

Pacific Journal of Mathematics

**LIE ALGEBRAS OF TYPE D_4 OVER ALGEBRAIC NUMBER
FIELDS**

HARRY P. ALLEN

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If $\tilde{\mathfrak{A}}$ is a nonassociative algebra over an algebraically closed field L , then the classification problem for $\tilde{\mathfrak{A}}$ is the determination of all algebras \mathfrak{A} over $\phi \subset L$ where $\tilde{\mathfrak{A}} \cong \mathfrak{A} \otimes_{\phi} L$. This brief note studies this problem for the case where \mathfrak{A} is the Lie algebra D_4 and ϕ is a (finite) algebraic number field. The main result is a type of Hasse principle which tells us that a Lie algebra \mathfrak{L} (over ϕ) of type D_4 has known type if the algebra \mathfrak{L}_{ϕ_p} has known type for every completion ϕ_p of ϕ . This is used in §3 to obtain canonical splitting fields for Lie algebras of type D_4 over ϕ . Although the results are inconclusive with regard to the existence or nonexistence of new algebras, it indicates a (twisted) construction, which if non-vacuous, would yield new exceptional algebras of type D_{4III} .¹

The notation will be the same as that in the author's "Jordan Algebras and Lie Algebras of Type D_4 " [2]. Throughout the present paper ϕ, F, E, K, P will denote algebraic number fields and $\Omega(X)$ will denote the complete set of primes on the algebraic number field X . Also, we shall adopt the following convention without further mention: if X is an algebraic number field, Y a subfield and $p \in \Omega(X)$, then we shall use p to represent $p|Y$ and Y_p for the completion of Y , at $p|Y$, in X_p . We begin with a field theoretic preliminary.

1. PROPOSITION 1. Let P/ϕ be a finite dimensional Galois extension with Galois group G , and let $p \in \Omega(P)$. Then P_p/ϕ_p is Galois and $G_p = g(P_p/\phi_p)$ is isomorphic to a subgroup of G .

Proof. If P is a splitting field for $f(\lambda) \in \phi[\lambda]$ over ϕ , then P_p is a splitting field for $f(\lambda)$ over ϕ_p and thus P_p/ϕ_p is Galois. If $P = \phi(\zeta)$, then $P_p = \phi_p(\zeta)$ and the correspondence $s_p \rightarrow s_p|P = s'_p$ is an injection of G_p in G .

G_p is called the local Galois group at p and we note that if E is the subfield of P/ϕ of G'_p -invariants, then $E_p = \phi_p$, for E is contained in the P_p invariants of G_p so $E_p \subseteq \phi_p$.

To avoid unnecessary complication we let Q be the field of rational numbers, \mathfrak{C}_0 the split Cayley algebra over Q , $\mathfrak{S} = \mathfrak{h}(\mathfrak{C}_0, 1)$ the split exceptional central simple Jordan algebra over Q and $\mathfrak{D} = \mathfrak{D}(\mathfrak{S}/\Sigma Qe_i)$ the split Lie algebra of type D_4 over Q . If X is any field of charac-

¹ The author has recently shown, in collaboration with J. Ferrar, that this construction can be carried out over algebraic number fields.

teristic 0, then $\mathfrak{D}_X = \mathfrak{D}(\mathfrak{S}_X/\Sigma X e_i)$ will be taken as the split Lie algebra of type D_4 over X .

Now let \mathfrak{L} be a Lie algebra of type D_4 over ϕ with P/ϕ a finite dimensional Galois splitting extension. If $p \in \Omega(P)$, then \mathfrak{L}_{ϕ_p} is a Lie algebra of type D_4 over ϕ_p split by P_p . We first determine the relationship between the pre-cocycle of G in $\text{Aut}_{\phi}(\mathfrak{D}_P)$ corresponding to \mathfrak{L} , and the pre-cocycle of G_p in $\text{Aut}_{\phi_p}(\mathfrak{D}_{P_p})$ corresponding to \mathfrak{L}_{ϕ_p} ([2] § 2).

Thus let $r \rightarrow \eta(r) \leftrightarrow C_r = [p(r), T(r)]$ be the pre-cocycle of G in $\text{Aut}_{\phi}(\mathfrak{D}_P)$ corresponding to \mathfrak{L} . If $h' = h_p | P \in G'_p$, then h' has a unique extension to G_p , viz., h_p . We let C_{h_p} be the h_p -semilinear extension of $C_{h'}$ to $GL_{\phi_p}(\mathfrak{S}_{P_p}/\Sigma P_p e_i)$. $C_{h_p} = [p(h_p | P), T(h_p)]$ where $T(h_p)$ is the h_p -semilinear extension of $T(h_p | P)$ ([3] p. 12).

We have $C_{h_p} C_{r_p} = C_{h_p r_p} \delta_{h', r'}$ where $C_{h'} C_{r'} = C_{h' r'} \delta_{h', r'}$. Thus if $\eta(h_p) \leftrightarrow C_{h_p}$, then $h_p \rightarrow \eta(h_p)$ is a pre-cocycle of G_p in $\text{Aut}_{\phi_p}(\mathfrak{D}_{P_p})$. The fixed ϕ_p -form of \mathfrak{D}_{P_p} associated with this pre-cocycle clearly contains \mathfrak{L}_{ϕ_p} , so it must be \mathfrak{L}_{ϕ_p} .

PROPOSITION 2. Let \mathfrak{L} be a Lie algebra of type D_4 over ϕ with P/ϕ a finite dimensional Galois splitting extension and F the canonical D_{4I} -field extension of \mathfrak{L} . If $p \in \Omega(P)$ then

- (i) The D_4 type of \mathfrak{L}_{ϕ_p} is the D_4 type of a canonical extension of \mathfrak{L} ([2] § 2).
- (ii) the canonical D_{4I} -field extension of \mathfrak{L}_{ϕ_p} is F_p .
- (iii) if L is exceptional then \mathfrak{L}_{ϕ_p} is exceptional if and only if $[F_p: \phi_p] \geq 3$.

Proof. (i) is a direct consequence of the preceding discussion. Let $F(p)$ be the canonical D_{4I} -field extension of \mathfrak{L}_{ϕ_p} and suppose that $F(p)$ is the invariants of $H_p \subset G_p$. If F' is the invariants of H'_p then $F \subset F'$ so $F_p \subseteq F'(p)$. But \mathfrak{L}_{F_p} is of type D_{4I} so $F_p \cong F'(p)$. If \mathfrak{L} is exceptional then this shows that \mathfrak{L}_{ϕ_p} is exceptional if and only if $[F_p: \phi_p] \geq 3$.

2. The classical results on central simple associative algebras and quadratic forms over algebraic number fields are used to deduce the next two important results.

THEOREM 1. *Let \mathfrak{L} be a Lie algebra of type D_4 over an algebraic number field ϕ . Then there exists a finite subset S of $\Omega(\phi)$ such that \mathfrak{L}_{ϕ_p} is a Jordan D_4 for all $p \in \Omega(\phi) - S$.*

Proof. First suppose that \mathfrak{L} is of type D_{4I} and let $\mathfrak{L}^* = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_3$ be its ϕ -enveloping algebra ([2] § 2). Let S be any finite subset of

$\Omega(\phi)$ such that $\mathfrak{A}_{i\phi_p} \sim 1, i = 1, 2, 3$ for all $p \in \Omega(\phi) - S$ ([1] Chap IX). Since $\mathfrak{L}_{\phi_p}^*$ is clearly the ϕ_p -enveloping algebra of \mathfrak{L}_{ϕ_p} for any $p \in \Omega(\phi)$, we see that for $p \in \Omega(\phi) - S$, $\mathfrak{L}_{\phi_p}^*$ is a sum of matrix algebras over ϕ_p . This implies that \mathfrak{L}_{ϕ_p} is a Jordan D_4 ([2] Th. I).

Now let \mathfrak{L} be an arbitrary Lie algebra of type D_4 and let F/ϕ be its canonical D_{4I} -field extension. Let T be any finite subset of $\Omega(F)$ such that $(\mathfrak{L}_F)_{F_p}$ is a Jordan D_4 for all $p \in \Omega(F) - T$, and choose S as the set of all traces of elements of T on ϕ . If $p \mid \phi \in \Omega(\phi) - S$ then $p \in \Omega(F) - T$ and $(\mathfrak{L}_{\phi_p})_{F_p} = \mathfrak{L}_{F_p} = (\mathfrak{L}_F)_{F_p}$ is a Jordan D_4 . Since F_p is the canonical D_{4I} -field extension of \mathfrak{L}_{ϕ_p} , \mathfrak{L}_{ϕ_p} is a Jordan D_4 ([1] Th. I).

THEOREM 2. *Let \mathfrak{L} be a Lie algebra of type D_4 over an algebraic number field ϕ . Then \mathfrak{L} is a Jordan D_4 if and only if \mathfrak{L}_{ϕ_p} is a Jordan D_4 for every $p \in \Omega(\phi)$.*

Proof. One direction is clear. For the other let F be the canonical D_{4I} -field extension of \mathfrak{L} and let $\mathfrak{L}_F^* = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_3$ be the F -enveloping algebra of \mathfrak{L}_F . Our hypothesis implies that $\mathfrak{L}_{F_p} = (\mathfrak{L}_{\phi_p})_{F_p}$ is a Jordan D_4 for every $p \in \Omega(F)$. Thus $\mathfrak{A}_{iF_p} \sim 1, i = 1, 2, 3$ and all $p \in \Omega(F)$, so $\mathfrak{A}_i \sim 1, i = 1, 2, 3$. ([1] Chap IX). Thus \mathfrak{L}_F is a Jordan D_{4I} and \mathfrak{L} is a Jordan D_4 .

COROLLARY. *\mathfrak{L} is split if and only if \mathfrak{L}_{ϕ_p} is split for all $p \in \Omega(\phi)$.*

Proof. One direction is trivial. For the other, Theorem 1 and Theorem 2 imply that \mathfrak{L} is a Jordan D_{4I} . If $\mathfrak{L} = \mathfrak{s}(\mathfrak{C}, n(\cdot))$, \mathfrak{C} a Cayley algebra over ϕ , then \mathfrak{L}_{ϕ_p} split for all p implies that \mathfrak{C}_{ϕ_p} is isotropic for all p . Thus \mathfrak{C} is isotropic, ([4], Th. 66.1) hence split, and \mathfrak{L} is split.

3. This last section is devoted to a proof of Proposition 3. Using this proposition we are able to give a fairly explicit description of pre-cocycles arising from algebras of type D_{4III} .

PROPOSITION 3. *Let \mathfrak{L} be a Lie algebra of type D_4 over an algebraic number field ϕ , and let F be the canonical D_{4I} -field extension of \mathfrak{L} . Then \mathfrak{L} is split by a Galois extension of degree at most $2[F:\phi]$.*

Proof. We will only give the argument when \mathfrak{L} is of type D_{4I} or D_{4III} . The other cases are similar. Let S be a finite subset of $\Omega(\phi)$ such that \mathfrak{L}_{ϕ_p} is a Jordan ${}_i\mathfrak{D}$ if $p \in \Omega(\phi) - S$. Without loss of generality we suppose that S contains all the real primes on ϕ . If $p \in S$, then \mathfrak{L}_{ϕ_p} is necessarily of type D_{4I} . By the local classification of D_4 's ([2] § 4), \mathfrak{L}_{ϕ_p} is split by a quadratic extension $K_{(p)}/\phi_p$. Let $K_{(p)}$ be a root field for $\lambda^2 + \alpha_p \in \phi[\lambda]$. By the approximation theorem,

since S consists of inequivalent primes, there exists an $\alpha \in \phi$ with each $|\alpha - \alpha_p|_p$ sufficiently small. Let K be a root field for $\lambda^2 + \alpha$ over ϕ and F the canonical D_{4I} -field extension of \mathfrak{L} . Note that $K_p = K_{(p)}$ for all $p \in S$.

If \mathfrak{L} is of type D_{4I} then it is easy to see that $(\mathfrak{L}_K)_{K_p}$ is split for all $p \in \Omega(K)$. Thus \mathfrak{L}_K is split. If \mathfrak{L} is of type D_{4III} then we claim that \mathfrak{L} is split by $P = K \otimes_{\phi} F$. Let $p \in \Omega(P)$. If $p | \phi \in \Omega(\phi) - S$, then \mathfrak{L}_{ϕ_p} is a Jordan D_4 . If p is complex, \mathfrak{L}_{ϕ_p} is clearly split whereas if p is discrete, \mathfrak{L}_{ϕ_p} is split by its canonical D_{4I} field extension ([2] § 4). In any event, $(\mathfrak{L}_P)_{P_p} = (\mathfrak{L}_{\phi_p})_{P_p} = ((\mathfrak{L}_{\phi_p})_{F_p})_{P_p}$ is split. If $p | \phi \in S$, then \mathfrak{L}_{ϕ_p} is split by $K_{(p)} = K_p \subset P_p$ so $(\mathfrak{L}_P)_{P_p}$ is split. Thus $(\mathfrak{L}_P)_{P_p}$ is split at every $p \in \Omega(P)$ and the corollary to Theorem 2 shows that \mathfrak{L}_P is split. Note that P is sixth degree cyclic.

Now let \mathfrak{L} be a Lie algebra of type D_{4III} over ϕ , with P/ϕ a cyclic sixth degree Galois splitting extension. P/ϕ contains a unique quadratic subfield E/ϕ . \mathfrak{L}_E is a D_{4III} split by P , so \mathfrak{L}_E is a Jordan D_4 . If $\mathfrak{L}_E = \mathfrak{D}(\mathfrak{Y}'/r)$, then since \mathfrak{Y}' is reduced and is split by P/E , a cubic extension, \mathfrak{Y}' is itself split. \mathfrak{f} of course is isomorphic to P . The isomorphism condition for Jordan D_4 's ([2] Th. II) implies that \mathfrak{L}_E is a Steinberg D_{4III} ([2] (10)).

Let $r \rightarrow p(r)$ be the anti-homomorphism of $g(P/\phi) = G$ onto A_3 determined by \mathfrak{L} , and choose s as a generator for G with $p(s) = (123)$. Let \mathfrak{C} be any Cayley algebra over ϕ , split by P , and let S be the s -semilinear automorphism of \mathfrak{C}_p which is one on \mathfrak{C} . Finally set $D_s = [(123), S]$ (cf. [2] (10)), and let $r \rightarrow \eta(r) \leftarrow C_r$ be the pre-cocycle of G in $\text{Aut}_{\phi}(\mathfrak{D}_P)$ corresponding to \mathfrak{L} . The preceding observation about \mathfrak{L}_E enables us to assume that $C_s^2 = D_s^2 \mu$, $\mu \in K$. By replacing C_s by $C_s \lambda$, for some suitable $\lambda \in K$, if necessary, we may assume that C_s and D_s^2 commute. This implies that $\mu^{s^2} = D_s^{-2} \mu D_s^2 = \mu$. If $C_s^6 = \delta \in K$, then $\delta = \mu^3$ and $\delta^s = C_s^{-1} \delta C_s = D_s^{-1} \delta D_s = \delta$ so $(\mu^3)^s = \mu^3$. Applying $\zeta(\cdot)$ to the relation $C_s^2 = D_s^2 \mu$ we obtain

$$(1) \quad \zeta(C_s)^s \zeta(C_s) = \mu^2.$$

But $\zeta(C_s)^s \zeta(C_s)$ is fixed under s since $\zeta(C_s)$ is fixed s^2 . Thus $(\mu^2)^s = \mu^2$. This, together with the previous relation shows that $\mu = \mu^s$.

For simplicity write $\zeta(C_s) = (\rho, p^{s^2}, \rho^{s^4}) \mu = (\beta, \beta^{s^2}, \beta^s)$, $\beta^{s^3} = \beta$. (1) is now equivalent to $\rho \rho^{s^3} = \beta^2$. Since $(\rho \beta^{-1})(\rho \beta^{-1})^{s^3} = 1$, $\rho = \lambda^{-1} \lambda^{s^3} \beta$ and $\rho \lambda^2 = \lambda \lambda^{s^3} \beta \in F$, the canonical D_{4I} -field extension of \mathfrak{L} . Replacing C_s by $\tilde{C}_s = C_s(\lambda, \lambda^{s^2} \lambda^{s^4})$ we again obtain $\zeta(\tilde{C}_s)^s = \zeta(\tilde{C}_s)$. But

$$\zeta(\tilde{C}_s) = (\lambda \lambda^{s^3} \beta, ((\lambda \lambda^{s^3} \beta)^s)^2, (\lambda \lambda^{s^3} \beta)^s)$$

and is fixed under s^3 . Thus $\zeta(\tilde{C}_s)^s = \zeta(\tilde{C}_s)$. We may affect a similar

alteration of C_s so that $\zeta(C_s) = (\alpha_1, \alpha_1^{s^2}, \alpha_1^s) = \alpha$ where $\alpha_1^{s^3} = \alpha_1$ and $\alpha_1 \alpha_1^s \alpha_1^{s^2} = 1$, i.e., \tilde{C}_s , in addition to the above is norm preserving. Calculating we see that $\tilde{C}_s^2 = D_s^2 \alpha$. Then $\tilde{C}_s^6 = \alpha^3$ and \mathfrak{L} is a Jordan D_4 if and only if $\alpha_1 \in N_{P|F}(P^*)$. Setting $\tilde{C}_s = D_s E$ we see that $D_s \alpha = E D_s E$. The simplest form of this equation occurs where E and D_s commute and we obtain $E^2 = \alpha$. Thus we are led to the following (possibly vacuous) construction.

Let \mathfrak{S} be a reduced exceptional central simple Jordan algebra over a field ϕ , P a cyclic sixth degree extension of ϕ and F a subfield of \mathfrak{S} isomorphic to the cubic subfield of P/ϕ . Then if there exists an $E \in GL(\mathfrak{S}/F)$ such that

- (i) $E \in GL(\mathfrak{S}_F/\{Pe_i\}_i)$,
- (ii) $\zeta(E) = (\alpha_1, \alpha_1^{s^2}, \alpha_1^s)$ where $\alpha_1 \notin N_{P|F}(P^*)$ and
- (iii) $E^2 = \zeta(E)$,

then the s -semilinear extension of E to \mathfrak{S}_P induces a pre-cocycle corresponding to a non-Jordan D_{4III} .

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Received January 17, 1967.

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