

Pacific Journal of Mathematics

A NOTE ON QUASI-FROBENIUS RINGS

EDGAR EARLE ENOCHS

A NOTE ON QUASI-FROBENIUS RINGS

EDGAR ENOCHS

Morita and Curtis proved independently that if A is a quasi-Frobenius ring and P_a^\vee finitely generated, projective, faithful, left A -module, then the ring of endomorphisms $B = \text{End}_A(P)$ is quasi-Frobenius and P is a finitely generated, projective, faithful, left B -module. It also turns out that $A \cong \text{End}_B(P)$. We prove a theorem implying that every quasi-Frobenius ring can be represented as such a ring of endomorphisms.

In fact the following holds:

THEOREM. *If A is a quasi-Frobenius ring there is a Frobenius ring B such that $B/\text{Rad}(B)$ is the product of a finite number of (not necessarily commutative) fields and a finitely generated, projective, faithful, left B -module P such that $A \cong \text{End}_B(P)$. If B' is another Frobenius ring such that $B'/\text{Rad}(B')$ is the product of a finite number of fields and P' a finitely generated, projective, faithful, left B' -module such that $A \cong \text{End}_{B'}(P')$ then there is a semilinear isomorphism of the B -module P into the B' -module P' .*

We note the results mentioned above appear in [2, pp. 405-406].

Proof. Let A_s be A considered as a left A -module. Let $A_s = E_1 + \cdots + E_n$ (direct) where each E_i is nonzero and indecomposable, and so has a simple socle. Consider the equivalence relation $E_i \cong E_j$ on the set $\{E_1, E_2, \dots, E_n\}$. Note $E_i \cong E_j$ if and only if $S_i \cong S_j$ where S_i is the socle of E_i for each i . Choose one representative from each equivalence class and let P be their direct sum. Then we easily see that P is a finitely generated, projective, faithful, left A -module. Let $B = \text{End}_A(P)$. Then by Morita and Curtis' result, B is a quasi-Frobenius ring and P is a finitely generated, projective, faithful, left B -module. We claim that if we show $B/\text{Rad}(B)$ is the product of a finite number of fields then it will follow that B is Frobenius. For in this case $B/\text{Rad}(B)$ is the direct sum of a finite number of simple pair-wise nonisomorphic left B -modules. But since B is quasi-Frobenius each simple left B -module is isomorphic to a submodule of B [2, p. 401, Corollary 58.13]. But to show $B/\text{Rad}(B)$ is a product of fields we only need note that $B/\text{Rad}(B) \cong \text{End}_A(T)$ where T is the socle of P . But by the construction of P , T is the direct sum of a finite number of pair-wise nonisomorphic simple left A -modules so $\text{End}_A(T)$ is the

product of a finite number of fields. But now as remarked above, $A \cong \text{End}_B(P)$ and P is a finitely generated, projective, faithful, left B -module.

Now suppose $A \cong \text{End}_{B'}(P')$ where B' is a Frobenius ring with $B'/\text{Rad}(B')$ the product of a finite number of fields and that P' is a finitely generated, projective, faithful, left B' -module. Then P' is a finitely generated, projective, faithful, left A -module and $B' \cong \text{End}_A(P')$. But then since A is quasi-Frobenius, $P' \cong \bigoplus_{i=1}^m E_{k_i}$ where $1 \leq k_i \leq n$ for each $i = 1, 2, \dots, m$ [2, p. 401, Corollary 58.13]. But P' is a faithful left A -module so it's easy to see that for each $j, 1 \leq j \leq n$, $E_{k_i} \cong E_j$ for some $i, 1 \leq i \leq m$. But now if T' is the socle of P' (as a left A -module), $B'/\text{Rad}(B') \cong \text{End}_A(T')$. But $B'/\text{Rad}(B')$ is the product of a finite number of fields so we see that T' is the direct sum of a finite number of pair-wise nonisomorphic simple left A -modules. Thus $P \cong P'$ (as left A -modules). But then

$$B \cong \text{End}_A(P) \cong \text{End}_A(P') \cong B' \quad \text{and}$$

we easily see that there is a semi-linear isomorphism from the B -module P to the B' -module P' .

We note that if A is a simple ring (i.e. left Artinian, without radical and having no nontrivial two sided ideals) we get the usual representation of A as the ring of matrices over a field (i.e. the endomorphism ring of a finite dimensional vector space) since in this case B is a field.

BIBLIOGRAPHY

1. C. W. Curtis, *Quasi-Frobenius rings and Galois theory*, Illinois J. Math. **3** (1959), 134-144.
2. C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, New York, New York. 1962.
3. K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Science reports of the Tokyo Kyoiku Daigaku, **6** (1958), 83-142.

Received February 28, 1967.

UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, SOUTH CAROLINA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN

Stanford University
Stanford, California

J. P. JANS

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

Department of Mathematics
Rice University
Houston, Texas 77001

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL ORDNANCE TEST STATION

Harry P. Allen, <i>Lie algebras of type D_4 over algebraic number fields</i>	1
Charles Ballantine, <i>Products of positive definite matrices. II</i>	7
David W. Boyd, <i>The spectral radius of averaging operators</i>	19
William Howard Caldwell, <i>Hypercyclic rings</i>	29
Francis William Carroll, <i>Some properties of sequences, with an application to noncontinuable power series</i>	45
David Fleming Dawson, <i>Matrix summability over certain classes of sequences ordered with respect to rate of convergence</i>	51
D. W. Dubois, <i>Second note on David Harrison's theory of preprimes</i>	57
Edgar Earle Enochs, <i>A note on quasi-Frobenius rings</i>	69
Ronald J. Ensey, <i>Isomorphism invariants for Abelian groups modulo bounded groups</i>	71
Ronald Owen Fulp, <i>Generalized semigroup kernels</i>	93
Bernard Robert Kripke and Richard Bruce Holmes, <i>Interposition and approximation</i>	103
Jack W. Macki and James Sai-Wing Wong, <i>Oscillation of solutions to second-order nonlinear differential equations</i>	111
Lothrop Mittenthal, <i>Operator valued analytic functions and generalizations of spectral theory</i>	119
T. S. Motzkin and J. L. Walsh, <i>A persistent local maximum of the pth power deviation on an interval, $p < 1$</i>	133
Jerome L. Paul, <i>Sequences of homeomorphisms which converge to homeomorphisms</i>	143
Maxwell Alexander Rosenlicht, <i>Liouville's theorem on functions with elementary integrals</i>	153
Joseph Goeffrey Rosenstein, <i>Initial segments of degrees</i>	163
H. Subramanian, <i>Ideal neighbourhoods in a ring</i>	173
Dalton Tarwater, <i>Galois cohomology of abelian groups</i>	177
James Patrick Williams, <i>Schwarz norms for operators</i>	181
Raymond Y. T. Wong, <i>A wild Cantor set in the Hilbert cube</i>	189