ (βZ denotes the β-compactification of Z) and define $F: \beta F \to Y$ by $F(x) = \{x\} \cap Y$. Clearly F is a perfect Y-separable map. However, while βZ is clearly separable it is well-known that Y is not separable (see, for example, exercise 60(1, 2) of [4] for an

easy proof of the fact that Y has $2^{\aleph_0}$ mutually disjoint open subsets which clearly implies that Y is not separable). Example (b). Let $X = [0, 2]$, $Y = X - \{1\}$ and define $F: X \to Y$ by $F(x) = \{x\} \cap Y)$.

As a final comment of this section, we note that the requirement that X and Y be completely regular in the statement of Theorem 3 (page 127) of Ponomarev [13] is much stronger than needed, as the ensuing result indicates:

THEOREM 3.5. *Let X and Y be regular spaces and $F: X \to Y$ be an onto perfect multivalued function. If X is paracompact (locally compact; countably paracompact; star-paracompact) then so is Y. The converse is also true if $F(x) \neq \emptyset$ for each $x \in X$.*

Proof. Since F is a usc-function then $F^{-1}(X)$ is a closed subset of X. Hence the "if" part follows immediately from Theorem 2.2 of [5] and Theorem 1 of [11], whose proofs clearly depend only on the fact that X and Y are regular spaces, and Theorem 2.6. The converse follows from Theorem 2.10.

4. **Multivalued quotient maps.** Since multivalued functions behave very much like single-valued functions, it seems imperative that one consider the extension of a *quotient map* for single-valued functions to multivalued functions. In so doing, one is immediately compelled to consider two distinct concepts of multivalued quotient maps in the same manner as with the continuity of multivalued maps. However, it turns out that the situation is not as simple as it seems. Therefore, let us first prove some results which will justify our Definition 4.2.

LEMMA 4.1. *Let $F: X \to Y$ be a multivalued function and let X be a topological space. Then*

(a) *$\{U \subset Y \mid F^{-1}(U)$ is open$\}$ is not necessarily a base for a topology on Y.*

(b) *$\{U \subset Y \mid F^{-1}(Y - U)$ is closed$\}$ is a base for a topology on Y.*

Proof. It is easily seen that

(i) $F^{-1}(\bigcup_{\alpha \in A} A_\alpha) = \bigcup_{\alpha \in A} F^{-1}(A_\alpha)$

(ii) $F^{-1}(\bigcup_{\alpha \in A} A_\alpha) \subset \bigcup_{\alpha \in A} F^{-1}(A_\alpha)$, and equality does not necessarily hold, for any collection $\{A_\alpha\}_{\alpha \in A}$ of subsets of Y.

In order to prove (a) let X be the real line with the usual topology, Y the set of all real numbers, and let $F: X \to Y$ be defined by

$$F(1) = [1, 2],$$
$$F(x) = Y - [1, 2] \text{ for all } x \neq 1.$$

Then $F^{-1}(\{y \in Y \mid y \geqq (3/2)\})$ and $F^{-1}(\{y \in Y \mid y \leqq (3/2)\})$ are open subsets of X, but $F^{-1}(\{3/2\})$ is not an open subset of X. Hence $\{U \subset Y \mid F^{-1}(U)$ is open$\}$ is not a base for a topology on Y.

Finally we prove (b): Let $y \in U \cap V$ such that $F^{-1}(Y - U)$ and $F^{-1}(Y - V)$ are closed subsets of X. Since

$$\begin{aligned} F^{-1}(Y - V \cap U) &= F^{-1}((Y - U) \cup (Y - V)) \\ &= F^{-1}(Y - U) \cup F^{-1}(Y - V) , \end{aligned}$$

due to (i), one easily sees that $\{U \subset Y \mid F^{-1}(Y - U)$ is closed$\}$ is a base for a topology on Y, which completes the proof.

While Lemma 4.1 dashes all hopes of developing a concept of "multivalued quotient topology," nonetheless one can define a *multivalued quotient map* $F: X \to Y$ in such a way that F is either a usc-function or a lsc-function.

DEFINITION 4.2. Let X and Y be topological spaces and $F: X \to Y$ an onto multivalued map. Then F is said to be a *us-quotient* (*ls-quotient*) map provided that a subset U of Y is closed (open) if and only if $F^{-1}(U)$ is a closed subset of X ($F^{-1}(U)$ is an open subset of X). F is said to be a quotient map whenever F is both a *us*-quotient map and a *ls*-quotient map.

We will now study the relationship between a multivalued function $F: X \to Y$ and the maps p_X and p_Y, as well as other relations.

PROPOSITION 4.3. Let X and Y be topological spaces and $F: X \to Y$ a multivalued function. Then

(a) if p_X is closed and F is a *us*-quotient map then p_Y is a quotient map,

(b) if F is a *ls*-quotient map then p_Y is a quotient map, and p_X is an open map,

(c) if F is a Y-compact *us*-quotient map then p_Y is a quotient map, and p_X is a perfect map.

Proof. Part (a) is straightforward by use of Lemma 2.5(c). Part (b) is straightforward by use of Lemma 2.5(c) and Theorem 2.8(a). Part (c) is an immediate consequence of (a) and Theorem 2.6(a).

Now we generalize Theorem 3 and the Corollary on page 695 of Stone [16] to multivalued quotient maps.

THEOREM 4.4. *Let $F: X \to Y$ be a multivalued Y-compact quotient map from the locally compact separable metrizable space X onto a Hausdorff first countable space Y with a G_δ-diagonal. Then Y is a*

locally compact separable metrizable space.

Proof. By Proposition 4.3, p_Y is a quotient map and p_X is an open perfect map. Hence grF is locally compact (due to Theorem 3.5), metrizable (due to Theorem 3.2), and separable (because p_X is open and each $p_X^{-1}(x)$ is separable-indeed each $p_X^{-1}(x)$ is compact metrizable). Since p_Y is a quotient map, we get that Y is a locally compact separable metrizable space, by Theorem 3 of [16].

Similarly one can prove the following result, by using the Corollary of page 695 of [16]:

THEOREM 4.5. *Let $F: X \to Y$ be a multivalued Y-compact quotient map from the separable metrizable space X onto a regular first countable T_1-space Y with a G_δ-diagonal. Then Y is a separable metrizable space.*

Finally we generalize Theorem 4 of Stone [16] to multivalued open maps.

THEOREM 4.6. *Let $F: X \to Y$ be an open, Y-compact, X-separable continuous function from a metrizable and locally separable space X onto a regular T_1-space Y with a G_δ-diagonal. Then Y is metrizable and locally separable.*

Proof. Similar to the proof of Theorem 4.4 except that we use, respectively, Theorem 2.8(b), Theorem 2.8(a) and 2.6(a), and Theorem 4 of [16].

5. **Continuous extensions.** We will now generalize Theorem 4.3 of [1] to multivalued functions. Throughout this section, we let $C_u(Y, Z)(C_l(Y, Z); C(Y, Z))$ be the space of all usc-functions (lsc-functions; continuous functions) from Y to Z, for any topological spaces Y and Z.

Before stating our main theorem we state the following straightforward but crucial result:

LEMMA 5.1. *Let $F: X \to Y$ be a multivalued function. Then*

(a) *F is a* usc-*function if and only if for each $x \in X$ and neighborhood V of $F(x)$ there exists a neighborhood U of x such that $F(U) \subset V$.*

(b) *F is a* lsc-*function if and only if for each $x \in X$ and open $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists a neighborhood U of x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.*

THEOREM 5.2. *Let X be a stratifiable space, A a closed subset of X and E any topological space. Then there exist mappings*

$$\varphi: C_u(A, E) \to C_u(X, E) ,$$
$$\psi: C_l(A, E) \to C_l(X, E) ,$$

satisfying the following conditions:

(a) *each $\varphi(f)$ and $\varphi(f)$ is an extension of f,*

(b) range $\varphi(f) =$ range f; range $\psi(f) =$ range f,

(c) $\varphi(f)(x) \neq \emptyset$ $(\psi(g)(x) \neq \emptyset)$ *for each $x \in X$ whenever $f(x) \neq \emptyset$ $(g(x) \neq \emptyset)$ for each $x \in A$.*

Proof. Without further comment we will frequently use the terminology of Definition 4.1 of [1] and the crucial results of Lemma 4.2 of [1], since our proof parallels that of Theorem 4.3 of [1], to a large extent. Let $W = X - A$, and let $W' = \{x \in W \mid x \in U_y$ for some $y \in A$ and open U containing $y\}$. For each $x \in W'$, let

$$m(x) = \max \{n(U, y) \mid y \in A \text{ and } x \in U_y\} .$$

It is easily seen that each $m(x) \leqq n(W, x) < \infty$.

Using the paracompactness of W, let $\mathscr{V}$ be an open locally finite (with respect to W) cover of W such that $\{V^- \mid V \in \mathscr{V}$ and V^- is the closure of V with respect to $W\}$ is a refinement of $\{W_x \mid x \in W\}$. For each $V \in \mathscr{V}$, pick $x_V \in W$ such that $V^- \subset W_{x_V}$. If $x_V \in W'$, pick $a_V \in A$ and open S_V containing a_V such that $x_V \in (S_V)_{a_V}$ and $n(S_V, a_V) = m(x_V)$; if $x_V \notin W'$, let a_V be the fixed point $a_0 \in A$.

We now define two functions $g: X \to E$ and $h: X \to E$ by

$$g(x) = f(x) = h(x) \qquad \text{if } x \in A ,$$
$$g(x) = \bigcup \{f(a_V) \mid x \in V^-\} \qquad \text{if } x \in W ,$$
$$h(x) = \bigcup \{f(a_V) \mid x \in V) \qquad \text{if } x \in W .$$

It is easily seen that g is a usc-function on W (Let $x \in W$ and let V be a neighborhood of $g(x)$. Then $U = W - \bigcup \{V^- \mid V \in \mathscr{V}$ and $x \notin V^-\}$ is a neighborhood of x, and clearly $g(U) \subset V$; indeed, for each $z \in U$, $g(z) \subset g(x)$). Similarly, one easily sees that h is a lsc-function on W (Let $x \in W$ and let U be an open subset of E such that $h(x) \cap U \neq \emptyset$. By the definition of $h(x)$ there exists some $V \in \mathscr{V}$ such that $f(a_V) \cap U \neq \emptyset$ and $x \in V$. Then $h(z) \cap U \neq \emptyset$ for each $z \in V$).

To show that g is a usc-function at $b \in A$, whenever $f \in C_u(A, E)$, let 0 be any open subset of E containing $f(b)$. Since f is a usc-function, there exists an open neighborhood N of b such that $f(A \cap N) \subset 0$.

Now it is not hard to see that

(a) If $x \in [(N_b)_b - A] \cap V^-$ with $V \in \mathscr{V}^\wedge$ and V^- the closure of V with respect to W, then $a_V \in N$. (A detailed proof of (a) can be found in the last paragraph of the proof of Theorem 4.3 of [1].) Consequently, $g((N_b)_b) \subset 0$, due to (a), and thus g is a usc-function.

To show that h is a lsc-function at $b \in A$, whenever $f \in C_l(A, E)$, let 0 be any open subset of E such that $f(b) \cap 0 \neq \varnothing$. Since f is a lsc-function, there exists an open neighborhood N of b such that $f(z) \cap 0 \neq \varnothing$ for each $z \in N \cap A$. It is easily seen that $x \in (N_b)_b) - A$ implies $h(x) \cap 0 \neq \varnothing$, due to (a) and the definition of $h(x)$. Consequently, $h(x) \cap 0 \neq \varnothing$ for each $x \in (N_b)_b$, and thus h is a lsc-function.

To complete the proof, we simply let $\varphi(f) = g$ and $\psi(f) = h$.

REMARK 5.3. It is easily seen that the function g is not necessarily a lsc-function on W (Let U be any subset of E and suppose $f(a_V) \cap U \neq \varnothing$ for some $V \in \mathscr{V}^\wedge$. Then $g(x) \cap U \neq \varnothing$ for each $x \in V^-$, since $x \in V^-$ implies $f(a_V) \subset g(x)$ by the definition of $g(x)$. Hence

$$g^{-1}(U) = \bigcup \{V^{-1} \mid f(a_V) \cap U \neq \varnothing\}$$

is a closed subset of W for any subset U of E (since $\mathscr{V}^\wedge$ is locally finite with respect to W)). Thus g is not a lsc-function, unless $g^{-1}(U)$ is clopen[3] for each open subset U of E. We have thus essentially proved the following result:

THEOREM 5.4. *Let X be a stratifiable space, A a closed subset of X such that* $\dim (X - A) = 0$ *(covering dimension). Then there exist mappings*

$$\varphi: C_u(A, E) \longrightarrow C_u(X, E)$$
$$\psi: C_l(A, E) \longrightarrow C_l(X, E)$$
$$\theta: C(A, E) \longrightarrow C(X, E)$$

satisfying the following

(a) *Each $\varphi(f)$, $\psi(f)$ and $\theta(f)$ is an extension of f,*

(b) range $\varphi(f) =$ range $\psi(f) =$ range $\theta(f) =$ range f.

Proof. The same as the proof of Theorem 5.2 except that we choose the open cover $\mathscr{V}^\wedge$ of $X - A$ so that each $V \in \mathscr{V}^\wedge$ is a clopen subset of $X - A$ (see Proposition 2(b) of [9]).

We end this section with the following two questions:

1. Does Theorem 5.2 remain valid if X is a paracompact space? (We conjecture a negative answer.)

[3] A subset U of a topological space X is said to be *clopen* provided that U is an open and closed subset of X.

2. In Theorem 5.4, does the existence of the mapping θ imply that for some single-valued quotient map q from X onto $\tilde{X}$, say, q maps A homomorphically onto a closed subset of $\tilde{X}$, $q(A)$ and $q(X - A)$ are disjoint, and $\dim(q(X - A)) = 0$?

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