Pacific Journal of Mathematics

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Vol. 23, No. 3 May 1967

PACIFIC JOURNAL OF MATHEMATICS
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Let λ be a central endomorphism of a group G in the sense that λ induces the identity map on the inner automorphism group of G. Despite the nearness of the situation to commutativity, it is not necessarily true that the central endomorphisms of G form a ring or even that the subset generated by λ be a ring. The displacement map τ, given by $\tau(g) = g^{-1}\lambda(g)$ for each $g \in G$, is an endomorphism with central values. We shall show (Theorem 1) that if τ satisfies a certain pair of simultaneous equations then λ or λ^2 is idempotent. Let P be a formal polynomial with integral coefficients, and let t be the sum of these coefficients. Then (Theorem 2) $P(\lambda)$ is an endomorphism if and only if t induces an integral endomorphism on G. If G is nilpotent of class 2 then (Theorem 3) $P(\lambda)$ is an endomorphism if and only if $t(t-1)/2$ is an exponent for the commutator subgroup Q of G.

Theorem 3 gives us an alternate proof of an older (essentially equivalent) result [2, Th. 7, Corollary]. If α and β are two maps in G^G, then $\gamma = \alpha + \beta$ is to mean the map given by $\gamma(g) = \alpha(g)\beta(g)$ for all $g \in G$. The symbol ι will be reserved for the identity map on G. By $\operatorname{diag}_m x$ we mean the m-by-m matrix with x repeated down the main diagonal and with zeros elsewhere. If 1_G is the unity of the group G, we say that an integer m is an exponent of G if $g^m = 1_G$ for each $g \in G$. An integer m is said to induce an integral endomorphism on a group G if $(xy)^m = x^m y^m$ for all $x, y \in G$.

1. Preliminaries.

Let τ be a center-endomorphism of a group G. That is, τ is an endomorphism of G, and $\operatorname{Im}\tau \leqq Z$, the center of G. The map $\lambda \in G^G$ given by $\lambda(x) = x\tau(x)$ for each $x \in G$ is a normal endomorphism of G in that it commutes with each inner automorphism of G. It is a central endomorphism in that $\lambda = \iota + \tau$ where τ is a center-endomorphism. See [3]. Each center-endomorphism of G is likewise a normal endomorphism; but if G is nonabelian, no such endomorphism is a central endomorphism. The central endomorphism $\lambda = \iota + \tau$ is said to be related to the center-endomorphism τ. The set of all center-endomorphisms of a group G is a ring $C(G)$ under endomorphism addition and composition.

If τ is a center-endomorphism of G with related central endomorphism λ, then, with multiplication proceeding from left to right with increasing i and with $C(n, i)$ as the usual binomial coefficient, we have

$$(A_n) \qquad \lambda^n(x) = x \prod_{i=1}^{n} \tau^i(x^{C(n,i)})$$

and

$$(B_n) \qquad \tau^n(x) = \left[x \prod_{i=1}^{n} \lambda^i (x^{(-1)^i C(n,i)}) \right]^{(-1)^n}$$

for each $x \in G$ and for each positive integer n. From (A_n), each λ^n is a central endomorphism related to $\sum_{i=1}^{n} C(n, i)\tau^i \in C(G)$ where λ is related to τ. One readily sees that λ is idempotent if and only if $-\tau$ is idempotent. Under this assumption, $\tau^{2j+1} = \tau = -\tau^{2j}$ for each positive integer j.

Observe that the 2^n factors on the right of (B_n) can be rearranged at will. In fact, if one considers the mapping $P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ where the a_i are integers with $a_n \neq 0$, where $\lambda^0 = \iota$, and where $P(\lambda)x = \prod_{i=0}^{n} \lambda^i(x^{a_i})$ for each $x \in G$, then the terms of $P(\lambda)$ can be rearranged in any way. Nevertheless, $P(\lambda)$ need not be an endomorphism. If, however, it is an endomorphism, then it is normal. Call n the degree of P.

THEOREM 1. *Let τ be a center-endomorphism with related central endomorphism λ on a group G.*

(a) *Suppose that there exist integers $m > 0$ and $k \geqq 0$ such that $\tau^{2m+k} + \tau^m = 0$. Then there exists a formal polynomial P with integral coefficients and of degree $2m + 2k$ for which λ is a zero.*

(b) *If there exists an integer $n \geqq 3$ such that $\tau + \tau^{n-1} = 0 = \tau^2 + \tau^{n-2}$, then λ is idempotent if n is odd; while if n is even, $\operatorname{Im}\tau$ is elementary 2-abelian, $\lambda^3 = \lambda^2$, and λ^2 is idempotent.*

Proof. (a) From $\tau^{2m+2k} + \tau^{m+k} = 0$ and the above remark on idempotents, the central endomorphism σ related to τ^{m+k} must be idempotent. From (B_{m+k}), σ must be of degree $m + k$ as a polynomial in λ. Let T be the formal polynomial corresponding to σ. Let $P = T^2 - T$.

(b) $\tau = \tau^3$ so that $\tau^2 = \tau^4$, all odd powers reducing to τ, even to τ^2. If n is odd, then $\tau^{n-1} = \tau^2$ while $\tau^{n-2} = \tau$, from which $\tau^2 = -\tau$ and $\lambda^2 = \lambda$. If n is even, $\tau^{n-1} = \tau$ whence $\tau^{n-1} + \tau = 0$ yields $\tau(x^2) = 1_G$ for every $x \in G$. At once, $\operatorname{Im}\tau$ is elementary 2-abelian. Now, (A_2) leads to $\lambda^2(x) = x\tau^2(x)$ in this case. Applying λ, $\lambda^3(x) = x\tau(x^2)\tau^2(x) = \lambda^2(x)$. Thus, $\lambda^3 = \lambda^2$, all higher powers reducing to λ^2. In particular, λ^2 is idempotent.

As an example of (b), take G to be the group of m-by-m non-singular real matrices, and, for each matrix A therein, let $\tau(A) = \operatorname{diag}_m(|\det A|^{-1/m})$. It is clear that τ is a center-endomorphism of G and that $\tau^2 + \tau = 0$. If we take $n = 3$, we have the situation in (b).

2. The sum of the coefficients. If P is a polynomial with integral coefficients, let $t(P)$ denote the sum of these coefficients.

LEMMA. *Let α be a center-endomorphism of a group G, and let β be a member of G^G. Then $\alpha + \beta$ is an endomorphism of G if and only if β is an endomorphism.*

Proof. $(\alpha + \beta)(xy) = \alpha(x)\alpha(y)\beta(xy)$ while $(\alpha + \beta)(x)(\alpha + \beta)(y) = \alpha(x)\beta(x)\alpha(y)\beta(y)$. Since $\alpha(y)$ is in the center, the result is clear.

If k is an integer, let $[k]$ be that member of G^G which is given by $[k]x = x^k$ for each $x \in G$. Observe that if τ is a center-endomorphism of G, then τ generates a subring $\{\tau\}$ of $C(G)$.

THEOREM 2. *Let τ be a center-endomorphism of a group G, and let λ be its related central endomorphism. Let P be a polynomial with integral coefficients.*

(a) *If $t(P) = 0$, then $P(\lambda)$ is a center-endomorphism, a member of $\{\tau\}$.*

(b) *If $t(P) = 1$, then $P(\lambda)$ is a central endomorphism related to some member of $\{\tau\}$.*

(c) *If G is noncommutative and if $t(P) = 2$, then $P(\lambda)$ is no endomorphism.*

(d) *If $t(P) \neq 0, 1, 2$, then $P(\lambda)$ is:* (1) *an endomorphism if and only, if $[t(P)]$ is an endomorphism on G;* (2) *a center-endomorphism if and only if $[t(P)]$ is a center-endomorphism on G;* (3) *a central endomorphism if and only if $[t(P) - 1]$ is a center-endomorphism on G.*

Proof. Suppose that $P(\lambda) = \sum_{i=0}^{n} a_i\lambda^i$ for integers a_i. Note that $\lambda^0 = \iota$ and that, from (A_i), $\lambda^i = \iota + \sum_{j=1}^{i} C(i, j)\tau^j$ if $i > 0$. Upon substitution, $P(\lambda) = \sum_{i=0}^{n} a_i(\iota + \sum_{j=1}^{i} C(i, j)\tau^j) = t(P)\iota + \sum_{i=1}^{n} q_i\tau^i$ for suitable integers q_i. (a) and (b) are now immediate. If $t(P) = 2$, the lemma says that $2\iota = [2]$ is an endomorphism of G if and only if $P(\lambda)$ is an endomorphism. But $[2]$ is an endomorphism if and only if G is abelian, establishing (c). For $t(P) \neq 0, 1, 2$, the lemma gives (d), (1) and (2), directly. Now $P(\lambda)$ is central and related to a center-endomorphism if and only if $P(\lambda) = \iota + \sigma$ for some center-endomorphism σ. Equivalently, $(t(P) - 1)\iota + \sum_{i=1}^{n} q_i\tau^i - \sigma = 0$; that is, $(t(P) - 1)\iota = [t(P) - 1]$ is a center-endomorphism on G, establishing (d), (3).

By (a) above, each $\lambda^n - \lambda$ is a center-endomorphism, $n = 1, 2, \cdots$. By (c), if G is noncommutative, no $\lambda^n + \lambda$ is an endomorphism, $n = 1, 2, \cdots$.

Recall that a group is (nilpotent) of class 2 if its inner automorphism group is abelian.

THEOREM 3. *Let G be a class 2 group, λ a central endomorphism of G, and P a polynomial with integral coefficients. Then $P(\lambda)$ is a normal endomorphism of G if and only if $(t(P)-1)t(P)/2$ is an exponent of Q.*

Proof. Note that $P(\lambda)=\sum_{i=0}^{n} a_i\lambda^i$ is a normal endomorphism if and only if it is an endomorphism. Each λ^i is central (by A_i). For $x, y \in G$, let w denote $[y^{-1}, x^{-1}] = y^{-1}x^{-1}yx$. For a class 2 group, recall that $y^b x^a = x^a y^b w^{ab}$ and that $(xy)^a = x^a y^a w^{a(a-1)/2}$ for all integers a and b. By the centrality of the powers of λ, $\lambda^i(y^b)\lambda^j(x^a) = \lambda^j(x^a)\lambda^i(y^b)w^{ab}$ for all $x, y \in G$, all nonnegative integers i and j, and all integers a and b. It is now easy to show that $P(\lambda)(xy) = P(\lambda)(x)P(\lambda)(y)w^E$ where the integer $E=\sum_{i=0}^{n} a_i(a_i-1)/2+\sum_{i<j} a_i a_j$. From a routine observation one sees that $E=(t(P)-1)t(P)/2$.

COROLLARY. [2, Th. 7, Corollary] *Let s be an integer $\neq 0, 1, 2$. Let G be a class 2 group for which $s(s-1)/2$ is an exponent for Q. Then $[s]$ is an integral endomorphism for Q.*

Proof. By the theorem, any polynomial P with integral coefficients and with coefficient-sum s has $P(\lambda)$ an endomorphism for each central endomorphism λ, and the set of all such λ is nonempty. By Theorem 2, (d), $[s]$ is an endomorphism on G.

As an example of this corollary, let F be a commutative ring of finite characteristic and with a unity. Suppose that the characteristic $k=s(s-1)/2$ for some integer $s>2$. Let G be the set of all ordered triples $\langle a, b, c\rangle$ over F with multiplication given by $\langle a, b, c\rangle\langle a', b', c'\rangle = \langle a+a', b+b', c+c'+ba'\rangle$. We have the well known class 2 group G of triangular matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}$$

where $Z=Q$ is the set of all $\langle 0, 0, x\rangle$. Since $\langle 0, 0, x\rangle^n = \langle 0, 0, nx\rangle$, the characteristic k is an exponent for Q. In general, $\langle a, b, c\rangle^n = \langle na, nb, nc+(n(n-1)/2)\,bc\rangle$ for each integer n. An easy calculation now shows that $(\langle a, b, c\rangle\langle a', b', c'\rangle)^s - \langle a, b, c\rangle^s\langle a', b', c'\rangle^s = \langle 0, 0, (s-s^2)ba'\rangle$. But $(s-s^2)ba' = -2kba' = 0$, so that $[s]$ is indeed an integral endomorphism of G.

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Received September 1, 1965. This research was supported jointly by the National Science Foundation and by Washington University under grants No. GP-89 and GP-3874.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics
Vol. 23, No. 3 May, 1967