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**SOME PROPERTIES OF SEQUENCES, WITH AN
APPLICATION TO NONCONTINUABLE POWER SERIES**

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SOME PROPERTIES OF SEQUENCES,
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 NONCONTINUABLE POWER SERIES

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For a real sequence $f = \{f(n)\}$ and positive integer N , let F^N denote the sequence of N -tuples $\{(f(n+1), \dots, f(n+N))\}$. A functional equation method due to Kemperman is used to obtain a sufficient condition on s in order that s^N have an independent N -tuple among its cluster points. If a bounded s has the latter property, and if $g = rs$, where $r(n) \rightarrow \infty$ and $r(n+1)/r(n) \rightarrow 1$ as $n \rightarrow \infty$, then there is a subsequence S of the sequence of positive integers such that, for almost all real α , the restriction of αg^N to S is uniformly distributed (mod 1) in the N -cube.

Let F be an analytic function whose Maclaurin series has bounded coefficients $\{a_n\}$ which satisfy the additional requirement

$$\liminf_{M \rightarrow \infty} \inf_{0 \leq k < \infty} \sum_{n=k}^{k+M} |a_n| = \infty .$$

If $a_n = |a_n| \exp\{2\pi i f(n)\}$, then the density (mod 1) of f^N for each N is sufficient in order that F have the unit circle as a natural boundary. Hence, a metric result for noncontinuable series is obtained from the results for sequences.

1. **Notation.** For x real, let $((x)) = x - [x]$, and $e(x) = \exp(2\pi i x)$. h_1, \dots, h_N will denote an N -tuple of integers, not all of which are zero. The sequence of nonnegative integers will be denoted by Z , and subsequences of Z by S_1, S_2 , etc. For a real sequence f , we denote by Δf the sequence $\{f(n+1) - f(n)\}$ and

$$\Delta^{j+1}f = \Delta(\Delta^j f) , \quad (j = 1, 2, \dots)$$

2. **The property (PN).**

DEFINITION. A bounded sequence s of real numbers will be said to have *property (PN)* if there is an independent N -tuple among the cluster points of s^N . In other words, s has property (PN) if there is a subsequence S of Z such that for every N -tuple h_1, \dots, h_N of integers not all zero, there holds

$$(2.1) \quad \lim_{n \rightarrow \infty} |h_1 s(n+1) + \dots + h_N s(n+N)| > 0 , \quad (n \in S) .$$

We shall be interested in sequences s of the following form:

$$(2.2) \quad s(n) = \varphi(\psi(n)) , \quad (n \in Z) ,$$

where φ is a function of period 1 with at most a nowhere dense set of points of discontinuity, and ψ has the property (QN).

(QN) There exists a subsequence S_1 of Z such that

$$(2.3) \quad \begin{aligned} \text{(i)} \quad & \Delta^j \psi(n) \text{ converges (mod 1) for } n \rightarrow \infty , \\ & n \in S_1 \quad (j = 2, \dots, N) \\ \text{(ii)} \quad & \{((\psi(n)), (\Delta \psi(n))): n \in S_1\} \text{ is not nowhere dense.} \end{aligned}$$

THEOREM 2.1. *Let s be of the form (2.2), where φ and ψ have the properties listed above. Then either s has property (PN), or else φ agrees on some interval $I \subset [0, 1]$ with a polynomial of degree $N-2$ at most.*

Proof. Under the conditions on φ and ψ , it is possible to obtain a subsequence S_2 of S_1 and an open disk D in the plane such that

$$(2.4) \quad \begin{aligned} \text{(i)} \quad & \lim_{n \rightarrow \infty} \Delta^j \psi(n) = \tau_j \pmod{1}, \quad (n \in S_2), \quad (j = 2, \dots, N), \\ \text{(ii)} \quad & \{((\psi(n)), (\Delta \psi(n))): n \in S_2\} \text{ is dense in } D, \\ \text{(iii)} \quad & \text{for every } (\tau_0, \tau_1) \text{ in } D, \text{ and} \\ & \text{every } p, 1 \leq p \leq N, \text{ the point} \end{aligned}$$

$$\tau_0 + p\tau_1 + \sum_{j=2}^p \binom{p}{j} \tau_j$$

is a point of continuity for φ .

For each (τ_0, τ_1) in D , a subsequence $S_3 = S_3(\tau_0, \tau_1)$ of S_2 can be chosen so that the corresponding subsequence of (2.4 (ii)) converges to (τ_0, τ_1) . In this case, as $n \rightarrow \infty$, $n \in S_3$, one has for every h_1, \dots, h_N ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{p=1}^N h_p s(n+p) &= \lim_{n \rightarrow \infty} \sum_{p=1}^N h_p \varphi(\psi(n) \\ &+ p\Delta \psi(n) + \sum_{j=2}^p \binom{p}{j} \Delta^j \psi(n)) \end{aligned}$$

so that

$$(2.5) \quad \lim_{n \rightarrow \infty} \sum_{p=1}^N h_p s(n+p) = \sum_{p=1}^N h_p \varphi\left(\tau_0 + p\tau_1 + \sum_{j=2}^p \binom{p}{j} \tau_j\right), \quad (n \in S_3) .$$

Suppose now that s does not have property (PN). Then for each (τ_0, τ_1) in D , there is an N -tuple h_1, \dots, h_N such that the right hand member of (2.5) is zero. Hence D is a countable union of closed sets

$$F = F(h_1, \dots, h_N) = \{(\tau_0, \tau_1) \in D: (2.5) \text{ vanishes}\} .$$

Some F , then, must contain an open subdisk D_1 , with center

(τ'_0, τ'_1) . That is, there exists an N -tuple h_1, \dots, h_N of integers not all zero with the property that for all sufficiently small positive h and k ,

$$\sum_{p=1}^N h_p \varphi \left(h + pk + \tau'_0 + p\tau'_1 + \sum_{j=2}^p \binom{p}{j} \tau_j \right) = 0 .$$

The assertion of the theorem follows upon taking

$$\varphi_p(x) = h_p \varphi \left(x + \tau'_0 + p\tau'_1 + \sum_{j=2}^p \binom{p}{j} \tau_j \right)$$

in the following lemma, a weak version of one due to Kemperman [4, p. 41]. The proof is included for completeness.

LEMMA. Let $a > 0$, and let $\varphi_1, \dots, \varphi_N$ be real functions, with φ_j defined and continuous on $I_j = (-(j+1)a, (j+1)a)$, ($j = 1, \dots, N$). Suppose that for all x, y in $(-a, a)$, there holds

$$(2.6) \quad \sum_{j=1}^N \varphi_j(x + jy) = 0 .$$

Then φ_j is equal on I_j to a polynomial of degree $N-2$ at most.

Proof. We may suppose that $N \geq 2$ (the case $N = 1$ is trivial), and that the lemma holds for $N - 1$. Let $0 < b < a$, and let $I'_j = (-(j+1)b, (j+1)b)$.

Next, we choose and keep fixed a number h , $0 < h < \min(b, a - b)$. For this h , and $j = 1, \dots, N$, let

$$\tilde{\varphi}_j(x) = \varphi_j(x + (1 - j/N)h) - \varphi_j(x), \quad (x \in I'_j) .$$

We note that each $\tilde{\varphi}_j$ is continuous, and $\tilde{\varphi}_N \equiv 0$. Moreover, if x, y are in $(-b, b)$, then $x, y, x + h$, and $y - h/N$ are in $(-a, a)$.

Thus, for all x, y in $(-b, b)$, we have

$$\sum_{j=1}^{N-1} \tilde{\varphi}_j(x + jy) = \sum_{j=1}^N \varphi_j(x + h + j(y - h/N)) - \sum_{j=1}^N \varphi_j(x + jy) = 0 .$$

The induction hypothesis implies that, for $j = 1, \dots, N - 1$, $\tilde{\varphi}_j$ is a polynomial of degree $N - 3$ at most on I'_j . Hence φ_j is, on I'_j , the sum of a polynomial of degree $N - 2$ at most and a function of period $(1 - j/N)h$. But such a representation is given for every sufficiently small positive h , which, with the continuity of φ_j , implies that φ_j is a polynomial of degree $N - 2$ at most on I'_j , ($1 \leq j \leq N - 1$). From the arbitrariness of b , φ_j is such a polynomial on I_j . Finally, (2.6) shows that φ_N is also such a polynomial on I_N .

In a previous paper [1], results of *v. d. Corput* were used to

obtain various sufficient conditions on a real sequence ψ in order that ψ satisfy condition (I):

(I) *There exists a sequence S such that $\lim \Delta^j \psi(n)$ ($n \in S$) exists for all $j \geq r$, while $\{(\psi(n), \Delta \psi(n), \dots, \Delta^{r-1} \psi(n)): n \in S\}$ is uniformly distributed (mod 1) in the r -dimensional unit cube.*

(I) clearly implies that ψ has property (QN) for every $N \geq 2$. The reader is referred to the paper for details and proofs.

3. A metric result for uniform distribution in the N -cube.

THEOREM 3.1 *Let $g = \{g(n): n \in Z\}$ be a sequence of real numbers. Let there exist a subsequence S_0 of Z such that, for every N -tuple h_1, \dots, h_N of integers not all zero there holds*

$$(3.1) \quad \lim \left| \sum_{p=1}^N h_p g(n+p) \right| = \infty, \quad \text{as } n \rightarrow \infty, \quad n \in S_0.$$

Then there exists a subsequence S of S_0 such that, for almost all real α , the sequence $(\alpha g^N) | S$ is uniformly distributed (mod 1) in the N -cube.

Proof. Let the set of all such N -tuples be ordered, with, say, h'_1, \dots, h'_N as the first. Let a subsequence $S_1 \subset S_0$ be taken such that

$$\sum_{p=1}^N h'_p \{g(n+p) - g(m+p)\}$$

is either greater than 1 for every n, m in S_1 , with $n > m$, or else is less than -1 for every such n and m . Successively extracting subsequences $S_1 \supset S_2 \supset \dots$ in this way, and then using a diagonal procedure, one finally obtains a sequence S such that, for every N -tuple h_1, \dots, h_N , there is an $m_0 = m_0(h_1, \dots, h_N)$ such that one has either

$$(3.2) \quad \sum_{p=1}^N h_p \{g(n+p) - g(m+p)\} \geq 1$$

for all n and m in S with $n > m \geq m_0$
or else

$$(3.3) \quad \sum_{p=1}^N h_p \{g(n+p) - g(m+p)\} \leq -1$$

for all such n and m .

By a well-known result of Weyl [6, p. 348], either condition (3.2) or (3.3) implies that, for almost all real α , the sequence

$$(3.4) \quad \alpha \sum_{p=1}^N h_p g(n+p) \quad (n \in S)$$

is uniformly distributed (mod 1). There being only countably many N -tuples, it follows that, for almost all α , (3.4) is uniformly distributed (mod 1) for every N -tuple h_1, \dots, h_N . But this shows [2, p. 66] that for almost all α the sequence $(\alpha g^N) \mid S$ is uniformly distributed (mod 1) in the N -cube.

It is easy to see that if $\theta > 1$ is a transcendental number and $g(n) = \theta^n$, then Theorem 3.1 is applicable. The next result shows the less obvious fact that Theorem 3.1 also applies if, for instance, $g(n) = n^3 \log n \sin n^2$.

THEOREM 3.2. *Let $g = \{g(n): n \in Z\}$ be of the form*

$$(3.5) \quad g(n) = r(n)s(n), \quad n \in Z,$$

where s has property (PN), while

$$(3.6) \quad \lim r(n) = \infty, \quad \lim (r(n+1)/r(n)) = 1.$$

Then there is a subsequence S_0 of Z such that (3.1) holds for every N -tuple h_1, \dots, h_N of integers not all zero.

Proof. For $p = 1, 2, \dots, N$, it follows from (3.6) that

$$r(n+p) = r(n)(1 + o(1)), \quad \text{as } n \rightarrow \infty.$$

Therefore we have

$$(3.7) \quad g(n+p) = r(n)s(n+p)(1 + o(1)), \quad \text{as } n \rightarrow \infty, \quad p = 1, \dots, N.$$

Since s has property (PN), there exists a subsequence S_0 of Z such that

$$(2.1) \quad \lim_{n \rightarrow \infty} |h_1 s(n+1) + \dots + h_N s(n+N)| > 0, \quad (n \in S_0)$$

for all N -tuples h_1, \dots, h_N of integers not all zero. But (3.6), (3.7), and (2.1) imply (3.1).

4. An application to noncontinuable power series. Perry [5] has proved that, for every real sequence $f = \{f(n): n \in Z\}$, there exists a sequence of moduli $\{|a_n|: n \in Z\}$ such that the power series

$$(4.1) \quad \sum_{n=0}^{\infty} |a_n| e(f(n))z^n$$

has radius of convergence 1 and the analytic function it represents can be continued analytically across a semicircle of the unit circle. However, if the additional requirements

$$(4.2) \quad |a_n| = o(1) \quad \text{as } n \rightarrow \infty$$

and

$$(4.3) \quad \liminf_{N \rightarrow \infty} \inf_{0 \leq k < \infty} \sum_{n=k+1}^{k+N} |a_n| = \infty$$

are imposed, then there are conditions on f sufficient that (4.1) represent a function with $|z| = 1$ as its natural boundary. Some such conditions were given in [1]. Theorem 4 gives a metric result in this direction.

THEOREM 4. *Let $\{|a_n|: n \in \mathbb{Z}\}$ satisfy (4.2) and (4.3). Let g be a real sequence which, for each N , satisfies the hypothesis of Theorem 3.1. For each real α , let*

$$(4.4) \quad F_\alpha(z) = \sum_{n=0}^{\infty} |a_n| e(\alpha g(n)) z^n, \quad |z| < 1.$$

Then the set of α for which F_α can be continued across an arc of the unit circle has measure zero.

Example. $\sum e(\alpha n \sin n^2) z^n$ has $|z| = 1$ as its natural boundary for almost all α .

For $N = 2, 3, \dots$, let A_N be the set of those real α for which αg^N is dense (mod 1) in the unit N -cube.

By Theorem 3.1, A_N contains almost all α , and it follows that almost all α are in A_N for every N . For each such α , and each $z_0 = e(\theta_0)$, there holds

$$(4.5) \quad \limsup_{k \rightarrow \infty} \left| \sum_{k+1}^{k+N} a_n e(\alpha g(n) + n\theta_0) \right| \geq \liminf_{k \rightarrow \infty} \sum_{k+1}^{k+N} |a_n|.$$

In view of (4.3), (4.5) shows that the partial sums of the series in (4.4) are unbounded at z_0 . By (4.2) and a well-known theorem of Fatou [3, p. 391], it follows that z_0 is a singularity for F .

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