AN EXTREMAL LENGTH CRITERION FOR THE PARABOLICITY OF RIEMANNIAN SPACES

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It is the purpose of this paper to show that a given Riemannian space satisfying a regularity condition is parabolic if and only if the extremal distance of a fixed ball in the space from the ideal boundary of the space is infinite.

We will also show that the harmonic modulus of a space bounded by two sets of boundary components coincides with the extremal distance between the two sets.

STATEMENTS OF MAIN RESULTS

1. Regularity condition. Throughout this paper we denote by $R$ a noncompact $C^\infty$ Riemannian space with the ideal boundary $\beta$. We always assume that $R$ is orientable and connected. Let $A$ be the complement of a regular subregion of $R$ with the relative boundary $\alpha$. We also assume that $A-\alpha$ is connected. We consider the following regularity condition for $R$ (more precisely, for $A$):

For any nonconstant harmonic function $u$ defined on a region $\Omega \subset A$, the set $\{x \in \Omega | |\nabla u(x)| = 0\}$ has zero capacity.

This condition is always satisfied if the dimension of $R$ is two. This is also true, for example, when the metric tensor $g_{ij}$ is real analytic on $A-\alpha$. A typical case is furnished by a locally flat $A-\alpha$.

In this paper we only consider those spaces $R$ for which the above regularity condition is met.

2. Extremal length. Let $\rho$ be a density, i.e. a nonnegative Borel function on $A$, and let $\Gamma$ be a family of curves $\gamma$ which issue from a point in $\alpha$ and lie in $A-\alpha$. We define the harmonic extremal length, or simply the extremal length of $\Gamma$, by

$$\lambda(\Gamma) = \sup_{\rho \neq 0} \frac{L(\Gamma, \rho)^2}{V(A, \rho)} ,$$

where $V(A, \rho) = \int_A \rho^2 dV$ and $L(\Gamma, \rho) = \inf_\gamma \int_{\gamma} \rho ds$. Here $dV$ and $ds$ are the volume and the line element.

We are particularly interested in the family $\Gamma_\beta \subset \Gamma$ of all curves $\gamma \in \Gamma$ terminating at $\beta$.

3. Parabolicity. We call $R$ parabolic, $R \in O_\sigma$, if $R$ carries no nonconstant positive superharmonic function. The main object of this
paper is to prove:

**Theorem 1.** The space $R$ is parabolic if and only if $\lambda(\Gamma_\beta) = \infty$.

4. Moduli. Let $\Omega$ be a regular subregion of $R$ with relative boundary $\beta_\Omega \subset A - \alpha$, and let $u_\alpha$ be the continuous function on $\bar{\Omega} \cap A$ which is harmonic in the interior of $\bar{\Omega} \cap A$ with $u_\alpha| = 0$ and $u_\alpha|\beta_\alpha = 1$. The constant $\mu_\alpha$ given by

\[
\log \mu_\alpha = 1/ \int_{\bar{\Omega} \cap A} du_\alpha \wedge \ast du_\alpha
\]

is called the harmonic modulus, or simply the modulus of $\bar{\Omega} \cap A$ with respect to $\alpha$. It is easy to see that

\[
\mu_\alpha \leq \mu_{\alpha'}
\]

for $\Omega \subset \Omega'$. Therefore, we can define $\mu_\alpha$, the harmonic modulus of $A$ with respect to $\alpha$, as the directed limit

\[
\mu_\alpha = \lim_{\alpha \to \beta} \mu_\alpha.
\]

It is again easy to see that $u_\alpha = \lim_{\alpha \to \beta} u_\alpha$ exists and is continuous on $A$, harmonic on $A - \alpha$ with $u_\alpha| \alpha = 0$. Moreover,

\[
\log \mu_\alpha = 1/ \int_A du_\alpha \wedge \ast du_\alpha
\]

It can be seen that $R \in 0_\alpha$ if and only if $\mu_\alpha = \infty$ (Glasner [3]). Thus Theorem 1 may be considered as a special case of

**Theorem 2.** The following identity is valid:

\[
\lambda(\Gamma_\beta) = \log \mu_\alpha.
\]

The proof will be given in 5-9.

**Parabolic Case**

5. A general inequality. We start with proving

\[
\lambda(\Gamma_\beta) \geq \log \mu_\alpha.
\]

Let $\Gamma_{\beta_\alpha}$ be the family of curves $\gamma \in \Gamma$ which lie in $\bar{\Omega} \cap A$ and terminate at a point of $\beta_\alpha$. Define $\rho$ as $(\log \mu_\alpha)| \nabla u_\alpha|$ in the interior of $\bar{\Omega} \cap A$ and as zero elsewhere in $R$. For $\gamma \in \Gamma_{\beta_\alpha}$,

\[
\int_{\gamma} \rho \, ds = \int_{\gamma} (\log \mu_\alpha)| \nabla u_\alpha| \, ds \geq (\log \mu_\alpha) \int_{\gamma} \frac{dh}{ds} \, ds = \log \mu_\alpha.
\]
Therefore
\[ L(\Gamma_{\beta\Omega}, \rho) = \inf_{\gamma} \int_{\gamma} \rho ds \geq \log \mu_{\Omega}. \]

By (2) we also obtain
\[ V(A, \rho) = \int_{\Omega \cap A} (\log \mu_{\Omega})^2 |F u_{\rho}|^2 dV = (\log \mu_{\Omega}) \int_{\Omega \cap A} du_{\rho} \wedge *du_{\rho} = \log \mu_{\Omega}, \]
and infer by (1) that
\[ \lambda(\Gamma_{\beta\Omega}) \geq \log \mu_{\Omega}. \]

Since every \( \gamma \in \Gamma_{\beta} \) contains a \( \gamma' \in \Gamma_{\beta\Omega} \), we can easily see that \( \lambda(\Gamma_{\beta}) \geq \gamma(\Gamma_{\beta\Omega}) \) (cf. Ahlfors-Sario [1, p. 222]). Thus (8) implies that \( \lambda(\Gamma_{\beta}) \geq \log \mu_{\Omega} \) for every \( \Omega \). On letting \( \Omega \to R \) we obtain (7).

6. Now suppose that \( R \in 0_{\Omega} \). Then since \( \mu_{R} = \infty \), (7) implies that
\[ \lambda(\Gamma_{\beta}) = \log \mu_{R} = \infty. \]

In order to complete the proofs of Theorems 1 and 2, we have only to show the validity of (6) under the assumption \( R \in 0_{\Omega} \). Note that in our discussion thus far we have not made any use of the regularity condition.

**HYPERBOLIC CASE**

7. \( u \)-lines. Hereafter we assume that \( R \in 0_{\Omega} \). Then \( u_{\Omega} \), to be denoted simply by \( u \), is not constant on \( A \). Since \( u|A - \alpha = 0 \) and \( u|A - \alpha > 0 \), we infer that \( |F u| \) can be extended continuously to all of \( A \) and that \( |F u| | \alpha \neq 0 \).

For each \( x \in \alpha \) we consider the unique curve \( l_{x} \) issuing from \( x \) and such that \( l_{x} - x \subset A - \alpha \), \( *du = 0 \) on \( l_{x}, |F u| \neq 0 \) on \( l_{x} \). Moreover we require that \( l_{x} \) either terminates at \( \beta \) or at a point of \( A \) at which \( |F u| = 0 \). Such an \( l_{x} \) will be called a \( u \)-line. As \( y \) traces \( l_{x}, u(y) \) increases. Thus we can classify points of \( \alpha \) as follows:
\[ \alpha_{0} = \{ x \in \alpha \mid \lim_{y \to \beta \gamma, y \in l_{x}} u(y) < 1 \}, \]
\[ \alpha_{1} = \{ x \in \alpha \mid \lim_{y \to \beta \gamma, y \in l_{x}} u(y) = 1 \}, \]
with
\[ \alpha = \alpha_{0} \cup \alpha_{1}. \]
8. Vanishing surface area. We denote by \( dS \) the surface element of \( \alpha \). We wish to show that

\[
S(\alpha_0) = \int_{\alpha_0} dS = 0.
\]

Let \( F_{-1} \) be the set of points \( x \in \alpha \) such that \( l_x \) terminates at some point of \( R \). Clearly \( F_{-1} \subset \alpha_0 \), and we set \( F_n = \alpha_0 - F_{-1} \). By the regularity condition in §1, we see that \( S(F_{-1}) = 0 \) (cf. Brelot-Choquet [2]). Therefore we only have to show that \( S(F_0) = 0 \). Let

\[
F_n = \left\{ x \in F_0 \mid \lim_{y \to x, y \in F_0} (1 - u(y)) \geq \frac{1}{n} \right\} \quad (n = 1, 2, \cdots).
\]

Since \( F_0 = \bigcup_n F_n \), it is sufficient to show that \( S(F_n) = 0 \).

We can find a positive harmonic function \( \omega \) in the interior of \( A \) with the following properties (cf. Nakai [4]): (a) \( \omega \) has the boundary values 0 on \( \alpha \), (b) \( \lim_{x \to \beta, y \in F_0} \omega(y) = \infty \) for \( x \in F_0 \), (c) \( \int_A |\nabla \omega|^2 dV \leq c \), with \( \omega_x = \min(\omega, c) \) for every positive number \( c \).

Fix a \( c > 0 \) arbitrarily and a point \( y_0 \in F_0 \) with \( \omega(y_0) = c \) for each \( x \in F_n \).

Set \( v = 1 - u \) on \( A \). In a neighborhood of a point in \( \alpha \) with respect to \( A \) we may incorporate \( v \) into a coordinate system, say \( v = x^1 \), while \( x^2, \cdots, x^m \) are \( m - 1 \) linearly independent parameters for \( \alpha \). Then

\[
|\Delta v|^2 = g^{11} \left( \frac{\partial v}{\partial x^i} \right)^2 = g^{11}.
\]

Since \( *dv = |Fv|dS = \sqrt{g^{ij}}dS \) on \( \alpha \), \( S(F_n) = 0 \) is equivalent to \( \int_{F_n} *dv = 0 \). Observe that

\[
c\int_{F_n} *dv \leq \int_{F_n} \left( \int_x^{y_0} \frac{\partial \omega_x}{\partial v} dv \right) *dv = \int_{F_n} \left( \int_x^{y_0} \frac{\partial \omega_x}{\partial v} dv \right) \quad dv \wedge *dv = \int_{F_n} \left( \int_x^{y_0} \frac{\partial \omega_x}{\partial v} dv \right) g^{ij}dV.
\]

By the Schwarz inequality we have

\[
\int_{F_n} \left( \int_x^{y_0} \frac{\partial \omega_x}{\partial v} dv \right) g^{ij}dV \leq \left( \int_{F_n} \left( \int_x^{y_0} \frac{\partial \omega_x}{\partial v} dv \right)^2 g^{ij}dV \right)^{1/2} \left( \int_{F_n} g^{ij}dV \right)^{1/2}.
\]
\[ \left( \int_A |F \omega_s|^2 dV \right)^{1/2} \left( \int_A |F V|^2 dV \right)^{1/2} \leq \sqrt{\frac{c}{\omega_s}} \left( \int_A du \wedge *du \right)^{1/2}. \]

From this we infer that
\[ \left| \int_{F_n} *dv \right| \leq \frac{1}{\sqrt{\mu_n c}}. \]

Since the number \( c \) can be arbitrarily large, we have \( \int_{F_n} *dv = 0 \), and (11) follows.

9. Let \( \rho \) be a density with \( \rho \neq 0 \) on \( A \). Since
\[ du \wedge *du = |F u| dV, \]
we can compute
\[
V(A, \rho) = \int_A \rho^2 dV = \int_A \frac{\rho^2}{|F u|^2} du \wedge *du
\]
\[ \geq \int_{\alpha_1} \left( \int_{I_x} \frac{\rho^2}{|F u|^2} du \right) *du \]
\[ = \int_{\alpha_1} \left( \int_{I_x} \frac{\rho^2}{|F u|^2} du \cdot \int_{I_x} 1^2 du \right) *du \]
\[ \geq \int_{\alpha_1} \left( \int_{I_x} \frac{\rho}{|F u|} du \right)^2 *du. \]

On \( l_s(x \in \alpha_1) \) we have \( du = |F u| ds \), and thus
\[ V(A, \rho) \geq \int_{\alpha_1} \left( \int_{I_x} \rho ds \right)^2 *du. \]

From \( l_x \in \Gamma_\beta \) for \( x \in \alpha_1 \) we obtain \( \int_{I_x} \rho ds \geq L(\Gamma_\beta, \rho) \), and therefore
\[ (12) \quad V(A, \rho) \geq L(\Gamma_\beta, \rho) \int_{\alpha_1} *du. \]

On the other hand, by (11), we have \( \int_{\alpha_1} *du = \int_{\alpha} *du \). Take an arbitrary regular region \( \Omega \) with \( \beta_\Omega \subset A - \alpha \). Then
\[ \int_{\alpha} *du = \lim_{\rho \to \Omega} \int_{\alpha_1} *du. \]

Here we see that
\[ \int_{\alpha} *du = \int_{\beta_\Omega} *du = \int_{\beta_\Omega - \alpha} u_\Omega *du_\Omega = \int_{\Omega \cap A} du_\Omega \wedge *du_\Omega, \]
and infer that
\[ \int_a^* du = \lim_{R \to \infty} \int_{\partial \Delta} du_\partial \wedge \ast du_\partial = \int_a du \wedge \ast du. \]

This together with (5) and (12) implies the inequality
\[ \log \mu_R \geq \frac{L(\Gamma'_{\beta}, \rho)^2}{V(A, \rho)}. \]

Since \( \rho \) was arbitrary, we now conclude that
\[ \log \mu_R \geq \lambda(\Gamma'_{\beta}). \]

We combine this with (7) and obtain (6).

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