SOME PROPERTIES OF SEQUENCES, WITH AN APPLICATION TO NONCONTINUABLE POWER SERIES

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For a real sequence \( f = \{f(n)\} \) and positive integer \( N \), let \( F^N \) denote the sequence of \( N \)-tuples \( \{(f(n + 1), \cdots, f(n+N))\} \).

A functional equation method due to Kemperman is used to obtain a sufficient condition on \( s \) in order that \( s^N \) have an independent \( N \)-tuple among its cluster points. If a bounded \( s \) has the latter property, and if \( g = rs \), where \( r(n) \to \infty \) and \( r(n + 1)/r(n) \to 1 \) as \( n \to \infty \), then there is a subsequence \( S \) of the sequence of positive integers such that, for almost all real \( \alpha \), the restriction of \( ag^N \) to \( S \) is uniformly distributed (mod 1) in the \( N \)-cube.

Let \( F \) be an analytic function whose Maclaurin series has bounded coefficients \( \{a_n\} \) which satisfy the additional requirement

\[
\lim \inf_{M \to \infty} \frac{1}{M} \sum_{n=M}^{M+k} |a_n| = \infty.
\]

If \( \alpha_n = |a_n| \exp\{2\pi if(n)\} \), then the density (mod 1) of \( f^N \) for each \( N \) is sufficient in order that \( F \) have the unit circle as a natural boundary. Hence, a metric result for noncontinuable series is obtained from the results for sequences.

1. Notation. For \( x \) real, let \( ((x)) = x - [x] \), and \( e(x) = \exp(2\pi ix) \).

\( h_1, \cdots, h_N \) will denote an \( N \)-tuple of integers, not all of which are zero. The sequence of nonnegative integers will be denoted by \( Z \), and subsequences of \( Z \) by \( S_1, S_2, \) etc.

For a real sequence \( f \), we denote by \( \Delta f \) the sequence \( \{f(n + 1) - f(n)\} \) and

\[
\Delta^{j+1}f = \Delta(\Delta^jf), \quad (j = 1, 2, \cdots)
\]

2. The property (PN).

Definition. A bounded sequence \( s \) of real numbers will be said to have property (PN) if there is an independent \( N \)-tuple among the cluster points of \( s^N \). In other words, \( s \) has property (PN) if there is a subsequence \( S \) of \( Z \) such that for every \( N \)-tuple \( h_1, \cdots, h_N \) of integers not all zero, there holds

\[
\lim_{n \to \infty} |h_1s(n + 1) + \cdots + h_Ns(n + N)| > 0, \quad (n \in S).
\]

We shall be interested in sequences \( s \) of the following form:
(2.2) \[ s(n) = \varphi(\psi(n)), \quad (n \in \mathbb{Z}), \]
where \( \varphi \) is a function of period 1 with at most a nowhere dense set of points of discontinuity, and \( \psi \) has the property (QN).

(QN) There exists a subsequence \( S_i \) of \( Z \) such that

\[
(2.3) \quad \begin{align*}
& (i) \quad \Delta^j \psi(n) \text{ converges (mod 1) for } n \to \infty, \\
& \quad \quad n \in S_i \quad (j = 2, \ldots, N) \\
& (ii) \quad \{((\psi(n)), ((\Delta \psi(n))): n \in S_i\} \text{ is not nowhere dense.}
\end{align*}
\]

**Theorem 2.1.** Let \( s \) be of the form (2.2), where \( \varphi \) and \( \psi \) have the properties listed above. Then either \( s \) has property (PN), or else \( \varphi \) agrees on some interval \( I \subset [0, 1] \) with a polynomial of degree \( N-2 \) at most.

**Proof.** Under the conditions on \( \varphi \) and \( \psi \), it is possible to obtain a subsequence \( S_2 \) of \( S_1 \) and an open disk \( D \) in the plane such that

\[
(2.4) \quad \begin{align*}
& (i) \quad \lim_{n \to \infty} \Delta^i \psi(n) = \tau_j \quad (\text{mod 1}), \quad (n \in S_2), \quad (j = 2, \ldots, N), \\
& (ii) \quad \{((\psi(n)), ((\Delta \psi(n))): n \in S_i\} \text{ is dense in } D, \\
& (iii) \quad \text{for every } (\tau_0, \tau_1) \text{ in } D, \text{ and every } p, 1 \leq p \leq N, \text{ the point}
\end{align*}
\]

\[
\tau_0 + p\tau_1 + \sum_{j=1}^{N} \left( \begin{array}{c} p \\ j \end{array} \right) \tau_j
\]

is a point of continuity for \( \varphi \).

For each \((\tau_0, \tau_1)\) in \( D \), a subsequence \( S_3 = S_2(\tau_0, \tau_1) \) of \( S_2 \) can be chosen so that the corresponding subsequence of (2.4 (ii)) converges to \((\tau_0, \tau_1)\). In this case, as \( n \to \infty \), \( n \in S_3 \), one has for every \( h_1, \ldots, h_N \),

\[
\lim_{n \to \infty} \sum_{p=1}^{N} h_p s(n + p) = \lim_{n \to \infty} \sum_{p=1}^{N} h_p \varphi(\psi(n)) + p\Delta \psi(n) + \sum_{j=1}^{N} \left( \begin{array}{c} p \\ j \end{array} \right) \Delta^j \psi(n))
\]

so that

\[
(2.5) \quad \lim_{n \to \infty} \sum_{p=1}^{N} h_p s(n + p) = \sum_{p=1}^{N} h_p \varphi(\tau_0 + p\tau_1 + \sum_{j=1}^{N} \left( \begin{array}{c} p \\ j \end{array} \right) \tau_j), \quad (n \in S_3).
\]

Suppose now that \( s \) does not have property (PN). Then for each \((\tau_0, \tau_1)\) in \( D \), there is an \( N \)-tuple \( h_1, \ldots, h_N \) such that the right hand member of (2.5) is zero. Hence \( D \) is a countable union of closed sets

\[
F = F(h_1, \ldots, h_N) = \{(\tau_0, \tau_1) \in D: (2.5) \text{ vanishes}\}.
\]

Some \( F \), then, must contain an open subdisk \( D_i \), with center
That is, there exists an \( N \)-tuple \( h_1, \ldots, h_N \) of integers not all zero with the property that for all sufficiently small positive \( h \) and \( k \),

\[
\sum_{p=1}^N h_p \phi \left( h + pk + \tau'_1 + p\tau'_2 + \sum_{j=3}^p \left( \frac{p}{j} \right) \tau_j \right) = 0.
\]

The assertion of the theorem follows upon taking

\[
\phi_p(x) = h_p \phi \left( x + \tau'_1 + p\tau'_2 + \sum_{j=3}^p \left( \frac{p}{j} \right) \tau_j \right)
\]

in the following lemma, a weak version of one due to Kemperman [4, p. 41]. The proof is included for completeness.

**Lemma.** Let \( \alpha > 0 \), and let \( \phi_1, \ldots, \phi_N \) be real functions, with \( \phi_j \) defined and continuous on \( I_j = (- (j + 1)a, (j + 1)a), (j = 1, \ldots, N) \). Suppose that for all \( x, y \) in \( (- \alpha, \alpha) \), there holds

\[
\sum_{j=1}^N \phi_j(x + jy) = 0.
\]

Then \( \phi_j \) is equal on \( I_j \) to a polynomial of degree \( N - 2 \) at most.

**Proof.** We may suppose that \( N \geq 2 \) (the case \( N = 1 \) is trivial), and that the lemma holds for \( N - 1 \). Let \( 0 < b < a \), and let \( I'_j = (- (j + 1)b, (j + 1)b) \).

Next, we choose and keep fixed a number \( h, 0 < h < \min (b, a - b) \).

For this \( h \), and \( j = 1, \ldots, N \), let

\[
\tilde{\phi}_j(x) = \phi_j(x + (1 - j/N)h) - \phi_j(x), \quad (x \in I'_j).
\]

We note that each \( \tilde{\phi}_j \) is continuous, and \( \tilde{\phi}_N = 0 \). Moreover, if \( x, y \) are in \( (-b, b) \), then \( x, y, x + h, \) and \( y - h/N \) are in \( (-a, a) \).

Thus, for all \( x, y \) in \( (-b, b) \), we have

\[
\sum_{j=1}^{N-1} \tilde{\phi}_j(x + jy) = \sum_{j=1}^N \phi_j(x + h + j(y - h/N)) - \sum_{j=1}^N \phi_j(x + jy) = 0.
\]

The induction hypothesis implies that, for \( j = 1, \ldots, N - 1 \), \( \tilde{\phi}_j \) is a polynomial of degree \( N - 3 \) at most on \( I'_j \). Hence \( \phi_j \) is, on \( I'_j \), the sum of a polynomial of degree \( N - 2 \) at most and a function of period \( (1 - j/N)h \). But such a representation is given for every sufficiently small positive \( h \), which, with the continuity of \( \phi_j \), implies that \( \phi_j \) is a polynomial of degree \( N - 2 \) at most on \( I'_j, (1 \leq j \leq N - 1) \). From the arbitrariness of \( b \), \( \phi_j \) is such a polynomial on \( I_j \). Finally, (2.6) shows that \( \phi_N \) is also such a polynomial on \( I_N \).

In a previous paper [1], results of v. d. Corput were used to
obtain various sufficient conditions on a real sequence $\psi$ in order that $\psi$ satisfy condition (I):

(I) There exists a sequence $S$ such that $\lim \Delta^j \psi(n) \ (n \in S)$ exists for all $j \geq r$, while $\{\psi(n), \Delta \psi(n), \cdots, \Delta^{r-1} \psi(n) : n \in S\}$ is uniformly distributed (mod 1) in the $r$-dimensional unit cube.

(I) clearly implies that $\psi$ has property (QN) for every $N \geq 2$.

The reader is referred to the paper for details and proofs.

3. A metric result for uniform distribution in the $N$-cube.

**Theorem 3.1** Let $g = \{g(n) : n \in \mathbb{Z}\}$ be a sequence of real numbers. Let there exist a subsequence $S_0$ of $\mathbb{Z}$ such that, for every $N$-tuple $h_1, \cdots, h_N$ of integers not all zero there holds

$$\lim_{n \to \infty} \left| \sum_{p=1}^{N} h_p g(n + p) \right| = \infty, \quad n \in S_0. \quad (3.1)$$

Then there exists a subsequence $S$ of $S_0$ such that, for almost all real $\alpha$, the sequence $(\alpha g^n) \mid S$ is uniformly distributed (mod 1) in the $N$-cube.

**Proof.** Let the set of all such $N$-tuples be ordered, with, say, $h'_1, \cdots, h'_N$ as the first. Let a subsequence $S_1 \subset S_0$ be taken such that

$$\sum_{p=1}^{N} h'_p [g(n + p) - g(m + p)]$$

is either greater than 1 for every $n, m$ in $S_1$, with $n > m$, or else is less than $-1$ for every such $n$ and $m$. Successively extracting subsequences $S_1 \supset S_2 \supset \cdots$ in this way, and then using a diagonal procedure, one finally obtains a sequence $S$ such that, for every $N$-tuple $h_1, \cdots, h_N$, there is an $m_0 = m_0(h_1, \cdots, h_N)$ such that one has either

$$\sum_{p=1}^{N} h_p [g(n + p) - g(m + p)] \geq 1 \quad (3.2)$$

for all $n$ and $m$ in $S$ with $n > m \geq m_0$ or else

$$\sum_{p=1}^{N} h_p [g(n + p) - g(m + p)] \leq -1 \quad (3.3)$$

for all such $n$ and $m$.

By a well-known result of Weyl [6, p. 348], either condition (3.2) or (3.3) implies that, for almost all real $\alpha$, the sequence

$$\alpha \sum_{p=1}^{N} h_p g(n + p) \quad (n \in S) \quad (3.4)$$
is uniformly distributed (mod 1). There being only countably many
N-tuples, it follows that, for almost all \( \alpha \), (3.4) is uniformly distributed
(mod 1) for every N-tuple \( h_1, \ldots, h_N \). But this shows [2, p. 66] that
for almost all \( \alpha \) the sequence \((\alpha g^n) / S\) is uniformly distributed (mod 1)
in the N-cube.

It is easy to see that if \( \theta > 1 \) is a transcendental number and
\( g(n) = \theta^n \), then Theorem 3.1 is applicable. The next result shows the
less obvious fact that Theorem 3.1 also applies if, for instance, \( g(n) = n^3 \log n \sin n^2 \).

**Theorem 3.2.** Let \( g = \{g(n) : n \in \mathbb{Z}\} \) be of the form
\[
(3.5) \quad g(n) = r(n)s(n), \quad n \in \mathbb{Z},
\]
where \( s \) has property (PN), while
\[
(3.6) \quad \lim r(n) = \infty, \quad \lim (r(n + 1)/r(n)) = 1.
\]
Then there is a subsequence \( S_0 \) of \( \mathbb{Z} \) such that (3.1) holds for every
N-tuple \( h_1, \ldots, h_N \) of integers not all zero.

**Proof.** For \( p = 1, 2, \ldots, N \), it follows from (3.6) that
\[
r(n + p) = r(n)(1 + o(1)), \quad \text{as } n \to \infty.
\]
Therefore we have
\[
(3.7) \quad g(n + p) = r(n)s(n + p)(1 + o(1)), \quad \text{as } n \to \infty, \quad p = 1, \ldots, N.
\]
Since \( s \) has property (PN), there exists a subsequence \( S_0 \) of \( \mathbb{Z} \) such that
\[
(2.1) \quad \lim_{n \to \infty} |h_1s(n + 1) + \cdots + h_Ns(n + N)| > 0, \quad (n \in S_0)
\]
for all N-tuples \( h_1, \ldots, h_N \) of integers not all zero. But (3.6), (3.7),
and (2.1) imply (3.1).

has proved that, for every real sequence \( f = \{f(n) : n \in \mathbb{Z}\} \), there exists
a sequence of moduli \( \{|a_n| : n \in \mathbb{Z}\} \) such that the power series
\[
(4.1) \quad \sum_{n=0}^{\infty} |a_n| e(f(n))z^n
\]
has radius of convergence 1 and the analytic function it represents
can be continued analytically across a semicircle of the unit circle.
However, if the additional requirements
\[
(4.2) \quad |a_n| = 0(1) \quad \text{as } n \to \infty
\]
and

\begin{equation}
(4.3) \quad \liminf_{N \to \infty} \sum_{n=k+1}^{k+N} |a_n| = \infty
\end{equation}

are imposed, then there are conditions on \( f \) sufficient that (4.1) represent a function with \(|z| = 1\) as its natural boundary. Some such conditions were given in [1]. Theorem 4 gives a metric result in this direction.

**Theorem 4.** Let \( \{ |a_n| : n \in \mathbb{Z} \} \) satisfy (4.2) and (4.3). Let \( g \) be a real sequence which, for each \( N \), satisfies the hypothesis of Theorem 3.1. For each real \( \alpha \), let

\begin{equation}
(4.4) \quad F_\alpha(z) = \sum_{n=0}^{\infty} |a_n| e(\alpha g(n))z^n, \quad |z| < 1.
\end{equation}

Then the set of \( \alpha \) for which \( F_\alpha \) can be continued across an arc of the unit circle has measure zero.

**Example.** \( \sum e(\alpha n \sin n^2) z^n \) has \(|z| = 1\) as its natural boundary for almost all \( \alpha \).

For \( N = 2, 3, \cdots \), let \( A_N \) be the set of those real \( \alpha \) for which \( \alpha g^N \) is dense (mod 1) in the unit \( N \)-cube.

By Theorem 3.1, \( A_N \) contains almost all \( \alpha \), and it follows that almost all \( \alpha \) are in \( A_N \) for every \( N \). For each such \( \alpha \), and each \( z_0 = e(\theta_0) \), there holds

\begin{equation}
(4.5) \quad \limsup_{k \to \infty} \left| \sum_{n=k+1}^{k+N} a_n e(\alpha g(n) + n\theta_0) \right| \geq \liminf_{k \to \infty} \sum_{n=k+1}^{k+N} |a_n|.
\end{equation}

In view of (4.3), (4.5) shows that the partial sums of the series in (4.4) are unbounded at \( z_0 \). By (4.2) and a well-known theorem of Fatou [3, p. 391], it follows that \( z_0 \) is a singularity for \( F \).

**References**


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The Ohio State University
Columbus, Ohio
Harry P. Allen, *Lie algebras of type $D_4$ over algebraic number fields* ........ 1
Charles Ballantine, *Products of positive definite matrices. II* ................. 7
David W. Boyd, *The spectral radius of averaging operators* ....................... 19
William Howard Caldwell, *Hypercyclic rings* ........................................ 29
Francis William Carroll, *Some properties of sequences, with an application to noncontinuable power series* ......................................................... 45
David Fleming Dawson, *Matrix summability over certain classes of sequences ordered with respect to rate of convergence* ......................... 51
D. W. Dubois, *Second note on David Harrison’s theory of preprimes* ........... 57
Edgar Earle Enochs, *A note on quasi-Frobenius rings* .............................. 69
Ronald J. Ensey, *Isomorphism invariants for Abelian groups modulo bounded groups* ................................................................. 71
Ronald Owen Fulp, *Generalized semigroup kernels* .................................... 93
Bernard Robert Kripke and Richard Bruce Holmes, *Interposition and approximation* ................................................................. 103
Jack W. Macki and James Sai-Wing Wong, *Oscillation of solutions to second-order nonlinear differential equations* ................................. 111
Lothrop Mittenthal, *Operator valued analytic functions and generalizations of spectral theory* ................................................................. 119
T. S. Motzkin and J. L. Walsh, *A persistent local maximum of the $p$th power deviation on an interval, $p < 1$* ............................................. 133
Jerome L. Paul, *Sequences of homeomorphisms which converge to homeomorphisms* ................................................................. 143
Maxwell Alexander Rosenlicht, *Liouville’s theorem on functions with elementary integrals* ................................................................. 153
Joseph Goeffrey Rosenstein, *Initial segments of degrees* ............................ 163
H. Subramanian, *Ideal neighbourhoods in a ring* ..................................... 173
Dalton Tarwater, *Galois cohomology of abelian groups* ............................ 177
James Patrick Williams, *Schwarz norms for operators* ................................ 181
Raymond Y. T. Wong, *A wild Cantor set in the Hilbert cube* ....................... 189