SOME PROPERTIES OF SEQUENCES, WITH AN APPLICATION TO NONCONTINUABLE POWER SERIES

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For a real sequence \( f = \{f(n)\} \) and positive integer \( N \), let \( F^N \) denote the sequence of \( N \)-tuples \( \{(f(n+1), \ldots, f(n+N))\} \). A functional equation method due to Kemperman is used to obtain a sufficient condition on \( s \) in order that \( s^N \) have an independent \( N \)-tuple among its cluster points. If a bounded \( s \) has the latter property, and if \( g = rs \), where \( r(n) \to \infty \) and \( r(n+1)/r(n) \to 1 \) as \( n \to \infty \), then there is a subsequence \( S \) of the sequence of positive integers such that, for almost all real \( \alpha \), the restriction of \( \alpha g^N \) to \( S \) is uniformly distributed (mod 1) in the \( N \)-cube.

Let \( F \) be an analytic function whose Maclaurin series has bounded coefficients \( \{a_n\} \) which satisfy the additional requirement

\[
\lim \inf_{N \to \infty} \sum_{k=0}^{k=M} |a_n| = \infty .
\]

If \( a_n = |a_n| \exp\{2\pi if(n)\} \), then the density (mod 1) of \( f^N \) for each \( N \) is sufficient in order that \( F \) have the unit circle as a natural boundary. Hence, a metric result for noncontinuable series is obtained from the results for sequences.

1. Notation. For \( x \) real, let \( ((x)) = x - [x] \), and \( e(x) = \exp(2\pi ix) \). \( h_1, \ldots, h_N \) will denote an \( N \)-tuple of integers, not all of which are zero. The sequence of nonnegative integers will be denoted by \( Z \), and subsequences of \( Z \) by \( S_1, S_2, \ldots \). For a real sequence \( f \), we denote by \( \Delta f \) the sequence \( \{f(n+1) - f(n)\} \) and

\[
\Delta^{j+1} f = \Delta(\Delta^j f) , \ (j = 1, 2, \ldots)
\]

2. The property (PN).

DEFINITION. A bounded sequence \( s \) of real numbers will be said to have property (PN) if there is an independent \( N \)-tuple among the cluster points of \( s^N \). In other words, \( s \) has property (PN) if there is a subsequence \( S \) of \( Z \) such that for every \( N \)-tuple \( h_1, \ldots, h_N \) of integers not all zero, there holds

\[
\lim_{n \to \infty} |h_1s(n + 1) + \cdots + h_Ns(n + N)| > 0 , \quad (n \in S).
\]

We shall be interested in sequences \( s \) of the following form:
(2.2) \[ s(n) = \varphi(\psi(n)), \quad (n \in \mathbb{Z}), \]
where \( \varphi \) is a function of period 1 with at most a nowhere dense set of points of discontinuity, and \( \psi \) has the property (QN).

(QN) There exists a subsequence \( S_1 \) of \( \mathbb{Z} \) such that

(2.3) \( \Delta^j \psi(n) \) converges (mod 1) for \( n \to \infty \),
\[ n \in S_1, \quad (j = 2, \ldots, N) \]

(i) \( \{(((\psi(n)), (\Delta \psi(n))): n \in S_1\} \) is not nowhere dense.

**Theorem 2.1.** Let \( s \) be of the form (2.2), where \( \varphi \) and \( \psi \) have the properties listed above. Then either \( s \) has property (PN), or else \( \varphi \) agrees on some interval \( I \subset [0, 1] \) with a polynomial of degree \( N-2 \) at most.

**Proof.** Under the conditions on \( \varphi \) and \( \psi \), it is possible to obtain a subsequence \( S_2 \) of \( S_1 \) and an open disk \( D \) in the plane such that

(2.4) \( \lim_{n \to \infty} \Delta^j \psi(n) = \tau_j \) \( \mod 1 \), \( (n \in S_2), \quad (j = 2, \ldots, N), \)

(i) \( \{(((\psi(n)), (\Delta \psi(n))): n \in S_2\} \) is dense in \( D \),

(ii) for every \( (\tau_0, \tau_1) \) in \( D \), and every \( p, 1 \leq p \leq N \), the point
\[ \tau_0 + p\tau_1 + \sum_{j=2}^{p} \left( \frac{p}{j} \right) \tau_j \]
is a point of continuity for \( \varphi \).

For each \( (\tau_0, \tau_1) \) in \( D \), a subsequence \( S_3 = S_3(\tau_0, \tau_1) \) of \( S_2 \) can be chosen so that the corresponding subsequence of (2.4 (ii)) converges to \( (\tau_0, \tau_1) \). In this case, as \( n \to \infty \), \( n \in S_3 \), one has for every \( h_1, \ldots, h_N \),

\[ \lim_{n \to \infty} \sum_{p=1}^{N} h_p s(n + p) = \lim_{n \to \infty} \sum_{p=1}^{N} h_p \varphi(\psi(n)) \]
\[ + p \Delta \psi(n) + \sum_{j=2}^{p} \left( \frac{p}{j} \right) \Delta^j \psi(n) \]
so that

(2.5) \[ \lim_{n \to \infty} \sum_{p=1}^{N} h_p s(n + p) = \sum_{p=1}^{N} h_p \varphi(\tau_0 + p\tau_1 + \sum_{j=2}^{p} \left( \frac{p}{j} \right) \tau_j), \quad (n \in S_3). \]

Suppose now that \( s \) does not have property (PN). Then for each \( (\tau_0, \tau_1) \) in \( D \), there is an \( N \)-tuple \( h_1, \ldots, h_N \) such that the right hand member of (2.5) is zero. Hence \( D \) is a countable union of closed sets

\[ F = F(h_1, \ldots, h_N) = \{(\tau_0, \tau_1) \in D: (2.5) \text{ vanishes}\}. \]

Some \( F \), then, must contain an open subdisk \( D_1 \), with center
That is, there exists an \( N \)-tuple \( h_1, \ldots, h_N \) of integers not all zero with the property that for all sufficiently small positive \( h \) and \( k \),
\[
\sum_{p=1}^{N} h_p \varphi\left(h + pk + \tau'_0 + p\tau'_1 + \sum_{j=2}^{p} \left( \frac{p}{j} \right)\tau_j \right) = 0.
\]

The assertion of the theorem follows upon taking
\[
\varphi_p(x) = h_p \varphi\left(x + \tau'_0 + p\tau'_1 + \sum_{j=2}^{p} \left( \frac{p}{j} \right)\tau_j \right)
\]
in the following lemma, a weak version of one due to Kemperman [4, p. 41]. The proof is included for completeness.

**Lemma.** Let \( a > 0 \), and let \( \varphi_1, \ldots, \varphi_N \) be real functions, with \( \varphi_j \) defined and continuous on \( I_j = ((j+1)a, (j+1)a), (j = 1, \ldots, N) \). Suppose that for all \( x, y \) in \((-a, a)\), there holds
\[
(2.6) \sum_{j=1}^{N} \varphi_j(x + jy) = 0.
\]

Then \( \varphi_j \) is equal on \( I_j \) to a polynomial of degree \( N-2 \) at most.

**Proof.** We may suppose that \( N \geq 2 \) (the case \( N = 1 \) is trivial), and that the lemma holds for \( N - 1 \). Let \( 0 < b < a \), and let \( I'_j = \left(-(j+1)b, (j+1)b\right) \).

Next, we choose and keep fixed a number \( h, 0 < h < \min(b, a-b) \). For this \( h \), and \( j = 1, \ldots, N \), let
\[
\tilde{\varphi}_j(x) = \varphi_j(x + (1 - j/N)h) - \varphi_j(x), \quad (x \in I'_j).
\]

We note that each \( \tilde{\varphi}_j \) is continuous, and \( \tilde{\varphi}_N \equiv 0 \). Moreover, if \( x, y \) are in \((-b, b)\), then \( x, y, x + h \), and \( y - h/N \) are in \((-a, a)\).

Thus, for all \( x, y \) in \((-b, b)\), we have
\[
\sum_{j=1}^{N-1} \tilde{\varphi}_j(x + jy) = \sum_{j=1}^{N} \varphi_j(x + h + j(y - h/N)) - \sum_{j=1}^{N} \varphi_j(x + jy) = 0.
\]

The induction hypothesis implies that, for \( j = 1, \ldots, N-1 \), \( \tilde{\varphi}_j \) is a polynomial of degree \( N-3 \) at most on \( I'_j \). Hence \( \varphi_j \) is, on \( I'_j \), the sum of a polynomial of degree \( N-2 \) at most and a function of period \((1 - j/N)h\). But such a representation is given for every sufficiently small positive \( h \), which, with the continuity of \( \varphi_j \), implies that \( \varphi_j \) is a polynomial of degree \( N-2 \) at most on \( I'_j \), \( (1 \leq j \leq N-1) \). From the arbitrariness of \( b \), \( \varphi_j \) is such a polynomial on \( I_j \). Finally, (2.6) shows that \( \varphi_N \) is also such a polynomial on \( I_N \).

In a previous paper [1], results of v. d. Corput were used to
obtain various sufficient conditions on a real sequence \( \psi \) in order that \( \psi \) satisfy condition (I):

(I) There exists a sequence \( S \) such that \( \lim_{j \to \infty} \psi(n) \) (\( n \in S \)) exists for all \( j \geq r \), while \( \{(\psi(n), \Delta^r \psi(n), \cdots, \Delta^{r-1} \psi(n)); n \in S\} \) is uniformly distributed \((\text{mod } 1)\) in the \( r \)-dimensional unit cube.

(I) clearly implies that \( \psi \) has property (QN) for every \( N \geq 2 \). The reader is referred to the paper for details and proofs.

3. A metric result for uniform distribution in the \( N \)-cube.

**Theorem 3.1** Let \( g = \{g(n): n \in \mathbb{Z}\} \) be a sequence of real numbers. Let there exist a subsequence \( S_0 \) of \( \mathbb{Z} \) such that, for every \( N \)-tuple \( h_1, \cdots, h_N \) of integers not all zero there holds

\[
\lim_{n \to \infty} \left| \sum_{p=1}^{N} h_p g(n + p) \right| = \infty, \quad n \in S_0.
\]

Then there exists a subsequence \( S \) of \( S_0 \) such that, for almost all real \( \alpha \), the sequence \( (\alpha g^n) | S \) is uniformly distributed \((\text{mod } 1)\) in the \( N \)-cube.

**Proof.** Let the set of all such \( N \)-tuples be ordered, with, say, \( h'_1, \cdots, h'_{N} \) as the first. Let a subsequence \( S_1 \subset S_0 \) be taken such that

\[
\sum_{p=1}^{N} h'_p \{g(n + p) - g(m + p)\}
\]

is either greater than \( 1 \) for every \( n, m \) in \( S_1 \), with \( n > m \), or else is less than \( -1 \) for every such \( n \) and \( m \). Successively extracting subsequences \( S_1 \supset S_2 \supset \cdots \) in this way, and then using a diagonal procedure, one finally obtains a sequence \( S \) such that, for every \( N \)-tuple \( h_1, \cdots, h_N \), there is an \( m_0 = m_0(h_1, \cdots, h_N) \) such that one has either

\[
\sum_{p=1}^{N} h_p \{g(n + p) - g(m + p)\} \geq 1
\]

for all \( n \) and \( m \) in \( S \) with \( n > m \geq m_0 \)

or else

\[
\sum_{p=1}^{N} h_p \{g(n + p) - g(m + p)\} \leq -1
\]

for all such \( n \) and \( m \).

By a well-known result of Weyl [6, p. 348], either condition (3.2) or (3.3) implies that, for almost all real \( \alpha \), the sequence

\[
\alpha \sum_{p=1}^{N} h_p g(n + p) \quad (n \in S)
\]
is uniformly distributed (mod 1). There being only countably many $N$-tuples, it follows that, for almost all $\alpha$, (3.4) is uniformly distributed (mod 1) for every $N$-tuple $h_1, \ldots, h_N$. But this shows [2, p. 66] that for almost all $\alpha$ the sequence $(\alpha g^n) \mid S$ is uniformly distributed (mod 1) in the $N$-cube.

It is easy to see that if $\theta > 1$ is a transcendental number and $g(n) = \theta^n$, then Theorem 3.1 is applicable. The next result shows the less obvious fact that Theorem 3.1 also applies if, for instance, $g(n) = n^2 \log n \sin n^2$.

**Theorem 3.2.** Let $g = \{g(n); n \in \mathbb{Z}\}$ be of the form
\begin{equation}
  g(n) = r(n)s(n), \quad n \in \mathbb{Z},
\end{equation}
where $s$ has property (PN), while
\begin{equation}
  \lim r(n) = \infty, \quad \lim (r(n + 1)/r(n)) = 1.
\end{equation}
Then there is a subsequence $S_0$ of $\mathbb{Z}$ such that (3.1) holds for every $N$-tuple $h_1, \ldots, h_N$ of integers not all zero.

**Proof.** For $p = 1, 2, \ldots, N$, it follows from (3.6) that
\[ r(n + p) = r(n)(1 + o(1)) \text{, as } n \to \infty. \]
Therefore we have
\begin{equation}
  g(n + p) = r(n)s(n + p)(1 + o(1)), \text{ as } n \to \infty, \quad p = 1, \ldots, N.
\end{equation}
Since $s$ has property (PN), there exists a subsequence $S_0$ of $\mathbb{Z}$ such that
\begin{equation}
  \lim_{n \to \infty} |h_1s(n + 1) + \cdots + h_Ns(n + N)| > 0, \quad (n \in S_0)
\end{equation}
for all $N$-tuples $h_1, \ldots, h_N$ of integers not all zero. But (3.6), (3.7), and (2.1) imply (3.1).

4. An application to noncontinuable power series. Perry [5] has proved that, for every real sequence $f = \{f(n); n \in \mathbb{Z}\}$, there exists a sequence of moduli $\{|a_n|; n \in \mathbb{Z}\}$ such that the power series
\begin{equation}
  \sum_{n=0}^{\infty} |a_n| e(f(n))z^n
\end{equation}
has radius of convergence 1 and the analytic function it represents can be continued analytically across a semicircle of the unit circle. However, if the additional requirements
\begin{equation}
  |a_n| = O(1) \text{ as } n \to \infty
\end{equation}
and

$$\liminf_{N \to \infty} \sum_{0 \leq k < \infty} \sum_{n=k+1}^{k+N} |a_n| = \infty$$

are imposed, then there are conditions on $f$ sufficient that (4.1) represent a function with $|z| = 1$ as its natural boundary. Some such conditions were given in [1]. Theorem 4 gives a metric result in this direction.

**Theorem 4.** Let $\{|a_n| : n \in \mathbb{Z}\}$ satisfy (4.2) and (4.3). Let $g$ be a real sequence which, for each $N$, satisfies the hypothesis of Theorem 3.1. For each real $\alpha$, let

$$F_\alpha(z) = \sum_{n=0}^{\infty} |a_n| e(\alpha g(n))z^n, \quad |z| < 1.\tag{4.4}$$

Then the set of $\alpha$ for which $F_\alpha$ can be continued across an arc of the unit circle has measure zero.

**Example.** $\sum e(\alpha n \sin n^2)z^n$ has $|z| = 1$ as its natural boundary for almost all $\alpha$.

For $N = 2, 3, \ldots$, let $A_N$ be the set of those real $\alpha$ for which $\alpha g^N$ is dense (mod 1) in the unit $N$-cube.

By Theorem 3.1, $A_N$ contains almost all $\alpha$, and it follows that almost all $\alpha$ are in $A_N$ for every $N$. For each such $\alpha$, and each $z_0 = e(\theta_0)$, there holds

$$\limsup_{k \to \infty} \sum_{k+1}^{k+N} |\sum_{n=0}^{\infty} a_n e(\alpha g(n) + n\theta_0)| \geq \liminf_{k \to \infty} \sum_{k+1}^{k+N} |a_n| .\tag{4.5}$$

In view of (4.3), (4.5) shows that the partial sums of the series in (4.4) are unbounded at $z_0$. By (4.2) and a well-known theorem of Fatou [3, p. 391], it follows that $z_0$ is a singularity for $F$.

**References**


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