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**MATRIX SUMMABILITY OVER CERTAIN CLASSES OF
SEQUENCES ORDERED WITH RESPECT TO RATE OF
CONVERGENCE**

DAVID FLEMING DAWSON

MATRIX SUMMABILITY OVER CERTAIN CLASSES OF SEQUENCES ORDERED WITH RESPECT TO RATE OF CONVERGENCE

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Let C_0 denote the set of all complex null sequences, and let S_0 denote the set of all sequences in C_0 which have at most a finite number of zero terms. If $a = \{a_p\} \in S_0$ and $b = \{b_p\} \in S_0$, we say that a converges faster than b , $a < b$, provided $\lim a_p/b_p = 0$. We say that a and b converge at the same rate, $a \sim b$, provided $0 < \liminf |a_p/b_p|$ and $\limsup |a_p/b_p| < \infty$. If $a \in S_0$, let $[a] = \{x \in S_0: x \sim a\}$. Let $E_0 = \{[x]: x \in S_0\}$. If $[a], [b] \in E_0$, then we say that $[a]$ is less than $[b]$, $[a] <' [b]$, provided $a < b$. We note that E_0 is partially ordered with respect to \leq' . In this paper we study matrix summability over subsets of S_0 and over elements of E_0 . Open intervals in S_0 will be denoted by (a, b) , $(a, -)$, and $(-, b)$, where $(a, -) = \{x \in S_0: a < x\}$ and $(-, b) = \{x \in S_0: x < b\}$. Some of our results characterize, for matrices, maximal summability intervals in S_0 . Such intervals are of the form $(-, b)$, never of the form $(-, b] = \{x \in S_0: \text{either } x < b \text{ or } x \sim b\}$.

Notational conveniences used are as follows. If $A = (a_{pq})$ is a matrix and b is a sequence such that for each positive integer p , the series $\sum_{q=1}^{\infty} a_{pq}b_q$ converges, then $A(b)$ will denote the sequence $\{\sum_{q=1}^{\infty} a_{pq}b_q\}_{p=1}^{\infty}$. We will use A_b to denote the matrix $(a_{pq}b_q)$. If each of a and b is a sequence, then ab will be used to denote the sequence $\{a_p b_p\}$.

Playing a basic role throughout the paper are the two classical Silverman-Toeplitz (abbreviated $S - T$) conditions which are necessary and sufficient for a matrix A to be convergence preserving over (abbreviated c.p.o.) C_0 . These conditions are

$$(1) \{a_{pq}\}_{p=1}^{\infty} \text{ converges, } q = 1, 2, 3, \dots,$$

and

$$(2) \text{ there exists } K \text{ such that } \sum_{q=1}^{\infty} |a_{pq}| < K, p = 1, 2, 3, \dots.$$

We note that the $S - T$ conditions are necessary and sufficient for a matrix A to be c.p.o. S_0 .

REMARK 1. A matrix sums every sequence in some interval $(-, b)$ if and only if it has convergent columns.

REMARK 2. If the matrix A is c.p.o. $[b]$ and c is a sequence such that $\lim c_p/b_p = 0$, then $A(c)$ is convergent.

REMARK 2'. If A is c.p.o. $[b]$, then A is c.p.o. $(-, b]$.

REMARK 3. If A is c.p.o. $(a, -)$, then A is c.p.o. C_0 .

LEMMA. Suppose K and L are countable subsets of S_0 such that if $x \in K$ and $y \in L$, then $x < y$. Then there exists $z \in S_0$ such that if $x \in K$ and $y \in L$, then $x < z < y$.

Proof. Our proof will be for the case that both K and L are infinite sets. Let $K = \{a^{(1)}, a^{(2)}, a^{(3)}, \dots\}$ and $L = \{b^{(1)}, b^{(2)}, b^{(3)}, \dots\}$. Let $\{n_p\}_{p=1}^\infty$ be an increasing sequence of positive integers such that if $i > n_p$, then

$$\left| \frac{b_i^{(j)}}{a_i^{(t)}} \right| > 2^p, \quad j, t = 1, 2, \dots, p.$$

Define

$$\begin{aligned} c_i &= b_i^{(1)}, \quad i = 1, 2, \dots, n_2, \\ c_i &= (1/p) \min [|b_i^{(1)}|, |b_i^{(2)}|, \dots, |b_i^{(p)}|], \\ & \quad n_p < i \leq n_{p+1}, \quad p = 2, 3, 4, \dots \end{aligned}$$

Let r be a positive integer. If $p > r$ and q is a positive integer such that $n_p < q \leq n_{p+1}$, then we have $|b_q^{(r)}/c_q| \geq p$, and, since $c_q = |b_q^{(t)}|/p$ for some $t \in \{1, 2, \dots, p\}$, we have $|c_q/a_q^{(r)}| > 2^p/p$. Thus $a^{(r)} < c < b^{(r)}$. This completes the proof.

THEOREM 1. If A is c.p.o. $[b]$, then there exists $b' \in S_0$ such that $b < b'$ and A is c.p.o. $[b']$.

Proof. Since A is c.p.o. $[b]$, then by Remarks 1 and 2', A has convergent columns. Let $a_q = \lim_{p \rightarrow \infty} a_{pq}$. By Remark 2, A sums every null sequence x such that $\lim x_p/b_p = 0$. Thus A_b sums every null sequence. Therefore from (2) of the $S - T$ conditions there exists M such that if n is a positive integer, then $\sum_{p=1}^\infty |a_{np}b_p| < M$. Clearly $\sum_{q=1}^\infty |a_qb_q| \leq M$. Let $C = (c_{pq})$ be the matrix defined by $c_{pq} = a_{pq}b_q - a_qb_q$. Let $D = (d_{pq})$ be the matrix defined by $d_{pq} = a_qb_q$. Then $A_b = C + D$. We wish to show that the sequence

$$(*) \quad \left\{ \sum_{p=1}^\infty |a_{np}b_p - a_p b_p| \right\}_{n=1}^\infty$$

converges to zero. We note that (*) is bounded. Suppose (*) has a subsequence which converges to $\mu > 0$. Note that each column of C converges to zero. Let n_1 be a positive integer such that

$$\left| \sum_{p=1}^\infty |c_{n_1 p}| - \mu \right| < \mu/8.$$

Let k_1 be a positive integer such that $\sum_{p=1}^{k_1} |c_{n_1 p}| > 7\mu/8$. Let $N_1 > n_1$ be an integer such that if $q > N_1$, then $\sum_{p=1}^{k_1} |c_{qp}| < \mu/8$. Let $n_2 > N_1$ be an integer such that

$$\left| \sum_{p=1}^{\infty} |c_{n_2 p}| - \mu \right| < \mu/8 .$$

Let $k_2 > k_1$ be an integer such that $\sum_{p=1}^{k_2} |c_{n_2 p}| > 7\mu/8$. Let $N_2 > n_2$ be an integer such that if $q > N_2$, then $\sum_{p=1}^{k_2} |c_{qp}| < \mu/8$. Continue the process to obtain increasing sequences $\{n_p\}_{p=1}^{\infty}$ and $\{k_p\}_{p=1}^{\infty}$ of positive integers. Define $t_{pq} = |c_{pq}|/c_{pq}$ if $c_{pq} \neq 0$, $t_{pq} = 1$ if $c_{pq} = 0$. Define

$$\begin{aligned} s_p &= 1, p = 1, 2, \dots, k_1, \\ s_p &= (-1)^{q+1} t_{n_q p}, k_{q-1} < p \leq k_q, q = 2, 3, 4, \dots . \end{aligned}$$

Suppose q is a positive even integer. Then

$$\begin{aligned} & \left| \sum_{p=1}^{\infty} c_{n_q p} s_p - (-\mu) \right| \\ &= \left| \sum_{p=1}^{k_{q-1}} c_{n_q p} s_p + \sum_{p=k_{q-1}+1}^{k_q} c_{n_q p} s_p + \sum_{p=k_q+1}^{\infty} c_{n_q p} s_p + \mu \right| \\ &\leq \sum_{p=1}^{k_{q-1}} |c_{n_q p}| + \sum_{p=k_q+1}^{\infty} |c_{n_q p}| + \left| \sum_{p=k_{q-1}+1}^{k_q} c_{n_q p} s_p + \mu \right| \\ &< \mu/8 + \mu/4 + \left| - \sum_{p=k_{q-1}+1}^{k_q} |c_{n_q p}| + \mu \right| \\ &< \mu/8 + \mu/4 + \mu/4 . \end{aligned}$$

Similarly, if q is a positive odd integer, then

$$\left| \sum_{p=1}^{\infty} c_{n_q p} s_p - \mu \right| < 5\mu/8 .$$

Thus $C(s)$ is divergent. But $A_b(s)$ is convergent since $A_b(s) = A(bs)$ and $bs \in [b]$. Clearly $D(s)$ is convergent. Hence $C(s)$ is convergent since $C(s) = A_b(s) - D(s)$. Therefore we have a contradiction. Thus (*) converges to zero since the assumption to the contrary leads to a contradiction.

Let j_1 be a positive integer such that if $q > j_1$, then $\sum_{p=1}^{\infty} |c_{pq}| < 1/4$. Let K be a number such that $\sum_{p=1}^{\infty} |c_{np}| < K, n = 1, 2, 3, \dots$. Let i_1 be a positive integer such that $\sum_{p=i_1+1}^{\infty} |c_{np}| < 1/4, n = 1, 2, \dots, j_1$. Let $j_2 > j_1$ be an integer such that if $q > j_2$, then $\sum_{p=1}^{\infty} |c_{qp}| < 1/4^2$. Let $i_2 > i_1$ be an integer such that $\sum_{p=i_2+1}^{\infty} |c_{np}| < 1/4^2, n = 1, 2, \dots, j_2$. Continue the process to obtain increasing sequences $\{j_p\}_{p=1}^{\infty}$ and $\{i_p\}_{p=1}^{\infty}$ of positive integers. Define

$$e_n = 1, n = 1, 2, \dots, i_1,$$

$$e_n = 2^t, i_t < n \leq i_{t+1}, t = 1, 2, 3, \dots$$

Consider the matrix C_e . If q is a positive integer, then

$$\begin{aligned} \sum_{p=1}^{\infty} |c_{qp}e_p| &= \sum_{p=1}^{i_1} |c_{qp}e_p| + \sum_{t=1}^{\infty} \left(\sum_{p=i_t+1}^{i_{t+1}} |c_{qp}e_p| \right) \\ &< K + \sum_{t=1}^{\infty} \left(2^t \cdot \sum_{p=i_t+1}^{i_{t+1}} |c_{qp}| \right) \\ &\leq K + \sum_{t=1}^{\infty} 2^t/4^t \\ &= K + 1. \end{aligned}$$

Let $\{r_p\}$ be an increasing sequence of positive integers such that

$$\sum_{p=r_q+1}^{\infty} |a_p b_p| < 1/4^q.$$

Define

$$f_p = 1, p = 1, 2, \dots, r_1,$$

$$f_p = 2^q, r_q < p \leq r_{q+1}, q = 1, 2, 3, \dots$$

Then

$$\begin{aligned} \sum_{p=1}^{\infty} |a_p b_p f_p| &= \sum_{p=1}^{r_1} |a_p b_p f_p| + \sum_{q=1}^{\infty} \left(\sum_{p=r_q+1}^{r_{q+1}} |a_p b_p f_p| \right) \\ &\leq M + \sum_{q=1}^{\infty} \left(2^q \cdot \sum_{p=r_q+1}^{r_{q+1}} |a_p b_p| \right) \\ &< M + \sum_{q=1}^{\infty} 2^q/4^q. \end{aligned}$$

Let $g_p = \min [e_p, f_p]$, $p = 1, 2, 3, \dots$. Then $g_p \rightarrow \infty$ as $p \rightarrow \infty$. Thus $b < bg$. If n is a positive integer, then

$$\begin{aligned} \sum_{p=1}^{\infty} |a_{np} b_p g_p| &\leq \sum_{p=1}^{\infty} |c_{np} g_p| + \sum_{p=1}^{\infty} |a_p b_p g_p| \\ &\leq \sum_{p=1}^{\infty} |c_{np} e_p| + \sum_{p=1}^{\infty} |a_p b_p f_p| \\ &< K + 1 + M + 1. \end{aligned}$$

Therefore the matrix A_{bg} sums every null sequence. Thus if $b < b' < bg$, then A is c.p.o. $[b']$. The existence of a sequence b' such that $b < b' < bg$ follows from the lemma. This completes the proof of the theorem.

REMARK 4. We note that the matrix A , defined by $a_{pq} = 1$ if $p \neq q$, $a_{pq} = 2^{p-1}$ if $p = q$, has a maximal interval $(-, b)$ over which it is convergence preserving. For example $b = \{1/2^{p-1}\}$.

On the other hand, the matrix A , defined by $a_{pq} = 0$ if $q > p$, $a_{pq} = 1$ if $p \geq q$, has no such maximal interval. This is easily shown by supposing that $(-, b)$ is a maximal summability interval for A . Then A_b is c.p.o. C_0 and hence satisfies the $S - T$ conditions. Thus $\sum_{p=1}^{\infty} |b_p|$ converges. It is easy to find $c \in S_0$ such that $b < c$ and $\sum_{p=1}^{\infty} |c_p|$ converges. Thus A_c satisfies the $S - T$ conditions and hence is c.p.o. C_0 . Therefore A is c.p.o. $(-, c)$.

It is easy to show that if there exist numbers r and R such that $0 < r < |a_{pq}| < R$, $p, q = 1, 2, 3, \dots$, then $A = (a_{pq})$ has no maximal summability interval. The proof will be omitted.

REMARK 5. Let Δ be a chain in S_0 unbounded above. If $a \in \Delta$, let $a' = \{a_1, 1/2, a_2, 1/4, a_3, 1/8, \dots\}$. Let $\Delta' = \{a' : a \in \Delta\}$. Then Δ' is a chain in S_0 which is unbounded above. Let $A = (a_{pq})$ be defined by $a_{pq} = 1/2^n$ if $q = 2n - 1$, $a_{pq} = 1$ if q is an even integer. Clearly if $a' \in \Delta'$, then A is c.p.o. $[a']$. But A is not c.p.o. C_0 .

THEOREM 2. If A is c.p.o. each of the sets $[b^{(1)}], [b^{(2)}], [b^{(3)}], \dots$, then there exists $d \in S_0$ such that $b^{(p)} < d$, $p = 1, 2, 3, \dots$, and A is c.p.o. $[d]$.

Proof. By Theorem 1 we can find $t^{(n)}$ in S_0 such that $t^{(n)} > b^{(n)}$ and A is c.p.o. $[t^{(n)}]$, $n = 1, 2, 3, \dots$. If n is a positive integer, let $\alpha^{(n)} \in [t^{(n)}]$ such that $0 < \alpha_p^{(n)} < 1$, $p = 1, 2, 3, \dots$. If n is a positive integer, let M_n be a number which exceeds $\sum_{q=1}^{\infty} |a_{pq}\alpha_q^{(n)}|$, $p = 1, 2, 3, \dots$. If n is a positive integer, let

$$\beta_p^{(n)} = \frac{\alpha_p^{(n)}}{2^n[M_n + 1]}, \quad p = 1, 2, 3, \dots$$

If p is a positive integer, let $c_p = \sum_{n=1}^{\infty} \beta_p^{(n)}$. We wish to show that $c \in S_0$. Let $\mu > 0$, and let k be a positive integer such that $2^{-k} < \mu/2$. Let R be a positive integer such that if $q > R$, then $\beta_q^{(p)} < \mu/2^{k+1}$, $p = 1, 2, \dots, k$. Then if $n > R$, we have

$$c_n = \sum_{p=1}^{\infty} \beta_n^{(p)} = \sum_{p=1}^k \beta_n^{(p)} + \sum_{p=k+1}^{\infty} \beta_n^{(p)} < \mu/2 + 2^{-k} < \mu.$$

Thus $c \in S_0$. If q is a positive integer, then, using the double sum theorem, we have

$$\begin{aligned}
\sum_{p=1}^{\infty} |a_{qp} c_p| &= \sum_{p=1}^{\infty} |a_{qp}| \left(\sum_{n=1}^{\infty} \beta_p^{(n)} \right) \\
&= \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} |a_{qp}| \beta_p^{(n)} \\
&= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} |a_{qp}| \beta_p^{(n)} \\
&= \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} |a_{qp}| \cdot \frac{\alpha_p^{(n)}}{2^n [M_n + 1]} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n [M_n + 1]} \sum_{p=1}^{\infty} |a_{qp}| \alpha_p^{(n)} \\
&< \sum_{n=1}^{\infty} 2^{-n}.
\end{aligned}$$

Thus A_c sums every null sequence. Therefore A sums every sequence $x \in S_0$ such that $x < c$. We note that if n is a positive integer, then $c_p / \beta_p^{(n)} > 1$, $p = 1, 2, 3, \dots$. Thus if n is a positive integer, then

$$\lim_{p \rightarrow \infty} \left| \frac{c_p}{b_p^{(n)}} \right| = \lim_{p \rightarrow \infty} \frac{c_p}{\beta_p^{(n)}} \cdot \frac{\beta_p^{(n)}}{t_p^{(n)}} \cdot \frac{t_p^{(n)}}{|b_p^{(n)}|} = \infty.$$

Hence $b^{(n)} < c$, $n = 1, 2, 3, \dots$. By the lemma, there exists $d \in S_0$ such that $b^{(n)} < d < c$, $n = 1, 2, 3, \dots$. A is c.p.o. $[d]$ since $d < c$ and A sums every sequence $x \in S_0$ such that $x < c$. This completes the proof of the theorem.

COROLLARY. *Suppose M is a countable set of matrices and L is a countable subset of E_0 such that if $A \in M$ and $[b] \in L$, then A is c.p.o. $[b]$. Then there exists $[c] \in E_0$ such that if $A \in M$ and $[b] \in L$, then $[b] <' [c]$ and A is c.p.o. $[c]$.*

Proof. The proof will be for the case that both M and L are infinite sets. Let $M = \{A^{(1)}, A^{(2)}, A^{(3)}, \dots\}$ and $L = \{[b^{(1)}], [b^{(2)}], [b^{(3)}], \dots\}$. By Theorem 2, if p is a positive integer, there exists $c^{(p)} \in S_0$ such that $b^{(n)} < c^{(p)}$, $n = 1, 2, 3, \dots$, and $A^{(p)}$ is c.p.o. $[c^{(p)}]$. Let $L' = \{[c^{(1)}], [c^{(2)}], [c^{(3)}], \dots\}$. By the lemma, there exists $c \in S_0$ such that if $b^{(s)} \in L$ and $c^{(t)} \in L'$, then $b^{(s)} < c < c^{(t)}$. If j is a positive integer, then by Remark 2', $A^{(j)}$ is c.p.o. $[c]$ since $A^{(j)}$ is c.p.o. $[c^{(j)}]$ and $[c] \subset (-, c^{(j)})$. This completes the proof.

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