CRITERIA FOR ZERO CAPACITY OF IDEAL BOUNDARY COMPONENTS OF RIEMANNIAN SPACES

WELLINGTON H. OW

Capacities of ideal boundary components of Riemannian spaces are introduced to measure their magnitude with respect to harmonic functions on the spaces. The main purpose of this paper is to find zero capacity criteria.

The modular criterion, well-known for Riemann surfaces, i.e. for 2-dimensional Riemannian spaces, is shown to be valid for general Riemannian spaces. The so-called metric criterion, however, brings forth entirely new aspects for higher dimensions.

CAPACITY OF A SUBBOUNDARY

1. Subboundaries. Throughout this paper we denote by $R$ a non-compact orientable connected $C^\infty$ Riemannian space. A relatively compact region whose relative boundary is smooth will be called a regular region. A sequence $\{R_n\}$ of regular regions $R_n \subset R$ such that $\overline{R_n} \cap R_{n+1}$ and $R = \bigcup_n R_n$ is called an exhaustion of $R$.

The ideal boundary component of Kerékjártó and Stoïlow may be related to $\{R_n\}$; here $R - \overline{R_n}$ can be assumed to consist of a finite number of relatively noncompact regions $F_{n,i}$ with corders $\beta_{n,i}$. Choose a sequence $F_1 = F_1, F_2 = F_2, \ldots$ such that $\overline{F_{n+1}} \subset F_n$. Then $\{F_n\}$ defines a boundary component $\gamma$. We denote by $\gamma_n$ the relative boundary $\partial F_n$ of $F_n$.

A subboundary, also to be denoted by $\gamma$, is a union of boundary components.

2. Capacity function. Let $B$ be a parametric ball about $a \in R$ with compact $\overline{B}$. Suppose $\gamma$ is a subboundary of $R$, and $\gamma_n$ the union of all $\partial F_n$ such that $\{F_n\}_n$ defines a boundary component belonging to $\gamma$.

Consider the family $P = \{p\}$ of harmonic functions $p$ on $R - a$ such that (a) $p = -g_s + h$ in $B$ where $g_s$ is the Green's function of $B$ with pole at $a$, and $h$ a harmonic function on $B$ with $h(a) = 0$, (b) $\int_{\gamma_n} *dp = 1$ and $\int_{\beta_{n,i}} *dp = 0$ for large $n$, where the $\beta_{n,i}$ are components of $\partial R_n$.

We use the conventional notations

$$\int_{\gamma} *dp = \lim_{n \to \infty} \int_{\gamma_n} *dp.$$
\[ \int_{\beta} p \star dp = \lim_{n \to \infty} \int_{\beta_n} p \star dp , \]

\( \beta \) being the entire boundary of \( R \).

Amalgamating the method of Sario [1] with the existence theorem of principal functions in Sario-Schiffer-Glasner [2], we can easily see that \( P \) is not empty and that there exists a function \( p_r \in P \) such that

\[ (1) \quad k_r = \min_p \int_{\beta} p \star dp = \int_{\beta} p_r \star dp_r . \]

Here \( -\infty < k_r \leq \infty \), and if \( k_r < \infty \), then \( p_r \) is unique. This follows from the identity

\[ (2) \quad \int_{\beta} p \star dp = D(p - p_r) + \int_{\beta} p_r \star dp_r , \]

where \( D \) indicates the Dirichlet integral.

The function \( p_r \) shall be referred to as a capacity function for \( \gamma \). The quantity \( c_r = e^{-k} \) for \( \dim R = 2 \), and \( k_r^{-(m-2)} \) for \( \dim R = m \geq 3 \) will be called the capacity of \( \gamma \).

**Modular Criterion**

3. **Moduli.** Let \( \Omega \) be a union of disjoint regular regions \( \Omega_j \), \( j = 1, \ldots, k \). Suppose that \( \partial \Omega_j \) consists of two nonempty disjoint sets \( \beta_j' \) and \( \beta_j'' \) which are unions of components of \( \partial \Omega_j \). Set \( \beta' = \bigcup_i \beta_j' \) and \( \beta'' = \bigcup_i \beta_j'' \). Let \( u_0 \) be the continuous function on \( \Omega \) which is harmonic on \( \Omega \) with \( u_0 | \beta' = 0, u_0 | \beta'' = \log \mu \), and \( \int_{\beta} * du = 1 \). The constant \( \mu > 1 \) is called the modulus of the configuration \( (\Omega, \beta', \beta'') \),

\[ \mu = \text{mod} (\Omega, \beta', \beta'') . \]

The function \( u_0 \) is referred to as the modulus function.

Consider the family \( U = U(\Omega, \beta', \beta'') \) of \( C^1 \)-functions \( u \) on \( \Omega \) which are harmonic on \( \Omega \) with \( \int_{\beta'} * du = 1 \). Then we have

\[ (3) \quad \min_u D_\delta(u) = D_\delta(u_0) . \]

This follows from the identity

\[ (4) \quad D_\delta(u) = D_\delta(u_0 - u) + D_\delta(u_0) \]

for every \( u \in U \).

4. **Modular criterion.** Let \( \bar{F}_n \) be the sum of those \( F_n \) for which \( \{F_n\} \) defines a boundary component in the subboundary \( \gamma \). Consider
\(E_n = (R^{n+1} - R_n) \cap R^n\) and set \(\gamma_n = \partial F_n, \gamma'_n = \partial E_n - \gamma_n\). In terms of
\[(5)\]
\[\mu_{n+1} = \text{mod} (E_n, \gamma_n, \gamma'_n)\]
we state:

**Theorem 1.** If there exists an exhaustion of \(R\) with
\[(6)\]
\[\prod_{n=1}^{\infty} \mu_{n+1} = \infty,\]
then the capacity of \(\gamma\) vanishes.

In fact, let \(p_n\) and \(k_n\) stand for \(p_r\) and \(k_r\) with respect to \(\gamma_n\) and \(R_n\). By (1) we infer that
\[D_{R_n+1 - R_n} (p_{n+1}) \leq k_{n+1} - k_n,\]
and by (3) that
\[
\log \mu_{n+1} \leq D_{R_n} (p_{n+1}).
\]
Therefore \(\log \mu_{n+1} \leq k_{n+1} - k_n\), and we conclude that (6) implies
\[
\lim_{n \to \infty} k_n = \infty.
\]

On the other hand it is not difficult to see that \(k_r = \lim_n k_n\), whence \(c_r = 0\).

**Metric Criterion**

5. Conformally equivalent metric. Let \(\lambda\) be a positive \(C^\infty\)-function on \(R\). The new metric
\[(7)\]
\[d\sigma = \lambda d\sigma\]
is conformally equivalent to the original metric \(d\sigma\) on \(R\). We fix a point \(a \in R\) and assume that
\[(8)\]
\[R(r) = \{x \in R \mid \sigma(x, a) < r\}\]
is relatively compact in \(R\) for \(0 < r < \infty\), with \(R = \bigcup_{0 < r < \infty} R(r)\). Consider the minimal union \(\gamma(r)\) of components of \(\beta(r) = \partial R(r)\) which separates \(\gamma\) from \(a\). Let
\[(9)\]
\[S_r(r) = \int_{\gamma(r)} dS_\sigma,\]
where \(dS_\sigma\) is the surface element induced by \(d\sigma\).

**Theorem 2.** If there exists an admissible \(\lambda\) such that
then $\gamma$ has vanishing capacity for $R$ with $\dim R = 2$. If, moreover, there exists a constant $M$ such that

$$0 < \frac{1}{M} \leq \lambda \leq M,$$

then the same conclusion holds regardless of the dimension of $R$.

For the proof we choose a sequence $\{r_n\}_{n=1}^\infty$ such that $\varepsilon < r_n < r_{n+1} < \infty$ and $\lim_n r_n = \infty$, with $R_n = R(r_n)$ regular. As in § 4 we define $E_n = (R_{n+1} - R_n) \cap \bar{E}_n$ and $\mu_{n_r}$. We also denote by $u_n$ the corresponding modulus function.

The proof in the case $\dim R = 2$ will be given in § 6 and that in the general case under the assumption (11), in § 7.

6. The case $\dim R = 2$. Observe that

$$\int_{E_n} |F_\sigma u_n|^2 dV_\sigma = \int_{r_n}^{r_{n+1}} \int_{\partial E_n} |F_\sigma u_n|^2 dS_\sigma \int_{\partial E_n} dS_\sigma \frac{dr}{S_\gamma(r)}.$$

By the Schwarz inequality we have

$$\int_{\partial E_n} |F_\sigma u_n|^2 dS_\sigma \int_{\partial E_n} dS_\sigma \geq \left( \int_{\partial E_n} *_\sigma d u_n \right)^2.$$

Since $*_\sigma = *$ and $|F_\sigma u_n|^2 dV_\sigma = |F u_n|^2 dV$, it is seen that (12), (13), and $D_{E_n}(u_n) = \log \mu_{n_r}$ imply

$$\log \prod_{k=1}^n \mu_{k\gamma} \geq \int_{r_1}^{r_{n+1}} \frac{dr}{S_\gamma(r)}.$$

We conclude that (10) implies (6), and consequently $c_r = 0$.

7. The case $\dim R = m > 2$. By (11) we see that

$$\int_{E_n} |F_\sigma u_n|^2 dV_\sigma \leq M^{m-2} D_{E_n}(u_n),$$

and

$$\int_{\partial E_n} *_\sigma d u_n \geq M^{-(m-2)} \int_{\partial E_n} * d u_n.$$

Therefore (14) must be modified to give

$$\log \prod_{k=1}^n \mu_{k\gamma} \geq M^{-(m-2)} \int_{r_1}^{r_{n+1}} \frac{dr}{S_\gamma(r)}.$$
But this sufficient to conclude that \( c_i = 0 \).

REMARK. Condition (11) cannot be suppressed in the case of higher dimensions.

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A. A. Aucoin, Diophantine systems ................................................. 419
Charles Ballantine, Products of positive definite matrices. I .................. 427
David Wilmot Barnette, A necessary condition for d-polyhedrality .......... 435
James Clark Beidleman and Tae Kun Seo, Generalized Frattini subgroups of finite groups ......................................................... 441
Carlos Jorge Do Rego Borges, A study of multivalued functions .............. 451
William Edwin Clark, Algebras of global dimension one with a finite ideal lattice ............................................................................. 463
Richard Brian Darst, On a theorem of Nikodym with applications to weak convergence and von Neumann algebras .......................... 473
George Wesley Day, Superatomic Boolean algebras .............................. 479
Lawrence Fearnley, Characterization of the continuous images of all pseudocircles ........................................................................ 491
Neil Robert Gray, Unstable points in the hyperspace of connected subsets .... 515
Franklin Haimo, Polynomials in central endomorphisms ........................ 521
John Sollion Hsia, Integral equivalence of vectors over local modular lattices ... 527
Jim Humphreys, Existence of Levi factors in certain algebraic groups ....... 543
E. Christopher Lance, Automorphisms of postliminal C*-algebras .......... 547
Sibe Mardesic, Images of ordered compacta are locally peripherally metric ..... 557
Albert W. Marshall, David William Walkup and Roger Jean-Baptiste Robert Wets, Order-preserving functions: Applications to majorization and order statistics ............................................................................. 569
Wellington Ham Ow, An extremal length criterion for the parabolicity of Riemannian spaces ................................................................ 585
Wellington Ham Ow, Criteria for zero capacity of ideal boundary components of Riemannian spaces .................................................. 591
J. H. Reed, Inverse limits of indecomposable continua ................................ 597
Joseph Gail Stampfli, Minimal range theorems for operators with thin spectra ... 601
Roy Westwick, Transformations on tensor spaces .................................. 613
Howard Henry Wicke, The regular open continuous images of complete metric spaces ........................................................................ 621
Abraham Zaks, A note on semi-primary hereditary rings ........................... 627
Thomas William Hungerford, Correction to: “A description of \( \text{Mult}_i(A^1, \ldots, A^n) \) by generators and relations” ................................................................. 629
Uppuluri V. Ramamohana Rao, Correction to: “On a stronger version of Wallis’ formula” ......................................................................... 629
Takesi Isiwata, Correction: “Mappings and spaces” ................................ 630
Henry B. Mann, Josephine Mitchell and Lowell Schoenfeld, Correction to: “Properties of differential forms in n real variables” .......... 631
James Calvert, Correction to: “An integral inequality with applications to the Dirichlet problem” .......................................................... 631
K. Srinivasacharyulu, Correction to: “Topology of some Kähler manifolds”.... 632