Pacific Journal of Mathematics

SOME PROPERTIES OF SEQUENCES, WITH AN APPLICATION TO NONCONTINUABLE POWER SERIES

FRANCIS WILLIAM CARROLL

Vol. 24, No. 1 May 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 1, 1968

SOME PROPERTIES OF SEQUENCES, WITH AN APPLICATION TO NONCONTINUABLE POWER SERIES

F. W. CARROLL

For a real sequence $f = \{f(n)\}$ and positive integer N, let F^N denote the sequence of N-tuples $\{(f(n+1), \cdots, f(n+N))\}$. A functional equation method due to Kemperman is used to obtain a sufficient condition on s in order that s^N have an independent N-tuple among its cluster points. If a bounded s has the latter property, and if $g = rs$, where $r(n) \to \infty$ and $r(n+1)/r(n) \to 1$ as $n \to \infty$, then there is a subsequence S of the sequence of positive integers such that, for almost all real α, the restriction of αg^N to S is uniformly distributed (mod 1) **in the N-cube.**

Let F be an analytic function whose Maclaurin series has bounded coefficients $\{a_n\}$ which satisfy the additional requirement

$$\lim_{M\to\infty} \inf_{0 \leq k < \infty} \sum_{n=k}^{k+M} |a_n| = \infty .$$

If $a_n = |a_n| \exp\{2\pi i f(n)\}$, then the density (mod 1) of f^N for each N is sufficient in order that F have the unit circle as a natural boundary. Hence, a metric result for noncontinuable series is obtained from the results for sequences.

1. Notation. For x real, let $((x)) = x - [x]$, and $e(x) = \exp(2\pi i x)$. $h_1, \cdots, h_N$ will denote an N-tuple of integers, not all of which are zero. The sequence of nonnegative integers will be denoted by Z, and subsequences of Z by S_1, S_2, etc. For a real sequence f, we denote by Δf the sequence $\{f(n+1) - f(n)\}$ and

$$\Delta^{j+1} f = \Delta(\Delta^j f) , \; (j = 1, 2, \cdots)$$

2. The property (PN).

DEFINITION. A bounded sequence s of real numbers will be said to have *property* (PN) if there is an independent N-tuple among the cluster points of s^N. In other words, s has property (PN) if there is a subsequence S of Z such that for every N-tuple $h_1, \cdots, h_N$ of integers not all zero, there holds

$$\lim_{n\to\infty} |h_1 s(n+1) + \cdots + h_N s(n+N)| > 0 , \qquad (n \in S) . \tag{2.1}$$

We shall be interested in sequences s of the following form:

$$(2.2) \qquad s(n) = \varphi(\psi(n)) , \qquad (n \in Z) ,$$

where φ is a function of period 1 with at most a nowhere dense set of points of discontinuity, and ψ has the property (QN).

(QN) There exists a subsequence S_1 of Z such that

(2.3) (i) $\Delta^j\psi(n)$ converges (mod 1) for $n \to \infty$, $n \in S_1$ $(j = 2, \cdots, N)$

(ii) $\{(((\psi(n)) , ((\Delta\psi(n))): n \in S_1\}$ is not nowhere dense.

Theorem 2.1. *Let s be of the form (2.2), where φ and ψ have the properties listed above. Then either s has property (PN), or else φ agrees on some interval $I \subset [0, 1]$ with a polynomial of degree N-2 at most.*

Proof. Under the conditions on φ and ψ, it is possible to obtain a subsequence S_2 of S_1 and an open disk D in the plane such that

(2.4) (i) $\lim_{n\to\infty} \Delta^j\psi(n) = \tau_j$ (mod 1) , $(n \in S_2)$, $(j = 2, \cdots, N)$,

(ii) $\{(((\psi(n)) , ((\Delta\psi(n))): n \in S_2\}$ is dense in D ,

(iii) for every (τ_0, τ_1) in D, and every p, $1 \leqq p \leqq N$, the point

$$\tau_0 + p\tau_1 + \sum_{j=2}^{p} \binom{p}{j} \tau_j$$

is a point of continuity for φ.

For each (τ_0, τ_1) in D, a subsequence $S_3 = S_3(\tau_0, \tau_1)$ of S_2 can be chosen so that the corresponding subsequence of (2.4 (ii)) converges to (τ_0, τ_1). In this case, as $n \to \infty$, $n \in S_3$, one has for every $h_1, \cdots, h_N$,

$$\lim_{n\to\infty} \sum_{p=1}^{N} h_p s(n + p) = \lim_{n\to\infty} \sum_{p=1}^{N} h_p \varphi(\psi(n) + p\Delta\psi(n) + \sum_{j=2}^{p} \binom{p}{j} \Delta^j\psi(n))$$

so that

$$(2.5) \qquad \lim_{n\to\infty} \sum_{p=1}^{N} h_p s(n + p) = \sum_{p=1}^{N} h_p \varphi\Big(\tau_0 + p\tau_1 + \sum_{j=2}^{p} \binom{p}{j} \tau_j\Big) , \qquad (n \in S_3) .$$

Suppose now that s does not have property (PN). Then for each (τ_0, τ_1) in D, there is an N-tuple $h_1, \cdots, h_N$ such that the right hand member of (2.5) is zero. Hence D is a countable union of closed sets

$$F = F(h_1, \cdots, h_N) = \{(\tau_0, \tau_1) \in D\colon (2.5) \text{ vanishes}\} .$$

Some F, then, must contain an open subdisk D_1, with center

(τ'_0, τ'_1). That is, there exists an N-tuple $h_1, \cdots, h_N$ of integers not all zero with the property that for all sufficiently small positive h and k,

$$\sum_{p=1}^{N} h_p \varphi\left(h + pk + \tau'_0 + p\tau'_1 + \sum_{j=2}^{p} \binom{p}{j} \tau_j\right) = 0 .$$

The assertion of the theorem follows upon taking

$$\varphi_p(x) = h_p \varphi\left(x + \tau'_0 + p\tau'_1 + \sum_{j=2}^{p} \binom{p}{j} \tau_j\right)$$

in the following lemma, a weak version of one due to Kemperman [4, p. 41]. The proof is included for completeness.

LEMMA. *Let $a > 0$, and let $\varphi_1, \cdots, \varphi_N$ be real functions, with φ_j defined and continuous on $I_j = (-(j+1)a, (j+1)a)$, $(j = 1, \cdots, N)$. Suppose that for all x, y in $(-a, a)$, there holds*

$$\sum_{j=1}^{N} \varphi_j(x + jy) = 0 . \tag{2.6}$$

Then φ_j is equal on I_j to a polynomial of degree N-2 at most.

Proof. We may suppose that $N \geqq 2$ (the case $N = 1$ is trivial), and that the lemma holds for $N - 1$. Let $0 < b < a$, and let $I'_j = (-(j+1)b, (j+1)b)$.

Next, we choose and keep fixed a number h, $0 < h < \min(b, a - b)$. For this h, and $j = 1, \cdots, N$, let

$$\tilde{\varphi}_j(x) = \varphi_j(x + (1 - j/N)h) - \varphi_j(x), \qquad (x \in I'_j) .$$

We note that each $\tilde{\varphi}_j$ is continuous, and $\tilde{\varphi}_N \equiv 0$. Moreover, if x, y are in $(-b, b)$, then $x, y, x + h$, and $y - h/N$ are in $(-a, a)$. Thus, for all x, y in $(-b, b)$, we have

$$\sum_{j=1}^{N-1} \tilde{\varphi}_j(x + jy) = \sum_{j=1}^{N} \varphi_j(x + h + j(y - h/N)) - \sum_{j=1}^{N} \varphi_j(x + jy) = 0 .$$

The induction hypothesis implies that, for $j = 1, \cdots, N - 1$, $\tilde{\varphi}_j$ is a polynomial of degree $N - 3$ at most on I'_j. Hence φ_j is, on I'_j, the sum of a polynomial of degree $N - 2$ at most and a function of period $(1 - j/N)h$. But such a representation is given for every sufficiently small positive h, which, with the continuity of φ_j, implies that φ_j is a polynomial of degree $N - 2$ at most on I'_j, $(1 \leqq j \leqq N - 1)$. From the arbitrariness of b, φ_j is such a polynomial on I_j. Finally, (2.6) shows that φ_N is also such a polynomial on I_N.

In a previous paper [1], results of *v. d.* Corput were used to obtain various sufficient conditions on a real sequence ψ in order that ψ satisfy condition (I):

(I) *There exists a sequence* S *such that* $\lim \Delta^j\psi(n)$ $(n \in S)$ *exists for all* $j \geqq r$, *while* $\{(\psi(n), \Delta\psi(n), \cdots, \Delta^{r-1}\psi(n)): n \in S\}$ *is uniformly distributed* (mod 1) *in the* r*-dimensional unit cube.*

(I) clearly implies that ψ has property (QN) for every $N \geqq 2$. The reader is referred to the paper for details and proofs.

3. A metric result for uniform distribution in the N-cube.

THEOREM 3.1 *Let* $g = \{g(n): n \in Z\}$ *be a sequence of real numbers. Let there exist a subsequence* S_0 *of* Z *such that, for every* N*-tuple* $h_1, \cdots, h_N$ *of integers not all zero there holds*

$$\lim \left| \sum_{p=1}^{N} h_p g(n+p) \right| = \infty , \qquad \text{as } n \to \infty , \quad n \in S_0 . \tag{3.1}$$

Then there exists a subsequence S *of* S_0 *such that, for almost all real* α, *the sequence* $(\alpha g^N) \mid S$ *is uniformly distributed* (mod 1) *in the* N*-cube.*

Proof. Let the set of all such N-tuples be ordered, with, say, $h'_1, \cdots, h'_N$ as the first. Let a subsequence $S_1 \subset S_0$ be taken such that

$$\sum_{p=1}^{N} h'_p\{g(n+p) - g(m+p)\}$$

is either greater than 1 for every n, m in S_1, with $n > m$, or else is less than -1 for every such n and m. Successively extracting subsequences $S_1 \supset S_2 \supset \cdots$ in this way, and then using a diagonal procedure, one finally obtains a sequence S such that, for *every* N-tuple $h_1, \cdots, h_N$, there is an $m_0 = m_0(h_1, \cdots, h_N)$ such that one has either

$$\sum_{p=1}^{N} h_p\{g(n+p) - g(m+p)\} \geqq 1 \tag{3.2}$$

for all n *and* m *in* S *with* $n > m \geqq m_0$
or else

$$\sum_{p=1}^{N} h_p\{g(n+p) - g(m+p)\} \leqq -1 \tag{3.3}$$

for all such n *and* m.

By a well-known result of Weyl [6, p. 348], either condition (3.2) or (3.3) implies that, for almost all real α, the sequence

$$\alpha \sum_{p=1}^{N} h_p g(n+p) \qquad (n \in S) \tag{3.4}$$

is uniformly distributed (mod 1). There being only countably many N-tuples, it follows that, for almost all α, (3.4) is uniformly distributed (mod 1) for every N-tuple $h_1, \cdots, h_N$. But this shows [2, p. 66] that for almost all α the sequence $(\alpha g^N) \mid S$ is uniformly distributed (mod 1) in the N-cube.

It is easy to see that if $\theta > 1$ is a transcendental number and $g(n) = \theta^n$, then Theorem 3.1 is applicable. The next result shows the less obvious fact that Theorem 3.1 also applies if, for instance, $g(n) = n^3 \log n \sin n^2$.

THEOREM 3.2. *Let* $g = \{g(n): n \in z\}$ *be of the form*

$$g(n) = r(n)s(n) , \qquad n \in Z , \tag{3.5}$$

where s *has property* (PN), *while*

$$\lim r(n) = \infty , \qquad \lim (r(n+1)/r(n)) = 1 . \tag{3.6}$$

Then there is a subsequence S_0 of Z *such that* (3.1) *holds for every* N*-tuple* $h_1, \cdots, h_N$ *of integers not all zero.*

Proof. For $p = 1, 2, \cdots, N$, it follows from (3.6) that

$$r(n+p) = r(n)(1 + 0(1)) , \quad \text{as } n \to \infty .$$

Therefore we have

$$g(n+p) = r(n)s(n+p)(1 + o(1)) , \quad \text{as } n \to \infty , \quad p = 1, \cdots, N. \tag{3.7}$$

Since s has property (PN), there exists a subsequence S_0 of Z such that

$$\lim_{n \to \infty} | h_1 s(n+1) + \cdots + h_N s(n+N) | > 0 , \qquad (n \in S_0) \tag{2.1}$$

for all N-tuples $h_1, \cdots, h_N$ of integers not all zero. But (3.6), (3.7), and (2.1) imply (3.1).

4. An application to noncontinuable power series.

Perry [5] has proved that, for every real sequence $f = \{f(n): n \in Z\}$, there exists a sequence of moduli $\{| a_n |: n \in Z\}$ such that the power series

$$\sum_{n=0}^{\infty} | a_N | e(f(n))z^n \tag{4.1}$$

has radius of convergence 1 and the analytic function it represents can be continued analytically across a semicircle of the unit circle. However, if the additional requirements

$$| a_n | = 0(1) \qquad \text{as } n \to \infty \tag{4.2}$$

and

$$\lim_{N\to\infty} \inf_{0\leq k<\infty} \sum_{n=k+1}^{k+N} |a_n| = \infty \tag{4.3}$$

are imposed, then there are conditions on f sufficient that (4.1) represent a function with $|z| = 1$ as its natural boundary. Some such conditions were given in [1]. Theorem 4 gives a metric result in this direction.

THEOREM 4. *Let* $\{|a_n|: n \in Z\}$ *satisfy* (4.2) *and* (4.3). *Let* g *be a real sequence which, for each* N, *satifies the hypothesis of Theorem* 3.1. *For each real* α, *let*

$$F_\alpha(z) = \sum_{n=0}^{\infty} |a_n| e(\alpha g(n))z^n, \qquad |z| < 1 . \tag{4.4}$$

Then the set of α *for which* F_α *can be continued across an arc of the unit circle has measure zero.*

Example. $\sum e(\alpha n \sin n^2) z^n$ has $|z| = 1$ as its natural boundary for almost all α.

For $N = 2, 3, \cdots$, let A_N be the set of those real α for which αg^N is dense (mod 1) in the unit N-cube.

By Theorem 3.1, A_N contains almost all α, and it follows that almost all α are in A_N for *every* N. For each such α, and each $z_0 = e(\theta_0)$, there holds

$$\limsup_{k\to\infty} \left| \sum_{k+1}^{k+N} a_n e(\alpha g(n) + n\theta_0) \right| \geqq \liminf_{k\to\infty} \sum_{k+1}^{k+N} |a_n| . \tag{4.5}$$

In view of (4.3), (4.5) shows that the partial sums of the series in (4.4) are unbounded at z_0. By (4.2) and a well-known theorem of Fatou [3, p. 391], it follows that z_0 is a singularity for F.

REFERENCES

1. F. W. Carroll, *On some classes of noncontinuable analytic functions*, Trans. Amer. Math. Soc. **94** (1960), 74-85.

2. J. W. Cassells, *An introduction to Diophantine approximation*, Cambridge Tract 45, Cambridge Univ. Press, 1957.

3. P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), 335-400.

4. J.H.B. Kemperman, *A general functional equation*, Trans. Amer. Math. Soc. **86** (1957), 28-56.

5. R. L. Perry, *A theorem on power series whose coefficients have given arguments*, J. Lond. Math. Soc. **35** (1960), 172-176.

6. H. Weyl, *Gleichverteilung von Zahlen mod Eins*, Math. Ann. **77** (1916), 313-352.

Received September 19, 1966. This research was supported by a grant from the National Science Foundation.

THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 24, No. 1 May, 1968