THE REGULAR OPEN CONTINUOUS IMAGES OF COMPLETE METRIC SPACES

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This article characterizes the regular $T_0$ open continuous images of complete metric spaces. These images are shown to be the regular $T_0$-spaces having monotonically complete bases of countable order. This follows from a theorem of Worrell and Wicke and a theorem below which shows that every regular $T_0$-space having a monotonically complete base of countable order is an open continuous image of a complete metric space.

The class of regular $T_0$-spaces having monotonically complete bases of countable order is equivalent to a class of spaces Aronszajn introduced axiomatically in [4]. This class includes the complete metric spaces and spaces satisfying R. L. Moore’s Axiom 1 [9]. Theorem 2 provides contrast to the theorem of Ponomarev [10]: every $T_0$ first countable space is an open continuous image of a metric space. A result related to Theorem 3 is Arhangel’skii’s characterization of the $T_i$ open compact continuous images of metrizable spaces as the metacompact developable $T_1$-spaces [2]. In connection with this result, it may be noted that an open compact continuous $T_1$ image of a regular $T_0$-space having a base of countable order also has a base of countable order [11], and a $T_i$ metacompact space having a base of countable order is developable [12].

For notation and terminology the reader is referred to [7], [9], and [12]. Space is used here to mean topological space. The null set convention is not used. A base for the topology of a space $S$ will be referred to as a base for $S$. Recall that a collection of sets is said to be perfectly decreasing [12], if and only if each of its elements properly includes an element of the collection; and that a base of countable order for a space [3], can be defined as a base $B$ for the space such that if $P$ is a point common to the elements of a perfectly decreasing subcollection $K$ of $B$, any open set containing $P$ includes an element of $K$; i.e., the elements of $K$ form a base at $P$. By a monotonically complete base [11], is meant a base $B$ such that the closures of the elements of any monotonic subcollection of $B$ have a point in common. Recall also that regular $T_0$-spaces are $T_1$, as Koutský remarked [5, p. 826].
2. Regular spaces having monotonically complete bases of countable order.

**Theorem 1.** A regular $T_0$-space $S$ has a monotonically complete base of countable order if and only if there exists a sequence $G_1, G_2, \ldots$ of bases for the topology of $S$ such that if $g_1, g_2, \ldots$ is a sequence such that, for each $n$, $g_n$ belongs to $G_n$ and $\overline{g_{n+1}}$ is a subset of $g_n$, then there exists a point $P$ in each $g_n$ such that the collection of terms of $g_1, g_2, \ldots$ is a base at $P$.

**Proof.** Suppose $V$ is a monotonically complete base of countable order for $S$. There exists a sequence $H_1, H_2, \ldots$ of well-ordered subcollections of $V$ covering $S$ such that these conditions are satisfied:

1. For each $n$ and $h$ in $H_n$ there exists a point $P_{n,h}$ belonging to $h$ such that no element of $H_n$ precedes $h$ and contains $P_{n,h}$.
2. If $n < k$, the closure of the first element $h$ of $H_k$ containing the point $P$ is a subset of the first element $h'$ of $H_n$ containing $P$. For some $k \geq j$, $g_j$ belongs to $H_k$. Let $P$ denote the point $P_{k,g_j}$. If $h$ is the first element of $H_n$ to contain $P$ then $h$ includes $g_j$. Thus $h$ does not precede $h_n$. Since $h_n$ contains $P$ it follows that $h = h_n$. Similarly, $h_{n+1}$ is the first element of $H_{n+1}$ to contain $P$ and thus $h_{n+1}$ is a subset of $h_n$. If $h_n = h_{n+1}$ for some $n$, then $h_n = \{P\}$ for some point $P$, and thus $g_k = \{P\}$ for some $k$, and $\{g_k\}$ is a base at $P$. If $h_n \neq h_{n+1}$, for any $n$, the terms of $h_1, h_2, \ldots$ form a monotonic subcollection of $V$ and thus there exists a point $P$ common to each $\overline{h_n}$. Since $\overline{h_{n+1}}$ is a subset of $h_n$, $P$ is in each $h_n$. If $D$ is open and contains $P$, there exists some $h_n$ which is a subset of $D$ and thus some $g_k$ is included in $D$. Hence $P$ is in $\overline{g_k}$ for all $k$, and since $\overline{g_k}$ is a subset of $g_{k-1}$ for all $k > 1$, it follows that $P$ is in each $g_k$. Since the $h_n$'s form a base at $P$ so do the $g_n$'s.

If $G_1, G_2, \ldots$ is a sequence as in the statement of Theorem 1 there exists a sequence $H_1, H_2, \ldots$ of well-ordered collections covering $S$ such that for each $n$: (1) $H_n$ is a subcollection of $G_n$. (2) Each element $h$ of $H_n$ contains a point belonging to no predecessor of $h$ in $H_n$. (3) If $n < k$ and $P$ is a point, the closure of the first element of $H_k$ containing $P$ is a subset of the first element of $H_n$ doing so. $V = H_1 + H_2 + \ldots$ is a base for $S$ and can be shown to be a base of
countable order by an argument used in Theorem 2 of [12]. A technique
similar to one employed there and also in the preceding paragraph,
shows that V is monotonically complete.

**Theorem 2.** A regular $T_0$-space having a monotonically complete
base of countable order is an open continuous image of a complete
metric space.

**Proof.** Let $S$ denote a regular $T_0$-space having a monotonically complete
base of countable order. By Theorem 1 there exists a sequence
$G_1, G_2, \cdots$ of bases for $S$ with the property stated in that theorem.
Form the Baire space $M$ [6] over the collections $G_1, G_2, \cdots$. The
elements of $M$ are sequences $\xi = (g_1, g_2, \cdots)$ where $g_n$ belongs to $G_n$. If $\xi = (g_1, g_2, \cdots)$ and $\xi' = (g_1', g_2', \cdots)$ the distance $\rho(\xi, \xi')$ is defined
to be $1/k$ if there exists a first positive integer $k$ such that $g_k \neq g_k'$. Otherwise
$\rho(\xi, \xi') = 0$. Designate by $O_{a_1\cdots a_k}$ the collection of all se-
quencies $(a'_1, a'_2, \cdots)$ such that $a_i = a'_i$, $i = 1, \cdots, k$. Let $W$ denote
the collection of all elements in $M$ of the form $(g_1, g_2, \cdots)$ where for
each $n$, $g_{n+1}$ is a subset of $g_n$. Then, by the condition on $G_1, G_2, \cdots$,
there exists a unique point $P$ common to the terms of $g_1, g_2, \cdots$. If
$\xi = (g_1, g_2, \cdots)$ is in $W$, define $f_\xi$ to be the unique point $P$ common
to the terms of $\xi$. Hence $f$ is a mapping of $W$ onto $S$. Suppose $W$
intersects the set $O_{a_1\cdots a_k}$. Then $g_{i+1}$ is a subset of $g_i$ for all $i \leq k - 1$. Clearly, $f(W \cdot O_{a_1\cdots a_k})$ is
a subset of $g_k$. If $P$ is an element of $g_k$, there exists $g_{k+1}, g_{k+2}, \cdots$
such that $g_{k+n}$ is a subset of $g_{k+n-1}$ for all $n \geq 1$. Hence $f(W \cdot O_{a_1\cdots a_k}) = g_k$. Since the collection of all sets $W \cdot O_{a_1\cdots a_k}$ is a base for $W$ and
by the property of $G_1, G_2, \cdots$, $f$ is open and continuous on $W$. (This argument is related to one used by Ponomarev [10].)

Suppose $P_1, P_2, \cdots$ is a sequence of points of $W$ satisfying the
Cauchy convergence criterion. For each $n$, there exists a positive
integer $m_n$ such that $\rho(P_k, P_j) < 1/n$, provided $k, j \geq m_n$. It may be
assumed that $m_{n+1} > m_n$ for every $n$. Let $a^n_1, a^n_2, \cdots, a^n_n$ denote the
first $n$ coordinates of $P_{m_n}$. Let $a_n$ denote $a^n_n$ for each $n$. Then if
$k \geq m_n$, the first $n$ coordinates of $P_k$ are $a_1, \cdots, a_n$. For if $n = 1,$
$a_1 = a^n_1$ is the first coordinate of $P_{m_1}$. If $k > m_n$, then $\rho(P_k, P_{m_n}) < 1,$
and thus $a_1$ is the first coordinate of $P_k$. Suppose the statement is
true for $n$. If $k \geq m_{n+1}$, then $\rho(P_k, P_{m_{n+1}}) < 1/(n + 1)$. Since
$m_{n+1} > m_n$, the first $n$ coordinates of $P_{m_{n+1}}$ are $a_1, \cdots, a_n$, by the as-
sumption, and the $(n + 1)^{st}$ coordinate is $a_{n+1}$. Let $P$ denote $(a_1, a_2, \cdots)$. It
follows that $P$ is the sequential limit point of $P_1, P_2, \cdots$. Moreover,
since $P_{m_n}$ is in $W$, the coordinates $a_1, a_2, \cdots, a_n$ satisfy the condition
that $\bar{a}_{k+1}$ is a subset of $a_k$ for all $k \leq n - 1$. Since this is true for all $n$, it follows that $P$ is in $W$, and thus $W$ is complete with respect to $\rho$.

**Remark.** From the proof of the above theorem it may be seen that the complete metric space of the theorem may be taken to be of zero dimension and of the same weight as the image space. (The weight of a topological space is the minimum cardinal number $m$ such that the space has a base of power $m$ [1].)

3. **The characterization theorem.** In [11] Worrell and Wicke define a $\lambda$-base for a topological space as a base $B$ of countable order for the space such that if $K$ is a perfectly decreasing monotonic sub-collection of $B$, there exists a point $P$ such that any open set containing $P$ includes an element of $K$. A regular $T_0$-space has a $\lambda$-base if and only if it has a monotonically complete base of countable order [11]. A principal theorem of [11] is that an open continuous (essentially) $T_1$ image of a space having a $\lambda$-base also has a $\lambda$-base.

**Theorem 3.** A regular $T_0$-space is an open continuous image of a complete metric space if and only if it has a monotonically complete base of countable order.

**Proof.** The sufficiency follows from Theorem 2. The necessity is a consequence of the theorems cited in the paragraph preceding the statement of Theorem 3, and the facts that a regular $T_0$-space is $T_1$ and that a complete metric space has a $\lambda$-base.

**Theorem 4.** The following conditions on a regular $T_0$-space are equivalent.

(a) The space has a monotonically complete base of countable order.

(b) The space satisfies Aronszajn's axiom [4, p. 231].

(c) The space has a $\lambda$-base.

(d) The space is an open continuous image of a complete metric space.

**Proof.** The equivalence of (a), (b), and (c) is stated in [11], and may be established by methods used in the proof of Theorem 1 above. Theorem 3 above shows the equivalence of (a) and (d).

By using techniques similar to those used above, the following theorem may be proved. (The sufficiency is a joint result of Worrell and Wicke given in [11].)
THEOREM 5. A $T_\gamma$-space $S$ has a base of countable order if and only if there exists a metric space $(M, d)$ and an open continuous mapping $f$ of $M$ onto $S$ such that for each $x$ in $S$, $f^{-1}(x)$ is complete with respect to the metric $d$.

This result and a theorem of Arhangel’skii [3] imply the following theorem of Michael [8]:

If $f$ is an open continuous mapping of a metric space $E$ onto a $T_1$ paracompact space $F$ such that $f^{-1}(y)$ is complete for every $y$ in $F$, then $F$ is metrizable.

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Received August 23, 1966. This work was supported by the United States Atomic Energy Commission.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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