Pacific Journal of Mathematics

# LIE ALGEBRAS OF TYPE $D_4$ OVER ALGEBRAIC NUMBER FIELDS

HARRY P. ALLEN

Vol. 24, No. 1 May 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 1, 1968

# LIE ALGEBRAS OF TYPE $D_4$ OVER ALGEBRAIC NUMBER FIELDS

H. P. ALLEN

**If $\widetilde{\mathfrak{A}}$ is a nonassociative algebra over an algebraically closed field $L$, then the classification problem for $\widetilde{\mathfrak{A}}$ is the determination of all algebras $\mathfrak{A}$ over $\Phi \subset L$ where $\widetilde{\mathfrak{A}} \cong \mathfrak{A} \otimes_\Phi L$. This brief note studies this problem for the case where $\mathfrak{A}$ is the Lie algebra $D_4$ and $\Phi$ is a (finite) algebraic number field. The main result is a type of Hasse principle which tells us that a Lie algebra $\mathfrak{L}$ (over $\Phi$) of type $D_4$ has known type if the algebra $\mathfrak{L}_{\Phi_p}$ has known type for every completion $\Phi_p$ of $\Phi$. This is used in §3 to obtain canonical splitting fields for Lie algebras of type $D_4$ over $\Phi$. Although the results are inconclusive with regard to the existence or nonexistence of new algebras, it indicates a (twisted) construction, which if nonvacuous, would yield new exceptional algebras of type $D_{4III}$.**[1]

The notation will be the same as that in the author's "Jordan Algebras and Lie Algebras of Type $D_4$" [2]. Throughout the present paper $\Phi, F, E, K, P$ will denote algebraic number fields and $\Omega(X)$ will denote the complete set of primes on the algebraic number field $X$. Also, we shall adopt the following convention without further mention: if $X$ is an algebraic number field, $Y$ a subfield and $p \in \Omega(X)$, then we shall use $p$ to represent $p \mid Y$ and $Y_p$ for the completion of $Y$, at $p \mid Y$, in $X_p$. We begin with a field theoretic preliminary.

1. PROPOSITION 1. Let $P/\phi$ be a finite dimensional Galois extension with Galois group $G$, and let $p \in \Omega(P)$. Then $P_p/\phi_p$ is Galois and $G_p = g(P_p/\phi_p)$ is isomorphic to a subgroup of $G$.

*Proof*. If $P$ is a splitting field for $f(\lambda) \in \phi[\lambda]$ over $\phi$, then $P_p$ is a splitting field for $f(\lambda)$ over $\phi_p$ and thus $P_p/\phi_p$ is Galois. If $P = \phi(\zeta)$, then $P_p = \phi_p(\zeta)$ and the correspondence $s_p \rightarrow s_p \mid P = s'_p$ is an injection of $G_p$ in $G$.

$G_p$ is called the local Galois group at $p$ and we note that if $E$ is the subfield of $P/\phi$ of $G'_p$-invariants, then $E_p = \phi_p$, for $E$ is contained in the $P_p$ invariants of $G_p$ so $E_p \subseteqq \phi_p$.

To avoid unnecessary complication we let $Q$ be the field of rational numbers, $\mathfrak{C}_0$ the split Cayley algebra over $Q$, $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_{03}, 1)$ the split exceptional central simple Jordan algebra over $Q$ and $\mathfrak{D} = \mathfrak{D}(\mathfrak{J}/\Sigma Qe_i)$ the split Lie algebra of type $D_4$ over $Q$. If $X$ is any field of charac-

[1] The author has recently shown, in collaboration with J. Ferrar, that this construction can be carried out over algebraic number fields.

teristic 0, then $\mathfrak{D}_X = \mathfrak{D}(\mathfrak{J}_X/\Sigma X e_i)$ will be taken as the split Lie algebra of type $D_4$ over $X$.

Now let $\mathfrak{L}$ be a Lie algebra of type $D_4$ over $\phi$ with $P/\phi$ a finite dimensional Galois splitting extension. If $p \in \Omega(P)$, then $\mathfrak{L}_{\phi_p}$ is a Lie algebra of type $D_4$ over $\phi_p$ split by $P_p$. We first determine the relationship between the pre-cocycle of $G$ in $\mathrm{Aut}_\phi(\mathfrak{D}_P)$ corresponding to $\mathfrak{L}$, and the pre-cocycle of $G_p$ in $\mathrm{Aut}_{\phi_p}(\mathfrak{D}_{P_p})$ corresponding to $\mathfrak{L}_{\phi_p}$ ([2] § 2).

Thus let $r \to \eta(r) \leftrightarrow C_r = [p(r), T(r)]$ be the pre-cocycle of $G$ in $\mathrm{Aut}_\phi(\mathfrak{D}_P)$ corresponding to $\mathfrak{L}$. If $h' = h_p \mid P \in G'_p$, then $h'$ has a unique extension to $G_p$, viz., $h_p$. We let $C_{h_p}$ be the $h_p$-semilinear extension of $C_{h'}$ to $\Gamma L_{\phi_p}(\mathfrak{J}_{P_p}/\Sigma P_p e_i)$. $C_{h_p} = [p(h_p \mid P), T(h_p)]$ where $T(h_p)$ is the $h_p$-semilinear extension of $T(h_p \mid P)$ ([3] p. 12).

We have $C_{h_p} C_{r_p} = C_{h_p r_p} \delta_{h',r'}$ where $C_{h'} C_{r'} = C_{h'r'} \delta_{h',r'}$. Thus if $\eta(h_p) \leftrightarrow C_{h_p}$, then $h_p \to \eta(h_p)$ is a pre-cocyle of $G_p$ in $\mathrm{Aut}_{\phi_p}(\mathfrak{D}_{P_p})$. The fixed $\phi_p$-form of $\mathfrak{D}_{P_p}$ associated with this pre-cocycle clearly contains $\mathfrak{L}_{\phi_p}$, so it must be $\mathfrak{L}_{\phi_p}$.

PROPOSITION 2. Let $\mathfrak{L}$ be a Lie algebra of type $D_4$ over $\phi$ with $P/\phi$ a finite dimensional Galois splitting extension and $F$ the canonical $D_{4I}$-field extension of $\mathfrak{L}$. If $p \in \Omega(P)$ then

(i) The $D_4$ type of $\mathfrak{L}_{\phi_p}$ is the $D_4$ type of a canonical extension of $\mathfrak{L}$ ([2] § 2).

(ii) the canonical $D_{4I}$-field extension of $\mathfrak{L}_{\phi_p}$ is $F_p$.

(iii) if $L$ is exceptional then $\mathfrak{L}_{\phi_p}$ is exceptional if and only if $[F_p\colon \phi_p] \geqq 3$.

*Proof.* (i) is a direct consequence of the preceding discussion. Let $F(p)$ be the canonical $D_{4I}$-field extension of $\mathfrak{L}_{\phi_p}$ and suppose that $F(p)$ is the invariants of $H_p \subset G_p$. If $F'$ is the invariants of $H'_p$ then $F \subset F'$ so $F_p \subseteqq F(p)$. But $\mathfrak{L}_{F_p}$ is of type $D_{4I}$ so $F_p \supseteqq F(p)$. If $\mathfrak{L}$ is exceptional then this shows that $\mathfrak{L}_{\phi_p}$ is exceptional if and only if $[F_p\colon \phi_p] \geqq 3$.

2. The classical results on central simple associative algebras and quadratic forms over algebraic number fields are used to deduce the next two important results.

THEOREM 1. *Let $\mathfrak{L}$ be a Lie algebra of type $D_4$ over an algebraic number field $\phi$. Then there exists a finite subset $S$ of $\Omega(\phi)$ such that $\mathfrak{L}_{\phi_p}$ is a Jordan $D_4$ for all $p \in \Omega(\phi) - S$.*

*Proof.* First suppose that $\mathfrak{L}$ is of type $D_{4I}$ and let $\mathfrak{L}^* = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_3$ be its $\phi$-enveloping algebra ([2] § 2). Let $S$ be any finite subset of

$\Omega(\phi)$ such that $\mathfrak{A}_{i\phi_p} \sim 1$, $i = 1, 2, 3$ for all $p \in \Omega(\phi) - S$ ([1] Chap IX). Since $\mathfrak{L}^*_{\phi_p}$ is clearly the $\phi_p$-enveloping algebra of $\mathfrak{L}_{\phi_p}$ for any $p \in \Omega(\phi)$, we see that for $p \in \Omega(\phi) - S$, $\mathfrak{L}^*_{\phi_p}$ is a sum of matrix algebras over $\phi_p$. This implies that $\mathfrak{L}_{\phi_p}$ is a Jordan $D_4$ ([2] Th. I).

Now let $\mathfrak{L}$ be an arbitrary Lie algebra of type $D_4$ and let $F/\phi$ be its canonical $D_{4I}$-field extension. Let $T$ be any finite subset of $\Omega(F)$ such that $(\mathfrak{L}_F)_{F_p}$ is a Jordan $D_4$ for all $p \in \Omega(F) - T$, and choose $S$ as the set of all traces of elements of $T$ on $\phi$. If $p \mid \phi \in \Omega(\phi) - S$ then $p \in \Omega(F) - T$ and $(\mathfrak{L}_{\phi_p})_{F_p} = \mathfrak{L}_{F_p} = (\mathfrak{L}_F)_{F_p}$ is a Jordan $D_4$. Since $F_p$ is the canonical $D_{4I}$-field extension of $\mathfrak{L}_{\phi_p}$, $\mathfrak{L}_{\phi_p}$ is a Jordan $D_4$ ([1] Th. I).

THEOREM 2. *Let $\mathfrak{L}$ be a Lie algebra of type $D_4$ over an algebraic number field $\phi$. Then $\mathfrak{L}$ is a Jordan $D_4$ if and only if $\mathfrak{L}_{\phi_p}$ is a Jordan $D_4$ for every $p \in \Omega(\phi)$.*

*Proof.* One direction is clear. For the other let $F$ be the canonical $D_{4I}$-field extension of $\mathfrak{L}$ and let $\mathfrak{L}^*_F = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_3$ be the $F$-enveloping algebra of $\mathfrak{L}_F$. Our hypothesis implies that $\mathfrak{L}_{F_p} = (\mathfrak{L}_{\phi_p})_{F_p}$ is a Jordan $D_4$ for every $p \in \Omega(F)$. Thus $\mathfrak{A}_{iF_p} \sim 1$, $i = 1, 2, 3$ and all $p \in \Omega(F)$, so $\mathfrak{A}_i \sim 1$, $i = 1, 2, 3$. ([1] Chap IX). Thus $\mathfrak{L}_F$ is a Jordan $D_{4I}$ and $\mathfrak{L}$ is a Jordan $D_4$.

COROLLARY. *$\mathfrak{L}$ is split if and only if $\mathfrak{L}_{\phi_p}$ is split for all $p \in \Omega(\phi)$.*

*Proof.* One direction is trivial. For the other, Theorem 1 and Theorem 2 imply that $\mathfrak{L}$ is a Jordan $D_{4I}$. If $\mathfrak{L} = \mathfrak{s}(\mathfrak{C}, n(\cdot))$, $\mathfrak{C}$ a Cayley algebra over $\phi$, then $\mathfrak{L}_{\phi_p}$ split for all $p$ implies that $\mathfrak{C}_{\phi_p}$ is isotropic for all $p$. Thus $\mathfrak{C}$ is isotropic, ([4], Th. 66.1) hence split, and $\mathfrak{L}$ is split.

3. This last section is devoted to a proof of Proposition 3. Using this proposition we are able to give a fairly explicit description of pre-cocycles arising from algebras of type $D_{4III}$.

PROPOSITION 3. Let $\mathfrak{L}$ be a Lie algebra of type $D_4$ over an algebraic number field $\Phi$, and let $F$ be the canonical $D_{4I}$-field extension of $\mathfrak{L}$. Then $\mathfrak{L}$ is split by a Galois extension of degree at most $2[F: \Phi]$.

*Proof.* We will only give the argument when $\mathfrak{L}$ is of type $D_{4I}$ or $D_{4III}$. The other cases are similar. Let $S$ be a finite subset of $\Omega(\phi)$ such that $\mathfrak{L}_{\phi_p}$ is a Jordan ${}_4\mathfrak{D}$ if $p \in \Omega(\phi) - S$. Without loss of generality we suppose that $S$ contains all the real primes on $\phi$. If $p \in S$, then $\mathfrak{L}_{\phi_p}$ is necessarily of type $D_{4I}$. By the local classification of $D_4$'s ([2] § 4), $\mathfrak{L}_{\phi_p}$ is split by a quadratic extension $K_{(p)}/\phi_p$. Let $K_{(p)}$ be a root field for $\lambda^2 + \alpha_p \in \phi[\lambda]$. By the approximation theorem, since $S$ consists of inequivalent primes, there exists an $\alpha \in \phi$ with each $|\alpha - \alpha_p|_p$ sufficiently small. Let $K$ be a root field for $\lambda^2 + \alpha$ over $\phi$ and $F$ the canonical $D_{4I}$-field extension of $\mathfrak{L}$. Note that $K_p = K_{(p)}$ for all $p \in S$.

If $\mathfrak{L}$ is of type $D_{4I}$ then it is easy to see that $(\mathfrak{L}_K)_{K_p}$ is split for all $p \in \Omega(K)$. Thus $\mathfrak{L}_K$ is split. If $\mathfrak{L}$ is of type $D_{4III}$ then we claim that $\mathfrak{L}$ is split by $P = K \otimes_\phi F$. Let $p \in \Omega(P)$. If $p \mid \phi \in \Omega(\phi) - S$, then $\mathfrak{L}_{\phi_p}$ is a Jordan $D_4$. If $p$ is complex, $\mathfrak{L}_{\phi_p}$ is clearly split whereas if $p$ is discrete, $\mathfrak{L}_{\phi_p}$ is split by its canonical $D_{4I}$ field extension ([2] § 4). In any event, $(\mathfrak{L}_P)_{P_p} = (\mathfrak{L}_{\phi_p})_{P_p} = ((\mathfrak{L}_{\phi_p})_{F_p})_{P_p}$ is split. If $p \mid \phi \in S$, then $\mathfrak{L}_{\phi_p}$ is split by $K_{(p)} = K_p \subset P_p$ so $(\mathfrak{L}_P)_{P_p}$ is split. Thus $(\mathfrak{L}_P)_{P_p}$ is split at every $p \in \Omega(P)$ and the corollary to Theorem 2 shows that $\mathfrak{L}_P$ is split. Not that $P$ is sixth degree cyclic.

Now let $\mathfrak{L}$ be a Lie algebra of type $D_{4III}$ over $\Phi$, with $P/\Phi$ a cyclic sixth degree Galois splitting extension. $P/\Phi$ contains a unique quadratic subfield $E/\phi$. $\mathfrak{L}_E$ is a $D_{4III}$ split by $P$, so $\mathfrak{L}_E$ is a Jordan $D_4$. If $\mathfrak{L}_E = \mathfrak{D}(\mathfrak{J}'/\mathfrak{k})$, then since $\mathfrak{J}'$ is reduced and is split by $P/E$, a cubic extension, $\mathfrak{J}'$ is itself split. $\mathfrak{k}$ of course is isomorphic to $P$. The isomorphism condition for Jordan $D_4$'s ([2] Th. II) implies that $\mathfrak{L}_E$ is a Steinberg $D_{4III}$ ([2] (10)).

Let $r \longrightarrow p(r)$ be the anti-homomorphism of $g(P/\phi) = G$ onto $A_3$ determined by $\mathfrak{L}$, and choose $s$ as a generator for $G$ with $p(s) = (123)$. Let $\mathfrak{C}$ be any Cayley algebra over $\phi$, split by $P$, and let $S$ be the $s$-semilinear automorphism of $\mathfrak{C}_P$ which is one on $\mathfrak{C}$. Finally set $D_S = [(123), S]$ (cf. [2] (10)), and let $r \longrightarrow \eta(r) \leftrightarrow C_r$ be the pre-cocycle of $G$ in $\mathrm{Aut}_\phi(\mathfrak{D}_P)$ corresponding to $\mathfrak{L}$. The preceding observation about $\mathfrak{L}_E$ enables us to assume that $C_s^2 = D_s^2\mu$, $\mu \in K$. By replacing $C_s$ by $C_S\lambda$, for some suitable $\lambda \in K$, if necessary, we may assume that $C_s$ and $D_s^2$ commute. This implies that $\mu^{s^2} = D_s^{-2}\mu D_s^2 = \mu$. If $C_s^6 = \delta \in K$, then $\delta = \mu^3$ and $\delta^s = C_s^{-1}\delta C_s = D_s^{-1}\delta D_s = \delta$ so $(\mu^3)^s = \mu^3$. Applying $\zeta(\cdot)$ to the relation $C_s^2 = D_s^2\mu$ we obtain

$$\zeta(C_s)^s\zeta(C_s) = \mu^2 . \tag{1}$$

But $\zeta(C_s)^s\zeta(C_s)$ is fixed under $s$ since $\zeta(C_s)$ is fixed $s^2$. Thus $(\mu^2)^s = \mu^2$. This, together with the previous relation shows that $\mu = \mu^s$.

For simplicity write $\zeta(C_s) = (\rho, p^{s^2}, \rho^{s^4})$ $\mu = (\beta, \beta^{s^2}, \beta^s)$, $\beta^{s^3} = \beta$. (1) is now equivalent to $\rho\rho^{s^3} = \beta^2$. Since $(\rho\beta^{-1})(\rho\beta^{-1})^{s^3} = 1$, $\rho = \lambda^{-1}\lambda^{s^3}\beta$ and $\rho\lambda^2 = \lambda\lambda^{s^3}\beta \in F$, the canonical $D_{4I}$-field extension of $\mathfrak{L}$. Replacing $C_s$ by $\widetilde{C}_s = C_s(\lambda, \lambda^{s^2}\lambda^{s^4})$ we again obtain $\zeta(\widetilde{C}_S)^{s^2} = \zeta(\widetilde{C}_s)$. But

$$\zeta(\widetilde{C}_s) = (\lambda\lambda^{s^3}\beta, ((\lambda\lambda^{s^3}\beta)^{s^2}, (\lambda\lambda^{s^3}\beta)^s)$$

and is fixed under $s^3$. Thus $\zeta(\widetilde{C}_s)^s = \zeta(\widetilde{C}_s)$. We may affect a similar alteration of $C_s$ so that $\zeta(C_s) = (\alpha_1, \alpha_1^{s^2}, \alpha_1^s) = \alpha$ where $\alpha_1^{s^3} = \alpha_1$ and $\alpha_1\alpha_1^s\alpha_1^{s^2} = 1$, i.e., $\tilde{C}_s$, in addition to the above is norm preserving. Calculating we see that $\tilde{C}_s^2 = D_s^2\alpha$. Then $\tilde{C}_s^6 = \alpha^3$ and $\mathfrak{L}$ is a Jordan $D_4$ if and only if $\alpha_1 \in N_{P/F}(P^*)$. Setting $\tilde{C}_s = D_sE$ we see that $D_s\alpha = ED_sE$. The simplest form of this equation occurs where $E$ and $D_S$ commute and we obtain $E^2 = \alpha$. Thus we are led to the following (possibly vacuous) construction.

Let $\mathfrak{J}$ be a reduced exceptional central simple Jordan algebra over a field $\phi$, $P$ a cyclic sixth degree extension of $\phi$ and $F$ a subfield of $\mathfrak{J}$ isomorphic to the cubic subfield of $P/\phi$. Then if there exists an $E \in GL(\mathfrak{J}/F)$ such that

(i) $E \in GL(\mathfrak{J}_P/\{Pe_i\}_i)$,

(ii) $\zeta(E) = (\alpha_1, \alpha_1^{s^2}, \alpha_1^s)$ where $\alpha_1 \notin N_{P/F}(P^*)$ and

(iii) $E^2 = \zeta(E)$,

then the $s$-semilinear extension of $E$ to $\mathfrak{J}_P$ induces a pre-cocycle corresponding to a non-Jordan $D_{4III}$.

## BIBLIOGRAPHY


1. A. A. Albert, *Structure of algebras*, A. M. S. Coll. (1939).
2. H. P. Allen, *Jordan algebras and Lie algebras of type $D_4$*, J. of Algebra (1967).
3. N. Jacobson, *Triality and Lie algebras of type $D_4$*, Rend. Cir. Mat. de Palermo.
4. O. T. O'Meara, *Introduction to Quadratic Forms*, Springer Verlag, 1963.


Received January 17, 1967.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

# Pacific Journal of Mathematics

Vol. 24, No. 1 May, 1968