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**AN EXTREMAL LENGTH CRITERION FOR THE
PARABOLICITY OF RIEMANNIAN SPACES**

WELLINGTON HAM OW

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It is the purpose of this paper to show that a given Riemannian space satisfying a regularity condition is parabolic if and only if the extremal distance of a fixed ball in the space from the ideal boundary of the space is infinite.

We will also show that the harmonic modulus of a space bounded by two sets of boundary components coincides with the extremal distance between the two sets.

STATEMENTS OF MAIN RESULTS

1. **Regularity condition.** Throughout this paper we denote by R a noncompact C^∞ Riemannian space with the ideal boundary β . We always assume that R is orientable and connected. Let A be the complement of a regular subregion of R with the relative boundary α . We also assume that $A-\alpha$ is connected. We consider the following regularity condition for R (more precisely, for A):

For any nonconstant harmonic function u defined on a region $\Omega \subset A$, the set $\{x \in \Omega \mid |\nabla u(x)| = 0\}$ has zero capacity.

This condition is always satisfied if the dimension of R is two. This is also true, for example, when the metric tensor g_{ij} is real analytic on $A-\alpha$. A typical case is furnished by a locally flat $A-\alpha$.

In this paper we only consider those spaces R for which the above regularity condition is met.

2. **Extremal length.** Let ρ be a density, i.e. a nonnegative Borel function on A , and let Γ be a family of curves γ which issue from a point in α and lie in $A-\alpha$. We define the *harmonic extremal length*, or simply the extremal length of Γ , by

$$(1) \quad \lambda(\Gamma) = \sup_{\rho \neq 0} \frac{L(\Gamma, \rho)^2}{V(A, \rho)},$$

where $V(A, \rho) = \int_A \rho^2 dV$ and $L(\Gamma, \rho) = \inf_\gamma \int_\gamma \rho ds$. Here dV and ds are the volume and the line element

We are particularly interested in the family $\Gamma_\beta \subset \Gamma$ of all curves $\gamma \in \Gamma$ terminating at β .

3. **Parabolicity.** We call R parabolic, $R \in O_G$, if R carries no nonconstant positive superharmonic function. The main object of this

paper is to prove:

THEOREM 1. *The space R is parabolic if and only if $\lambda(\Gamma_\beta) = \infty$.*

4. Moduli. Let Ω be a regular subregion of R with relative boundary $\beta_\Omega \subset A - \alpha$, and let u_Ω be the continuous function on $\bar{\Omega} \cap A$ which is harmonic in the interior of $\bar{\Omega} \cap A$ with $u_\Omega|_\alpha = 0$ and $u_\Omega|_{\beta_\Omega} = 1$. The constant μ_Ω given by

$$(2) \quad \log \mu_\Omega = 1 / \int_{\bar{\Omega} \cap A} du_\Omega \wedge * du_\Omega$$

is called the *harmonic modulus*, or simply the modulus of $\bar{\Omega} \cap A$ with respect to α . It is easy to see that

$$(3) \quad \mu_\Omega \leq \mu_{\Omega'}$$

for $\Omega \subset \Omega'$. Therefore, we can define μ_R , the *harmonic modulus* of A with respect to α , as the directed limit

$$(4) \quad \mu_R = \lim_{\Omega \rightarrow R} \mu_\Omega.$$

It is again easy to see that $u_R = \lim_{\Omega \rightarrow R} u_\Omega$ exists and is continuous on A , harmonic on $A - \alpha$ with $u_R|_\alpha = 0$. Moreover,

$$(5) \quad \log \mu_R = 1 / \int_A du_R \wedge * du_R.$$

It can be seen that $R \in 0_G$ if and only if $\mu_R = \infty$ (Glasner [3]). Thus Theorem 1 may be considered as a special case of

THEOREM 2. *The following identity is valid:*

$$(6) \quad \lambda(\Gamma_\beta) = \log \mu_R.$$

The proof will be given in 5-9.

PARABOLIC CASE

5. A general inequality. We start with proving

$$(7) \quad \lambda(\Gamma_\beta) \geq \log \mu_R.$$

Let Γ_{β_Ω} be the family of curves $\gamma \in \Gamma$ which lie in $\bar{\Omega} \cap A$ and terminate at a point of β_Ω . Define ρ as $(\log \mu_\Omega) |\nabla u_\Omega|$ in the interior of $\bar{\Omega} \cap A$ and as zero elsewhere in R . For $\gamma \in \Gamma_{\beta_\Omega}$,

$$\int_\gamma \rho ds = \int_\gamma (\log \mu_\Omega) |\nabla u_\Omega| ds \geq (\log \mu_\Omega) \int_\gamma \frac{dh}{ds} ds = \log \mu_\Omega.$$

Therefore

$$L(\Gamma_{\beta\Omega}, \rho) = \inf_{\Gamma} \int_{\gamma} \rho ds \geq \log \mu_{\Omega} .$$

By (2) we also obtain

$$V(A, \rho) = \int_{\bar{\alpha} \cap A} (\log \mu_{\Omega})^2 |\nabla u_{\Omega}|^2 dV = (\log \mu_{\Omega})^2 \int_{\bar{\alpha} \cap A} du_{\Omega} \wedge * du_{\Omega} = \log \mu_{\Omega} ,$$

and infer by (1) that

$$(8) \quad \lambda(\Gamma_{\beta\Omega}) \geq \log \mu_{\Omega} .$$

Since every $\gamma \in \Gamma_{\beta}$ contains a $\gamma' \in \Gamma_{\beta\Omega}$, we can easily see that $\lambda(\Gamma_{\beta}) \geq \lambda(\Gamma_{\beta\Omega})$ (cf. Ahlfors-Sario [1, p. 222]). Thus (8) implies that $\lambda(\Gamma_{\beta}) \geq \log \mu_{\Omega}$ for every Ω . On letting $\Omega \rightarrow R$ we obtain (7).

6. Now suppose that $R \in 0_G$. Then since $\mu_R = \infty$, (7) implies that

$$(9) \quad \lambda(\Gamma_{\beta}) = \log \mu_R = \infty .$$

In order to complete the proofs of Theorems 1 and 2, we have only to show the validity of (6) under the assumption $R \notin 0_G$. Note that in our discussion thus far we have not made any use of the regularity condition.

HYPERBOLIC CASE

7. *u*-lines. Hereafter we assume that $R \notin 0_G$. Then u_R , to be denoted simply by u , is not constant on A . Since $u|_{\alpha} = 0$ and $u|_{A - \alpha} > 0$, we infer that $|\nabla u|$ can be extended continuously to all of A and that $|\nabla u| |_{\alpha} \neq 0$.

For each $x \in \alpha$ we consider the unique curve l_x issuing from x and such that $l_x - x \subset A - \alpha$, $*du = 0$ on l_x , $|\nabla u| \neq 0$ on l_x . Moreover we require that l_x either terminates at β or at a point of A at which $|\nabla u| = 0$. Such an l_x will be called a *u*-line. As y traces l_x , $u(y)$ increases. Thus we can classify points of α as follows:

$$\alpha_0 = \{x \in \alpha \mid \lim_{y \rightarrow \beta, y \in l_x} u(y) < 1\} ,$$

$$\alpha_1 = \{x \in \alpha \mid \lim_{y \rightarrow \beta, y \in l_x} u(y) = 1\} ,$$

with

$$(10) \quad \alpha = \alpha_0 \cup \alpha_1 .$$

8. **Vanishing surface area.** We denote by dS the surface element of α . We wish to show that

$$(11) \quad S(\alpha_0) = \int_{\alpha_0} dS = 0 .$$

Let F_{-1} be the set of points $x \in \alpha$ such that l_x terminates at some point of R . Clearly $F_{-1} \subset \alpha_0$, and we set $F_0 = \alpha_0 - F_{-1}$. By the regularity condition in §1, we see that $S(F_{-1}) = 0$ (cf. Brelot-Choquet [2]). Therefore we only have to show that $S(F_0) = 0$. Let

$$F_n = \left\{ x \in F_0 \mid \lim_{y \rightarrow \beta, y \in l_x} (1 - u(y)) \geq \frac{1}{n} \right\} \quad (n = 1, 2, \dots) .$$

Since $F_0 = \bigcup_1^\infty F_n$, it is sufficient to show that $S(F_n) = 0$.

We can find a positive harmonic function ω in the interior of A with the following properties (cf. Nakai [4]): (a) ω has the boundary values 0 on α , (b) $\lim_{y \rightarrow \beta, y \in l_x} \omega(y) = \infty$ for $x \in F_0$, (c) $\int_A |\nabla \omega_c|^2 dV \leq c$, with $\omega_c = \min(\omega, c)$ for every positive number c .

Fix a $c > 0$ arbitrarily and a point $y_x \in l_x$ with $\omega_c(y_x) = c$ for each $x \in F_n$.

Set $v = 1 - u$ on A . In a neighborhood of a point in α with respect to A we may incorporate v into a coordinate system, say $v = x^1$, while x^2, \dots, x^m are $m - 1$ linearly independent parameters for α . Then

$$|\Delta v|^2 = g^{11} \left(\frac{\partial v}{\partial x^1} \right)^2 = g^{11} .$$

Since $*dv = |\nabla v| dS = \sqrt{g^{11}} dS$ on α , $S(F_n) = 0$ is equivalent to $\int_{F_n} *dv = 0$. Observe that

$$\begin{aligned} c \int_{F_n} *dv &\leq \int_{F_n} \left(\int_x^{y_x} \frac{\partial \omega_c}{\partial v} dv \right) *dv \\ &= \int_{F_n} \int_x^{y_x} \left| \frac{\partial \omega_c}{\partial v} \right| dv \wedge *dv \\ &= \int_{F_n} \int_x^{y_x} \left| \frac{\partial \omega_c}{\partial x^1} \right| g^{11} dV . \end{aligned}$$

By the Schwarz inequality we have

$$\int_{F_n} \int_x^{y_x} \left| \frac{\partial \omega_c}{\partial x^1} \right| g^{11} dV \leq \left(\int_{F_n} \int_x^{y_x} \left| \frac{\partial \omega_c}{\partial x^1} \right|^2 g^{11} dV \right)^{1/2} \left(\int_{F_n} \int_x^{y_x} g^{11} dV \right)^{1/2}$$

$$\begin{aligned} &\cong \left(\int_A |\nabla \omega_c|^2 dV \right)^{1/2} \left(\int_A |\nabla v|^2 dV \right)^{1/2} \\ &\leq \sqrt{c} \left(\int_A du \wedge *du \right)^{1/2}. \end{aligned}$$

From this we infer that

$$\left| \int_{F_n} *dv \right| \leq 1/\sqrt{\mu_R c}.$$

Since the number c can be arbitrarily large, we have $\int_{F_n} *dv = 0$, and (11) follows.

9. Let ρ be a density with $\rho \not\equiv 0$ on A . Since

$$du \wedge *du = |\nabla u| dV,$$

we can compute

$$\begin{aligned} V(A, \rho) &= \int_A \rho^2 dV = \int_A \frac{\rho^2}{|\nabla u|^2} du \wedge *du \\ &\cong \int_{\alpha_1} \left(\int_{l_x} \frac{\rho^2}{|\nabla u|^2} du \right) *du \\ &= \int_{\alpha_1} \left(\int_{l_x} \frac{\rho^2}{|\nabla u|^2} du \cdot \int_{l_x} 1^2 du \right) *du \\ &\cong \int_{\alpha_1} \left(\int_{l_x} \frac{\rho}{|\nabla u|} du \right)^2 *du. \end{aligned}$$

On $l_x (x \in \alpha_1)$ we have $du = |\nabla u| ds$, and thus

$$V(A, \rho) \geq \int_{\alpha_1} \left(\int_{l_x} \rho ds \right)^2 *du.$$

From $l_x \in \Gamma_\beta$ for $x \in \alpha_1$ we obtain $\int_{l_x} \rho ds \geq L(\Gamma_\beta, \rho)$, and therefore

$$(12) \quad V(A, \rho) \geq L(\Gamma_\beta, \rho)^2 \int_{\alpha_1} *du.$$

On the other hand, by (11), we have $\int_{\alpha_1} *du = \int_\alpha *du$. Take an arbitrary regular region Ω with $\beta_\Omega \subset A - \alpha$. Then

$$\int_\alpha *du = \lim_{\Omega \rightarrow R} \int_\alpha *du_\Omega.$$

Here we see that

$$\int_\alpha *du_\Omega = \int_{\beta_\Omega} *du_\Omega = \int_{\beta_\Omega - \alpha} u_\Omega *du_\Omega = \int_{\partial \Omega \cap A} du_\Omega \wedge *du_\Omega,$$

and infer that

$$\int_{\alpha} * du = \lim_{\rho \rightarrow R} \int_{\bar{\alpha} \cap A} du_{\rho} \wedge * du_{\rho} = \int_A du \wedge * du .$$

This together with (5) and (12) implies the inequality

$$\log \mu_R \geq \frac{L(\Gamma_{\beta}, \rho)^2}{V(A, \rho)} .$$

Since ρ was arbitrary, we now conclude that

$$\log \mu_R \geq \lambda(\Gamma_{\beta}) .$$

We combine this with (7) and obtain (6).

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