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Let $\mathscr{A}$ be the category of Abelian groups, let $\mathscr{B}$ be the class of bounded Abelian groups, and form the quotient category $\mathscr{A}/\mathscr{B}$. The principal goal of this paper is a complete set of invariants for direct sums of countable reduced p-groups, such groups considered as objects of the category $\mathscr{A}/\mathscr{B}$. Specifically, it will be shown that two direct sums of countable reduced p-groups G and H are isomorphic in $\mathscr{A}/\mathscr{B}$ if and only if there is an integer $k \geqq 0$ such that for all ordinal numbers α and all integers $r \geqq 0$

$$\sum_{j=0}^{r} f_G(\alpha + k + j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha + j)$$

and

$$\sum_{j=0}^{r} f_H(\alpha + k + j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha + j)$$

where $f_G(\beta)$ and $f_H(\beta)$ denote the βth Ulm invariants of G and H, respectively. Thus a complete set of $\mathscr{A}/\mathscr{B}$-isomorphism invariants for such groups is an equivalence class of Ulm invariants, the equivalence relation being given by these two inequalities.

The concept of quasi-isomorphism as defined by B. Jonsson [11] has come to play a significant role in the theory of torsion-free Abelian groups. However, nothing of significance had been done for quasi-isomorphism of torsion groups until R. Beaumont and R. Pierce [1] gave necessary and sufficient conditions on the Ulm invariants for two countable reduced p-groups to be quasi-isomorphic. This lack of results for p-groups was pointed out by E. A. Walker [14], who in fact submitted that the definition of quasi-isomorphism should be isomorphism in the category $\mathscr{A}/\mathscr{B}$. The importance of quasi-isomorphism for torsion-free groups motivates the study of Abelian groups as objects of $\mathscr{A}/\mathscr{B}$.

2. Preliminaries.

The word group will always mean Abelian group, the category of Abelian groups will be denoted by $\mathscr{A}$, and the class of bounded Abelian groups by $\mathscr{B}$. The term Abelian category will be used in the sense of MacLane [13]. For G and H in an Abelian category $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(G, H)$ will denote the group of maps from G to H. $\mathrm{Hom}_{\mathscr{A}}(G, H)$ will be written simply as $\mathrm{Hom}(G, H)$. Any unexplained notation or terminology will conform with that of Fuchs [4] or MacLane [13].

A nonempty class ζ of $\mathscr{A}$ is a Serre class of $\mathscr{A}$ if for each exact sequence

$$0 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 0$$

of groups, H is in ζ if and only if G and K are in ζ. It is clear that $\mathscr{B}$ is a Serre class of $\mathscr{A}$, and it is the quotient category $\mathscr{A}/\mathscr{B}$ as defined by Grothendieck [6] with which this paper is concerned.

The objects of $\mathscr{A}/\mathscr{B}$ are just the objects of $\mathscr{A}$, and $\mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(G,H)$ is the direct limit of the groups $\mathrm{Hom}(G', H/H')$, the limit being taken over all pairs (G', H') with $G' \subseteqq G$, $H' \subseteqq H$, and $G/G', H' \in \mathscr{B}$. To define composition let $\bar{f} \in \mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(G, H)$ and $\bar{g} \in \mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(H, K)$. Then $\bar{f}$ comes from an $f\colon G' \to H/H'$ and $\bar{g}$ comes from a $g\colon H'' \to K/K'$ where $G/G', H', H/H'', K' \in \mathscr{B}$. Let $G'' = f^{-1}((H'' + H')/H')$. Then $G'/G'' \in \mathscr{B}$, and since

$$0 \longrightarrow G'/G'' \longrightarrow G/G'' \longrightarrow G/G' \longrightarrow 0$$

is exact, $G/G'' \in \mathscr{B}$. Let $K''/K' = g(H' \cap H'')$. Then $K'' \in \mathscr{B}$ since $H', K' \in \mathscr{B}$. Now let h be the composition

$$G'' \xrightarrow{f'} (H'' + H')/H' \simeq H''/H' \cap H'' \xrightarrow{g'} K/K''$$

where f' is the restriction of f to G'' and g' is induced by g. Since $G/G'', K'' \in \mathscr{B}$, h determines an element $\bar{h} \in \mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(G, K)$. It is straightforward that $\bar{h}$ is uniquely determined by $\bar{f}$ and $\bar{g}$, and that $\mathscr{A}/\mathscr{B}$ is a category with the definition $\bar{h} = \bar{g} \circ \bar{f}$. That $\mathscr{A}/\mathscr{B}$ is an Abelian category can be found in [5].

Let $f\colon G' \to H/H'$ and $g\colon G'' \to H/H''$ where $G/G', H', G/G'', H'' \in \mathscr{B}$. Let $\bar{f}$ and $\bar{g}$ be the elements of $\mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(G, H)$ determined by f and g, respectively. The following facts are taken from [14].

2.1. $\bar{f} = 0$ if and only if $\mathrm{Im}\, f \in \mathscr{B}$.

2.2. $\bar{f}$ is an epimorphism if and only if $\mathrm{Coker}\, f \in \mathscr{B}$.

2.3. $\bar{f}$ is a monomorphism if and only if $\mathrm{Ker}\, f \in \mathscr{B}$.

2.4. $\bar{f} = \bar{g}$ if and only if there are subgroups $S \subseteqq G$ and $T \subseteqq H$ such that

(a) $S \subseteqq G' \cap G''$,

(b) $G/S \in \mathscr{B}$,

(c) $H' + H'' \subseteqq T$,

(d) $T \in \mathscr{B}$,

(e) the maps from S to H/T induced by $\bar{f}$ and $\bar{g}$ are the same.

From 2.2 and 2.3 it follows that G and H are isomorphic in $\mathscr{A}/\mathscr{B}$ if and only if there exist subgroups S and A of G, and T and B of H such that $S/A \simeq T/B$ in $\mathscr{A}$ and $G/S, A, H/T, B \in \mathscr{B}$.

Two groups G and H are quasi-isomorphic, and we write $G \doteq H$, if there exist isomorphic subgroups $S \subseteqq G$ and $T \subseteqq H$ such that G/S

and H/T are bounded. It is clear that in general quasi-isomorphism implies isomorphism in $\mathscr{A}/\mathscr{B}$ and that the two concepts are equivalent if we restrict ourselves to torsion-free groups.

3. Fundamentals of isomorphism in $\mathscr{A}/\mathscr{B}$.

THEOREM 3.1. *The following are equivalent.*

(i) $G \simeq H$ *in* $\mathscr{A}/\mathscr{B}$.

(ii) *There exists a bounded subgroup* $B \subseteqq H$ *such that* $G \doteq H/B$.

(iii) *There exists a positive integer* k *and homomorphisms* $f: G \to H$ *and* $g: H \to G$ *such that* $gf = k$ *and* $fg = k$.

Proof. Suppose $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Then there is a homomorphism $f: G' \to H/H'$ such that G/G', H', $\operatorname{Ker} f$, $\operatorname{Coker} f$ are bounded. By 2.4, we can take $G' = mG$ and $H' = H[m]$ for some integer $m > 0$. Let $n > 0$ be a bound for $\operatorname{Ker} f$ and $\operatorname{Coker} f$, and let $B/H[m] = f((mG)[n])$. Let α be the composition

$$mnG \longrightarrow (mG)/(mG)[n] \longrightarrow (H/H[m])/(B/H[m]) \longrightarrow H/B$$

where

$$mnG \longrightarrow (mG)/(mG)[n] \quad \text{and} \quad (H/H[m])/(B/H[m]) \longrightarrow H/B$$

are natural isomorphisms and

$$(mG)/(mG)[n] \longrightarrow (H/H[m])/(B/H[m])$$

is the monomorphism induced by f. Now α is a monomorphism with cokernel bounded by n. Thus $G \doteq H/B$ and since B is bounded by mn, (i) implies (ii).

Suppose $G \doteq H/B$ where $nB = 0$, $n > 0$. Then there exist isomorphic subgroups $S \subseteqq G$ and $T/B \subseteqq H/B$ such that G/S and

$$(H/B)/(T/B) \simeq H/T$$

are bounded. Let $m > 0$ with $mG \subseteqq S$ and $mH \subseteqq T$, and let θ be the isomorphism $S \simeq T/B$. Let f be the composition

$$G \xrightarrow{m} mG \subseteqq S \xrightarrow{\theta} T/B \subseteqq H/B \xrightarrow{\pi_1} H/H[n] \xrightarrow{\alpha} nH \subseteqq H$$

where $H/B \xrightarrow{\pi_1} H/H[n]$ is the projection induced by $B \subseteqq H[n]$ and $H/H[n] \xrightarrow{\alpha} nH$ is the natural isomorphism. Let g be the composition

$$H \xrightarrow{\pi_2} H/B \xrightarrow{m} (mH + B)/B \subseteqq T/B \xrightarrow{\theta^{-1}} S \subseteqq G$$

where $H \xrightarrow{\pi_2} H/B$ is the natural projection. Let $x \in G$ with $\theta(mx) =$

$y+B$. Then

$$x \xrightarrow{m} mx \xrightarrow{\theta} y + B \xrightarrow{\pi_1} y + H[n] \xrightarrow{\alpha} ny$$
$$\xrightarrow{\pi_2} ny + B \xrightarrow{m} mny + B \xrightarrow{\theta^{-1}} m^2ny\ .$$

Let $y \in H$ with $\theta(x) = my + B$. Then

$$y \xrightarrow{\pi_2} y + B \xrightarrow{m} my + B \xrightarrow{\theta^{-1}} x \xrightarrow{m} mx$$
$$\xrightarrow{\theta} m^2y + B \xrightarrow{\pi_1} m^2y + H[n] \xrightarrow{\alpha} m^2ny\ .$$

Thus $gf = m^2n$ and $fg = m^2n$. Hence (ii) implies (iii).

Suppose there exists a positive integer k and homomorphisms $f: G \to H$ and $g: H \to G$ such that $fg = k$ and $gf = k$. Since $gf = k$, Ker f is bounded by k and since $fg = k$, Coker f is bounded by k. Hence $G \cong H$ in $\mathscr{A}/\mathscr{B}$ and (iii) implies (i).

COROLLARY 3.2. *Let S and T be subgroups of G and H, respectively, such that $f(S) \subseteq T$ and $g(T) \subseteq S$ whenever $f: G \to H$ and $g: H \to G$. Let $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Then $S \simeq T$ in $\mathscr{A}/\mathscr{B}$ and $G/S \simeq H/T$ in $\mathscr{A}/\mathscr{B}$.*

Proof. There exists an integer $k > 0$ and homomorphisms $f: G \to H$ and $g: H \to G$ such that $gf = k$ and $fg = k$. Clearly $S \simeq T$ in $\mathscr{A}/\mathscr{B}$. Let $\bar{f}: G/S \to H/T$ and $\bar{g}: H/T \to G/S$ be the homomorphisms induced be f and g, respectively. Then $\bar{g}\bar{f} = k$ and $\bar{f}\bar{g} = k$. Hence $G/S \simeq H/T$ in $\mathscr{A}/\mathscr{B}$.

Let G_t be the torsion subgroup of G. If $G \simeq H$ in $\mathscr{A}/\mathscr{B}$, then $G_t \simeq H_t$ in $\mathscr{A}/\mathscr{B}$ and $G/G_t \doteq H/H_t$. Moreover, if G and H split, $G_t \simeq H_t$ in $\mathscr{A}/\mathscr{B}$, and $G/G_t \doteq H/H_t$, then $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. We now reduce the study of isomorphism in $\mathscr{A}/\mathscr{B}$ from torsion groups to primary groups.

PROPOSITION 3.3. Let G and H be torsion groups, and let G_p and H_p be the p components of G and H, respectively. Then $G \simeq H$ in $\mathscr{A}/\mathscr{B}$ if and only if $G_p \simeq H_p$ in $\mathscr{A}/\mathscr{B}$ for all primes p and $G_p \simeq H_p$ for almost all primes p.

Proof. Suppose $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Then there exists an integer $k > 0$ and homomorphisms $f: G \to H$ and $g: H \to G$ such that $gf = k$ $fg = k$. Since $f(G_p) \subseteq H_p$ and $g(H_p) \subseteq G_p$, $G_p \simeq H_p$ in $\mathscr{A}/\mathscr{B}$ for each prime p. Let p be a prime which does not divide k. Let $f_p = f \mid G_p$. Suppose $f_p(x) = 0$ for some $x \in G_p$. Let $o(x) = p^m$ and write

$1 = ak + bp^m$. Now $x = akx = agf_p(x) = 0$. Thus f_p is a monomorphism. Let $y \in H_p$, $o(y) = p^n$. Write $1 = a'k + b'p^n$. Now $f_pg(a'y) = a'f_pg(y) = a'ky = y$. Thus f_p is an epimorphism. Hence $G_p \simeq H_p$ for each prime p not dividing k.

The converse is clear.

Let G be a divisible p-group and $A = \sum\langle a_\alpha\rangle$ a bounded subgroup of G. Embed each $\langle a_\alpha\rangle$ in a summand $G_\alpha \simeq Z(p^\infty)$ of G. Then $G_\alpha/\langle a_\alpha\rangle \simeq G_\alpha$. Thus $G/A \simeq G$. Given two divisible groups D and E, it now follows from 3.1 that $D \simeq E$ in $\mathscr{A}/\mathscr{B}$ implies that $D \simeq E$. An application of 3.2 yields the next proposition.

PROPOSITION 3.4. Let $G = D \oplus R$ and $H = D' \oplus R'$ where D and D' are divisible and R and R' are reduced. Then $G \simeq H$ in $\mathscr{A}/\mathscr{B}$ if and only if $D \simeq D'$ in $\mathscr{A}$ and $R \simeq R'$ in $\mathscr{A}/\mathscr{B}$.

Let G be a group. Then $G^1 = \bigcap_{0<n>\omega} nG$ is a subgroup of G called the elements of infinite height of G. If G is a p-group, then $G^1 = \bigcap_{0<n<\omega} p^nG$.

To see that the property of being a direct sum of cyclic groups is not preserved under isomorphism in $\mathscr{A}/\mathscr{B}$, we need only consider the Prufer group, P, which is generated by elements $a_1, a_2, \cdots, a_n, \cdots$ with the relations

$$pa_1 = 0,\ a_1 = pa_2 = p^2a_3 = \cdots = p^na_{n+1} = \cdots .$$

Now $B = \sum_{n\geqq 2}\langle a_n - pa_{n+1}\rangle$ is a basic subgroup of P, $P^1 = \langle a_1\rangle \simeq C(p)$, and $P/P^1 \simeq B$. (See [4]). So $P \simeq B$ in $\mathscr{A}/\mathscr{B}$. We shall point out some other properties which are related to direct sums of cyclic groups and direct sums of countable groups and which are not preserved by isomorphisms in $\mathscr{A}/\mathscr{B}$. First a further look at direct sums of cyclic groups.

PROPOSITION 3.5. The following are equivalent for a p-group G.

(i) G is isomorphic in $\mathscr{A}/\mathscr{B}$ to a direct sum of cyclic groups.

(ii) G/A is a direct sum of cyclic groups for some bounded subgroup $A \subseteqq G$.

(iii) There is a direct sum of cyclic groups C and a bounded subgroup $B \subseteqq C$ such that $G \simeq C/B$.

(iv) Every subgroup of G is isomorphic in $\mathscr{A}/\mathscr{B}$ to a direct sum of cyclic groups.

(v) $\mathrm{Ext}(G, H)^1$ is bounded for all p-groups H.

Proof. It is clear that (ii), (iii), and (iv) each imply (i) and that (iii) implies (iv). That (i) implies (ii) follows from 3.1., and (ii) if and only if (v) is Theorem 27′ of [10].

Suppose $G \simeq H$ in $\mathscr{A}/\mathscr{B}$, H a direct sum of cyclic groups. Then there exists a bounded subgroup $B \subseteqq H$, an integer $n \geqq 0$, and a monomorphism $f: p^nG \to H/B$ with bounded cokernel. So $p^nG \simeq T/B$ where $p^kH \subseteqq T \subseteqq H$ for some $k \geqq 0$. By 50.1 in [4], there exists a group $T \subseteqq C$ such that the isomorphism $p^nG \simeq T/B$ can be extended to an isomorphism $G \simeq C/B$. Thus $p^nG \subseteqq T \subseteqq H$ and C is a direct sum of cyclic groups. Hence (i) implies (iii).

PROPOSITION 3.6. Let G be a p-group. If G is isomorphic in $\mathscr{A}/\mathscr{B}$ to a direct sum of cyclic groups, then G^1 is bounded. If G is a direct sum of countable groups, the converse holds.

Proof. The first statement follows from (ii) of 3.5. If G is a direct sum of countable groups and $p^kG^1 = 0$, then

$$p^kG \simeq (G/G^1)/(G[p^k]/G^1)$$

and G/G^1 is a direct sum of cyclic groups. The result follows from (iii) of 3.5.

Let G be a group. If every infinite subgroup of G can be embedded in a summand of the same cardinality, then G is Fuchs five [9]. It is easy to see that if G is a direct sum of countable groups, then G is Fuchs five. If G has no elements of infinite height and if for every infinite subgroup H of G, $|(G/H)^1| \leqq |H|$, then G is a Q-group [9]. If whenever G/H is divisible, H a subgroup of G, we have $|H| = |G|$, then G is starred [9]. Finally, G is fully starred if every subgroup of G is starred. F. Richman and J. Irwin [9] showed that if

$$0 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 0$$

is exact and if G and K are starred, then H is starred. Also in [9], it is shown that if G is Fuchs five and $G^1 = 0$, then G is a Q-group, and if G is a Q-group, then G is fully starred.

PROPOSITION 3.7. Let $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Then G is fully starred if and only if H is fully starred.

Proof. Suppose H is fully starred. Let A be a bounded subgroup of G such that $nH \simeq S/A$ where $mG \subseteqq S \subseteqq G$, m and n positive integers. Then S/A is fully starred and A, being a direct sum of cyclic groups, is starred. Since

$$0 \longrightarrow A \longrightarrow S \longrightarrow S/A \longrightarrow 0$$

is exact, it follows that S is starred. Now

$$0 \longrightarrow S \longrightarrow G \longrightarrow G/S \longrightarrow 0$$

is exact with G/S bounded. Thus G is starred. Let K be a subgroup of G. Then $(mK + A)/A$ is a subgroup of S/A and so $(mK + A)/A \simeq mK/mK \cap A$ is starred. By the exactness of

$$0 \longrightarrow mK \cap A \longrightarrow mK \longrightarrow mK/mK \cap A \longrightarrow 0$$

and

$$0 \longrightarrow K[m] \longrightarrow K \longrightarrow mK \longrightarrow 0$$

it now follows that K is starred. Hence G is fully starred.

The following example shows that a p-group can have no elements of infinite height, be isomorphic in $\mathscr{A}/\mathscr{B}$ to a direct sum of cyclic groups, and not be a Q-group.

EXAMPLE 3.8. Let $B \simeq C(p) \oplus C(p^2) \oplus \cdots$ where $C(p^n)$ is cyclic of order p^n. Then $\bar{B}$ is uncountable, has no elements of infinite height and is not fully starred. Let C be countable with $C^1 \neq 0$. In [9] it is shown that $\operatorname{Tor}(\bar{B}, C)$ is fully starred, has no elements of infinite height, and is not a Q-group. The exact sequence

$$0 \longrightarrow C(p) \longrightarrow P \longrightarrow B \longrightarrow 0$$

with P the Prufer group and $C(p) = P^1$ yields the exact sequence

$$0 \longrightarrow \operatorname{Tor}(\bar{B}, C(p)) \longrightarrow \operatorname{Tor}(\bar{B}, P) \longrightarrow \operatorname{Tor}(\bar{B}, B) .$$

But $\operatorname{Tor}(\bar{B}, C(p))$ is p-bounded and $\operatorname{Tor}(\bar{B}, B)$ is a direct sum of cyclic groups. Thus $\operatorname{Tor}(\bar{B}, P)$ is the desired group.

4. Ulm's theorem in $\mathscr{A}/\mathscr{B}$. All groups considered in this section are reduced primary Abelian groups for a fixed prime p. Let G be such a group and let α be an ordinal number. We define $p^\alpha G$ inductively as follows: $p^0 G = G$, $p^\alpha G = p(p^{\alpha-1}G)$ if $\alpha - 1$ exists, and $p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$ if α is a limit ordinal. The dimension $f_G(\alpha)$ of the vector space $(p^\alpha G)[p]/(p^{\alpha+1}G)[p]$ is called the αth Ulm invariant of G. If G is a direct sum of cyclic groups, then $f_G(n)$ is just the number of cyclic summands of G of order p^{n+1}. We define G inductively as follows: $G^0 = G$, $G^\alpha = p^\omega(G^{\alpha-1})$ if $\alpha - 1$ exists, and $G^\alpha = \bigcap_{\beta<\alpha} G^\beta$ if α is a limit ordinal. Since G is reduced, there is a smallest ordinal τ, $|\tau| \leqq |G|$, such that $G^\tau = 0$. τ is called the length of G. The quotient groups $G^\alpha/G^{\alpha+1}$, $\alpha < \tau$, are called the Ulm factors of G. All the Ulm factors except possibly the last, if it exists, are unbounded. Moreover, if G is a direct sum of countable groups, then $G^\alpha/G^{\alpha+1}$ is a direct sum of cyclic groups for each $\alpha < \tau$. The relationship between

$p^\alpha G$ and G^α is given by $G^\alpha = p^{\omega\alpha}G$. Finally an element $x \in G$ is said to have height α in G if $x \in p^\alpha G$, $x \notin p^{\alpha+1}G$, and we write $H_G(x) = \alpha$.

One of the most important theorems in the theory of Abelian groups is Ulm's theorem, which states that two countable reduced p-groups are isomorphic if and only if their corresponding Ulm factors are isomorphic, or equivalently, if and only if they have the same Ulm invariants. This result has been extended to direct sums of countable reduced p-groups by G. Kolettis [12]. Another proof is given by P. Hill [7].

The main result in this section is an "Ulm's theorem" for direct sums of countable reduced p-groups in the category $\mathscr{A}/\mathscr{B}$. In the course of the proof of this theorem, several subsidiary results are obtained. These will be stated as they arise.

Let G and H be reduced p-groups. Suppose

(I) there is an integer $k \geqq 0$ such that for all integers $n \geqq 0$ and $r \geqq 0$

$$\sum_{j=0}^{r} f_G(n+k+j) \leqq \sum_{j=0}^{r+2k} f_H(n+j),\ \sum_{j=0}^{r} f_H(n+k+j) \leqq \sum_{j=0}^{r+2k} f_G(n+j)\ ,$$

and

(II) $f_G(\alpha) = f_H(\alpha)$ for all $\alpha \geqq \omega$.

In [1] Beaumont and Pierce show that (I) and (II) are necessary and sufficient for two countable reduced p-groups to be quasi-isomorphic. In [2] Beaumont and Pierce show that (I) is necessary and sufficient for two direct sums of cyclic p-groups to be quasi-isomorphic. In view of Kolettis extension of Ulm's theorem, it is natural to ask if (I) and (II) are necessary and sufficient for two direct sums of countable reduced p-groups to be quasi-isomorphic. The answer is in the affirmative, and the proof is straightforward since $nG \subseteqq S \subseteqq G$ and G a direct sum of countable groups implies that S is a direct sum of countable groups. (For a proof of this latter fact, see Irwin and Richman [9].) In [8] P. Hill proves the following more general statement. If G and H are reduced p-groups with G/G^1 and H/H^1 direct sums of cyclic groups, then (I) and (II) are necessary and sufficient for G and H to be quasi-isomorphic. Hill also includes in [8] a short and informative proof of the previously mentioned result by Beaumont and Pierce [2].

While the following lemma does not appear in [8], it is an immediate consequence of a result in [8]. Moreover, it has an application later on.

LEMMA 4.1. *Let G and H be direct sums of cyclic p-groups which satisfy* (I). *Suppose $f_G(0) = |G|\,|H|\,\aleph_0 = f_H(0)$. Then*

$$G = G_0 \oplus G_1 \oplus \cdots \oplus G_{2k} \quad \text{and} \quad H = H_0 \oplus H_1 \oplus \cdots \oplus H_{2k}$$

where

$$p^kG_0 \simeq H_0,\ p^{k-1}G_1 \simeq H_1,\ \cdots,\ G_k \simeq H_k,\ G_{k+1} \simeq pH_{k+1},\ \cdots,\ G_{2k} \simeq p^kH_{2k}\ .$$

Proof. For each integer $n \geqq 0$, let A_n be the initial segment of ordinals less than the first ordinal of cardinality $f_G(n)$. Similarly, let B_n be the initial segment of ordinals less than the first ordinal of cardinality $f_H(n)$. Define

$$A = \{(n, \alpha) \mid 0 \leqq n < \omega, \alpha \in A_n\}\ ,$$
$$B = \{(n, \beta) \mid 0 \leqq n < \omega, \beta \in B_n\}\ .$$

Write $G = \sum_{n<\omega} G_n$ where G_n is a direct sum of cyclic groups of order p^{n+1}. Then $G_n = \sum_{\alpha \in A_n} \langle a_{n,\alpha}\rangle$ and $(n, \alpha) \longrightarrow a_{n,\alpha}$ is an equivalence between A and a basis of G. Write $H = \sum_{n<\omega} \sum_{\beta \in B_n} \langle b_{n,\beta}\rangle$ where $\langle b_{n,\beta}\rangle \simeq C(p^{n+1})$ for each $\beta \in B_n$. Then $b_{n,\beta} \longrightarrow (n, \beta)$ is an equivalence between B and a basis of H. In [8] Hill shows that there is an equivalence between B and A which alters indicies by no more than k, the index of the element (n, γ) of A or B being n. Thus we have an equivalence π between a basis of H and a basis of G which alters exponents by no more than k. For $0 \leqq i \leqq 2k$, define

$$G_i = \sum \{\langle a_{n,\alpha}\rangle \mid \pi(b_{m,\beta}) = a_{n,\alpha},\ n - m = k - i\}\ ,$$
$$H_i = \sum \{\langle b_{m,\beta}\rangle \mid \pi(b_{m,\beta}) = a_{n,\alpha},\ n - m = k - i\}\ .$$

The lemma follows.

Let G be a group. A subgroup H of G is essential in G if $K \subseteqq G$ and $K \cap H = 0$ imply $K = 0$. A group X is an n-extension of G if G is essential in X and $nX = G$. Every group has an n-extension. (See Walker [15].) In fact G^n is an n-extension of G if $G^n/G = (D/G)[n]$ where D is a divisible hull of G. This notion is used in the proof of the following corollary.

COROLLARY 4.2. *Let G and H be direct sums of cyclic p-groups which satisfy* (I). *Then there are subgroups $S \subseteqq G$ and $T \subseteqq H$ such that $p^kG \subseteqq S$, $p^kH \subseteqq T$, and $S \simeq T$.*

Proof. If $k = 0$, then $f_G(n) = f_H(n)$ for all integers $n \geqq 0$ and so $G \simeq H$. Assume $k > 0$ and let G^* and H^* be such that $pG^* = G$, $pH^* = H$, and $f_{G^*}(0) = |G^*|\,|H^*|\,\aleph_0 = f_{H^*}(0)$. For example, take p-extensions of G and H, respectively, and then add on summands $\sum_{\aleph} C(p)$ where $\aleph = |G^*|\,|H^*|\,\aleph_0$. Write $k = m + 1$. Then

$$\sum_{j=0}^{r} f_{G^*}(n + k + j) = \sum_{j=0}^{r} f_{pG^*}(n + m + j) \leqq \sum_{j=0}^{r+2m} f_{pH^*}(n + j)$$
$$= \sum_{j=0}^{r+2m} f_{H^*}(n + 1 + j) \leqq \sum_{j=0}^{r+2k} f_{H^*}(n + j)\ .$$

for all integers $n \geqq 0$ and $r \geqq 0$. Similarly

$$\sum_{j=0}^{r} f_{H^*}(n+k+j) \leqq \sum_{j=0}^{r+2k} f_{G^*}(n+j) .$$

Now write $G^* = G_0 \oplus \cdots \oplus G_{2k}$ and $H^* = H_0 \oplus \cdots \oplus H_{2k}$ according to 4.1. Let $S^* = p^k G_0 \oplus p^{k-1} G_1 \oplus \cdots \oplus G_k \oplus \cdots \oplus G_{2k}$ and let $T^* = H_0 \oplus \cdots \oplus H_k \oplus pH_{k+1} \oplus \cdots \oplus p^k H_{2k}$. Then $p^k G^* \subseteqq S^*$, $p^k H^* \subseteqq T^*$, and $S^* \simeq T^*$. Now $p^k G = p^{k+1} G^* \subseteqq pS^* \subseteqq pG^* = G$, $p^k H \subseteqq pT^* \subseteqq H$, and $pS^* \simeq pT^*$.

LEMMA 4.3. *Let H be a reduced p-group and n and r positive integers. Then*

$$\dim ((p^n H)[p]/(p^{n+r+1}H)[p]) = \sum_{j=0}^{r} f_H(n+j) .$$

Proof. Since

$$(p^n H)[p]/(p^{n+2}H)[p] \simeq ((p^{n+1}H)[p]/(p^{n+2}H)[p]) \oplus X$$

where $X \simeq (p^n H)[p]/(p^{n+1}H)[p]$, the lemma follows from induction and the definition of f_H.

LEMMA 4.4. *Let G be a reduced p-group, $k \geqq 0$, and $A \subseteqq G[p^k]$. Then for all integers $n \geqq 0$ and $r \geqq 0$,*

$$\sum_{j=0}^{r} f_{G/A}(n+j) \leqq \sum_{j=0}^{r+k} f_G(n+j) \quad \text{and} \quad \sum_{j=0}^{r} f_G(n+k+j) \leqq \sum_{j=0}^{r+k} f_{G/A}(n+j) .$$

Proof. Let B be a basic subgroup of G, and write $B = \sum_{i=1} B_i$ with each B_i a direct sum of cyclic groups of order p^i. Let

$$\{x_i + A + ((p^{n+r+1}G + A)/A)[p]\}_{i \in I}$$

be a basis of $((p^n G + A)/A)[p]/((p^{n+r+1}G + A)/A)[p]$, with each $x_i \in p^n G$, $x_i \notin p^{n+r+1}G$. Applying 4.3 to the group G/A, we have

$$|I| = \sum_{j=0}^{r} f_{G/A}(n+j) .$$

Since $n \leqq H_G(x_i) < n+r+1$ and $E(x_i) \leqq k+1$, $x_i = c_i + d_i$ where $0 \neq c_i \in B_{n+1} \oplus \cdots \oplus B_{n+k+r+1}$, $c_i \notin p^{n+r+1}G$, and $d_i \in p^{n+r+1}G$. Let $C = B_{n+1} \oplus \cdots \oplus B_{n+k+r+1}$. Now

$$\frac{((C + p^{n+r+1}G + A)/A)[p]}{((p^{n+r+1}G + A)/A)[p]} \supseteqq \frac{((p^n G + A)/A)[p]}{((p^{n+r+1}G + A)/A)[p]} .$$

Hence

$$\begin{aligned}
\sum_{j=0}^{r} f_{G/A}(n+j) &= |I| \\
&\leqq \dim(((C + p^{n+r+1}G + A)/A)[p]/((p^{n+r+1}G + A)/A)[p]) \\
&\leqq \dim((((C + p^{n+r+1}G + A)/A)/((p^{n+r+1}G + A)/A))[p]) \\
&= \dim(((C + p^{n+r+1}G + A)/p^{n+r+1}G + A))[p]) \\
&= \dim((C/(C(p^{n+r+1}G + A)))[p]) \\
&\leqq \dim(C[p]) = \sum_{j=0}^{r+k} f_B(n+j) = \sum_{j=0}^{r+k} f_G(n+j) .
\end{aligned}$$

Since $f_{p^kG}(m) = f_G(m+k)$ for all $m \geqq 0$, and $G[p^k]/A \subseteqq (G/A)[p^k]$, applying the first inequality to G/A and the subgroup $G[p^k]/A$ yields

$$\begin{aligned}
\sum_{j=0}^{r} f_G(n+k+j) &= \sum_{j=0}^{r} f_{p^kG}(n+j) \\
&= \sum_{j=0}^{r} f_{(G/A)/(G[p^k]/A)}(n+j) \leqq \sum_{j=0}^{r+k} f_{G/A}(n+j) .
\end{aligned}$$

This completes the proof.

Corollary 4.5. *Let G and H be reduced p-groups with $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Then G and H satisfy* (I).

Proof. By 3.1., there is a $k \geqq 0$ and a homomorphism $f\colon G \to H$ such that $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are bounded by p^k. Let $A = \operatorname{Ker} f$ and $T = \operatorname{Im} f$. By 4.4,

$$\sum_{j=0}^{r} f_{G/A}(n+j) \leqq \sum_{j=0}^{r+k} f_G(n+j) \quad \text{and} \quad \sum_{j=0}^{r} f_G(n+k+j) \leqq \sum_{j=0}^{r+k} f_{G/A}(n+j)$$

for all $n \geqq 0$ and all $r \geqq 0$. Now

$$p^kH \subseteqq T \subseteqq H,\ (p^{n+k}H)[p]/(p^{n+r+k+1}H)[p] \subseteqq (p^nT)[p]/(p^{n+r+k+1}H)[p]$$

and the latter is an image of $(p^nT)[p]/(p^{n+r+k+1}T)[p]$. Also

$$(p^nT)[p]/(p^{n+r+k+1}T)[p]\ (p^nH)[p]/(p^{n+r+k+1}T)[p]$$

and the latter is an image of $(p^nH)[p]/(p^{n+r+2k+1}H)[p]$. Thus, applying 4.3 and 4.4,

$$\sum_{j=0}^{r} f_H(n+k+j) \leqq \sum_{j=0}^{r+k} f_T(n+j) = \sum_{j=0}^{r+k} f_{G/A}(n+j) \leqq \sum_{j=0}^{r+2k} f_G(n+j)$$

and

$$\sum_{j=0}^{r} f_G(n+k+j) \leqq \sum_{j=0}^{r+k} f_{G/A}(n+j) = \sum_{j=0}^{r+k} f_T(n+j) \leqq \sum_{j=0}^{r+2k} f_H(n+j) .$$

Corollary 4.6. *Let G and H be reduced p-groups with basic* *subgroups B and C, respectively. Then $B \doteq C$ whenever $G \simeq H$ in $\mathscr{A}/\mathscr{B}$.*

Proof. B and C satisfy (I) whenever G and H do. Now apply the result in [2].

COROLLARY 4.7. *Let $\bar{B}$ and $\bar{C}$ be closed p-groups with basic subgroups B and C, respectively. Then $\bar{B} \simeq \bar{C}$ in $\mathscr{A}/\mathscr{B}$ if and only if $\bar{B} \doteq \bar{C}$.*

Proof. $\bar{B} \doteq \bar{C}$ in $\mathscr{A}/\mathscr{B}$ implies $B \doteq C$ by 4.6, and $B \doteq C$ implies $\bar{B} \simeq \bar{C}$ is Theorem 3.11 in [3].

COROLLARY 4.8. *Let G and H be reduced p-groups and suppose G/G^1 and H/H^1 are direct sums of cyclic groups. If $G \simeq H$ in $\mathscr{A}/\mathscr{B}$ and $G^1 \simeq H^1$, then $G \doteq H$.*

Proof. In [8], Hill shows that $G \doteq H$ if G and H satisfy (I) and (II) and G/G^1 and H/H^1 are direct sums of cyclic groups. The corollary now follows from 4.5.

In view of 4.6 and 4.7, it is natural to ask the following question. If $G \simeq H$ in $\mathscr{A}/\mathscr{B}$, is $G \doteq H$ whenever G and H are reduced p-groups without elements of infinite height? The answer is no and 3.8 furnishes us with a counterexample. Specifically, there is a p-group G and a subgroup $A \subseteqq G[p]$ such that

(i) $G^1 = 0$,

(ii) G is not a direct sum of cyclic groups,

(iii) $(G/A)^1 = 0$,

(iv) G/A is a direct sum of cyclic groups.

Now $G \simeq G/A$ in $\mathscr{A}/\mathscr{B}$, but $G \doteq G/A$ contradicts (ii) and (iv).

LEMMA 4.9. *Let G and H be direct sums of cyclic p-groups such that $f_G(0) = |G|\,|H|\,\aleph_0 = f_H(0)$. Suppose there is an integer $k \geqq 0$ such that for all integers $n \geqq 0$ and $r \geqq 0$*

$$\sum_{j=0}^{r} f_H(n+j) \leqq \sum_{j=0}^{r+k} f_G(n+j) \quad \textit{and} \quad \sum_{j=0}^{r} f_G(n+k+j) \leqq \sum_{j=0}^{r+k} f_H(n+j)\ .$$

Then there is an equivalence of a basis of G onto a basis of H such that exponents are never increased and are not decreased by more than k.

Proof. For each integer $n \geqq 0$, define A_n and B_n as in the proof of 4.1. Also define A and B as in 4.1. Call the first component of the element (n, γ) of A or B the index of the element. It was pointed out in the proof of 4.1 that A is in one-to-one correspondence with a basis of G such that if the index of an element of A is n, then the exponent of the corresponding basis element of G is $n+1$. There is a similar one-to-one correspondence between B and a basis of H. Hill [8] has shown that (I) implies the existence of an equivalence between A and B which alters indicies by no more than k. By a close examination of Hill's proof, it is seen that the stronger inequalities in the hypothesis of 4.9 imply the existence of an equivalence θ from A onto B with the index of $\theta(n, \alpha)$ between $n-k$ and n. Thus there is an equivalence from a basis of G onto a basis of H such that exponents are never increased and are not decreased by more than k.

COROLLARY 4.10. *Let G and H be direct sums of cyclic p-groups and k a positive integer such that for all integers $n \geqq 0$ and $r \geqq 0$*

$$\sum_{j=0}^{r} f_H(n+j) \leqq \sum_{j=0}^{r+k} f_G(n+j) \quad \text{and} \quad \sum_{j=0}^{r} f_G(n+k+j) \leqq \sum_{j=0}^{r+k} f_H(n+j) .$$

Then there is a subgroup $S \subseteqq G$ such that $p^kG \subseteqq S$ and $S \simeq H$.

Proof. If $k = 0$, then $f_G(n) = f_H(n)$ for all $n \geqq 0$ and so $G \simeq H$. Assume $k \geqq 1$, let $pG^* = G$, $pH^* = H$, and $f_{G^*}(0) = |G^*|\,|H^*|\,\aleph_0 = f_{H^*}(0)$. Write $k = m+1$. Let $n \geqq 0$ and $r \geqq 0$. Then

$$\sum_{j=0}^{r} f_{G^*}(n+k+j) = \sum_{j=0}^{r} f_{pG^*}(n+m+j) \leqq \sum_{j=0}^{r+m} f_{pH^*}(n+j) \leqq \sum_{j=0}^{r+k} f_{H^*}(n+j) .$$

If $n = 0$, then

$$\sum_{j=0}^{r} f_{H^*}(j) = f_{H^*}(0) = f_{G^*}(0) = \sum_{j=0}^{r+k} f_{G^*}(j) .$$

If $n \geqq 1$, say $n = m+1$, then

$$\sum_{j=0}^{r} f_{H^*}(n+j) = \sum_{j=0}^{r} f_{pH^*}(m+j) \leqq \sum_{j=0}^{r+k} f_{pG^*}(m+j) = \sum_{j=0}^{r+k} f_{G^*}(n+j) .$$

Thus G^* and H^* satisfy the hypothesis of 4.9. Let A be a basis of G^* and B a basis of H^*. Let π be an equivalence of A onto B determined by 4.9. For $0 \leqq i \leqq k$, define

$$G_i = \sum \{\langle a \rangle \mid a \in A,\ \pi(a) = b,\ E(a) - E(b) = k - i\} ,$$
$$H_i = \sum \{\langle b \rangle \mid b \in B,\ \pi(a) = b,\ E(a) - E(b) = k - i\} .$$

Clearly $G^* = G_0 \oplus \cdots \oplus G_k$, $H^* = H_0 \oplus \cdots \oplus H_k$, $p^{k-i}G_i \simeq H_i$ for $i = 1, \cdots, k$. Let $S^* = p^kG_0 \oplus p^{k-1}G_1 \oplus \cdots \oplus G_k$. Then $S^* \simeq H^*$

and $p^k G^* \subseteqq S^* \subseteqq G$. Let $S = pS^*$. Now $S \simeq H$ and $p^k G = p^{k+1}G^* \subseteqq pS^* \subseteqq pG^* = G$. This completes the proof.

COROLLARY 4.11. *Let G be a direct sum of cyclic p-groups, $k \geqq 0$, and $A \subseteqq G[p^k]$. Suppose G/A is a direct sum of cyclic groups. Then there is a subgroup $S \subseteqq G$ such that $p^k G \subseteqq S$ and $G/A \simeq S$.*

Proof. Apply 4.4 to see that G and G/A satisfy the hypothesis of 4.10 with $H = G/A$.

Corollary 4.11 is a generalization of the well-known fact that the quotient of a finite group G is isomorphic to a subgroup of G.

PROPOSITION 4.12. Let G and H be reduced p-groups with $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Then $p^\alpha G \simeq p^\alpha H$ in $\mathscr{A}/\mathscr{B}$, and $G^\alpha/G^{\alpha+1} \simeq H^\alpha/H^{\alpha+1}$ in $\mathscr{A}/\mathscr{B}$.

Proof. This is a corollary of 3.2.

THEOREM 4.13. *Let G and H be reduced p-groups with $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Then there exists an integer $k \geqq 0$ such that for all ordinals α and for all $r \geqq 0$*

$$\sum_{j=0}^{r} f_G(\alpha + k + j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha + j)$$

and

$$\sum_{j=0}^{r} f_H(\alpha + k + j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha + j) .$$

Proof. Since $p^\alpha G \simeq p^\alpha H$ in $\mathscr{A}/\mathscr{B}$, $p^\alpha G$ and $p^\alpha H$ satisfy (I). Moreover, by 3.1 and 4.5, k is the same for each α. Since $f_G(\alpha + n) = f_{p^\alpha G}(n)$, the theorem follows.

Let G and H be countable reduced p-groups and suppose $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. By 4.12 and 4.6, we see that the corresponding Ulm factors of G and H are quasi-isomorphic. The following example shows that the converse does not hold.

EXAMPLE 4.14. Let G and H be countable groups whose respective Ulm factors are

$$G_n \simeq C(p^{n+1}) \oplus C(p^{n+2}) \oplus \cdots \qquad \text{for} \quad n \geqq 0,\ G_\omega = 0$$

and

$$H_n \simeq C(p^{2n+2}) \oplus C(p^{2n+3}) \oplus \cdots \qquad \text{for} \quad n \geqq 0,\ H_\omega = 0 .$$

Clearly $G_n \doteq H_n$. In fact $p^{n+1}H_n \simeq G_n$. Suppose $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. By 3.1, there is an integer $n \geqq 0$ and homomorphisms $f: G \to H$ and $g: H \to G$ such that $gf = p^n$ and $fg = p^n$. Now f and g induce homomorphisms $f: G_n \to H_n$ and $g: H_n \to G_n$ such that $gf = p^n$ and $fg = p^n$. Let x generate a summand of G_n of order p^{n+1}. Then $f(x) = p^{n+1}y$ for some $y \in H_n$. Thus $0 \neq p^n x = gf(x) = p^{n+1}g(y)$ which is a contradiction. Hence G and H are not isomorphic in $\mathscr{A}/\mathscr{B}$.

In view of Example 4.14, we make the following definition.

DEFINITION 4.15. Let $\{G_\alpha\}_{\alpha \in I}$ and $\{H_\alpha\}_{\alpha \in I}$ be families of groups over the same index set. These families are *uniformly quasi-isomorphic* if there is an integer $k \geqq 0$ and for each $\alpha \in I$, subgroups $S_\alpha \subseteqq G_\alpha$ and $T_\alpha \subseteqq H_\alpha$ such that $p^k G_\alpha \subseteqq S_\alpha$, $p^k H_\alpha \subseteqq T_\alpha$, and $S_\alpha \simeq T_\alpha$.

COROLLARY 4.16. *Let G and H be direct sums of countable reduced p-groups. If $G \simeq H$ in $\mathscr{A}/\mathscr{B}$, then the corresponding Ulm factors of G and H are uniformly quasi-isomorphic.*

Proof. Apply 4.13 to see that the corresponding Ulm factors of G and H satisfy (I) with k fixed. Now apply 4.2.

LEMMA 4.17. *Let G be a reduced p-group. Then $(G/A)^\beta = G^\beta/A$ for all $\beta \leqq \alpha$ whenever $A \subseteqq G^\alpha$.*

Proof. The proof is by induction on β. For $\beta = 0$, the lemma is clear. Suppose $\beta = 1$ and let $x + A \in (G/A)^1$. Then $x + A = p^n x_n + A$ for each $n \geqq 0$. So $x = p^n x_n + a_n \in p^n G$ for each $n \geqq 0$. Thus $x + A \in G^1/A$. Now let $\beta > 1$ and assume $(G/A)^\gamma = G^\gamma/A$ for all $\gamma < \beta$. If $\beta = \gamma + 1$, then $(G/A)^\beta = (G^\gamma/A)^1 = G^{\gamma+1}/A = G^\beta/A$. Suppose β is a limit ordinal. Then

$$(G/A)^\beta = \bigcap_{\gamma<\beta} (G/A)^\gamma = \bigcap_{\gamma<\beta} (G^\gamma/A) = (\bigcap_{\gamma<\beta} G^\gamma)/A = G^\beta/A .$$

The proof is complete.

LEMMA 4.18. *Let G be a reduced p-group and $k \geqq 0$. Then $(G/G^\alpha[p^k])^\beta = G^\beta/G^\alpha[p^k]$ for $\beta \leqq \alpha$ and $(G/G^\alpha[p^k])^\alpha \simeq p^k G^\alpha$.*

Proof. The lemma follows from 4.17.

COROLLARY 4.19. *Let G be a reduced p-group, α an ordinal, and $k \geqq 0$. Then $G/G^\alpha[p^k]$ has Ulm factors $G^\beta/G^{\beta+1}$ for $\beta < \alpha$, $p^k G^\alpha/G^{\alpha+1}$, and $G^\beta/G^{\beta+1}$ for $\beta > \alpha$.*

Proof. Let $\beta < \alpha$. Then $\beta + 1 \leqq \alpha$ and by 4.17,

$$(G/G^\alpha[p^k])^\beta/(G/G^\alpha[p^k])^{\beta+1} = (G^\beta/G^\alpha[p^k])/(G^{\beta+1}/G^\alpha[p^k]) \simeq G^\beta/G^{\beta+1} .$$

By 4.18 and since $X^\alpha/X^{\alpha+1} \simeq Y^\alpha/Y^{\alpha+1}$ whenever X and Y are isomorphic p-groups, $(G/G^\alpha[p^k])^\alpha/(G/G^\alpha[p^k])^{\alpha+1} \simeq p^kG^\alpha/G^{\alpha+1}$. Furthermore, if $\beta > \alpha$, say $\beta = \alpha + \delta$, then

$$\begin{aligned}(G/G^\alpha[p^k])^\beta/(G/G^\alpha[p^k])^{\beta+1} &\simeq (G^\alpha/G^\alpha[p^k])^\delta/(G^\alpha/G^\alpha[p^k])^{\delta+1} \\ &\simeq (p^kG^\alpha)^\delta/(p^kG^\alpha)^{\delta+1} = G^{\alpha+\delta}/G^{\alpha+\delta+1} = G^\beta/G^{\beta+1} .\end{aligned}$$

LEMMA 4.20. *Let G be a countable reduced p-group of length τ and with Ulm factors G_α. For each $\alpha < \tau$, let $G_\alpha = S_\alpha \oplus T_\alpha$ where both S_α and T_α are unbounded whenever G_α is. Let H be a countable reduced p-group of length τ and with Ulm factors $p^{n_\alpha}S_\alpha \oplus T_\alpha$ where $0 \leqq n_\alpha \leqq k$, k fixed. (Such a group exists by Zippin's theorem.) Then $G/A \simeq H$ for some $A \subseteqq G[p^k]$.*

Proof. For $\tau = 0$, there is nothing to prove. If $\tau = 1$, G is a direct sum of cyclic groups and the lemma is clear. Assume $\tau > 1$. For the present, we will assume $G_{\tau-1}$ is unbounded if $\tau - 1$ exists. For each $\alpha < \tau$, let $T_\alpha = \sum_{\beta<\tau} G_{\alpha\beta}$ where $G_{\alpha\beta}$ is unbounded for $\beta \neq \alpha$ and $G_{\alpha\alpha} = 0$. For each $\alpha < \tau$, let K_α and L_α be countable reduced p-groups of length τ whose αth Ulm factors are $p^{n_\alpha}S_\alpha$ and S_α, respectively, and whose βth Ulm factors, $\beta \neq \alpha$, are $G_{\beta\alpha}$. Now $G \simeq \sum_{\alpha<\tau} L_\alpha$ and applying 4.18, $L_\alpha/L_\alpha^\alpha[p^{n_\alpha}] \simeq K_\alpha$ for each $\alpha < \tau$. Let $K = \sum_{\alpha<\tau} K_\alpha$. Then $G/A \simeq K$ where $A = \sum_{\alpha<\tau} L_\alpha^\alpha[p^{n_\alpha}]$. Moreover, K has Ulm factors $p^{n_\alpha}S_\alpha \oplus T_\alpha$. Hence $G/A \simeq K \simeq H$.

Now assume $\tau - 1$ exists and $G_{\tau-1}$ is bounded. For $\alpha < \tau - 1$, Let T_α be as above. Let K_0 be a countable reduced p-group whose Ulm factors are $p^{n_0}S_0$, $G_{\beta 0}$ for $0 < \beta < \tau - 1$, and $T_{\tau-1}$. Let L_0 be a countable reduced p-group whose Ulm factors are S_0, $G_{\beta 0}$ for $0 < \beta < \tau - 1$, and $T_{\tau-1}$. For $0 < \alpha < \tau - 1$, let K_α and L_α be countable reduced p-groups of length $\tau - 1$ whose αth Ulm factors are $p^{n_\alpha}S_\alpha$ and S_α, respectively, and whose βth Ulm factors, $\beta \neq \alpha$, are $G_{\beta\alpha}$. Let $K_{\tau-1}$ and $L_{\tau-1}$ be countable reduced p-groups whose Ulm factors are $G_{\beta,\tau-1}$ for $0 \leqq \beta < \tau - 1$ and whose $(\tau - 1)$ st Ulm factors are $p^{n_{\tau-1}}S_{\tau-1}$ and $S_{\tau-1}$, respectively. Let $K = \sum_{\alpha<\tau} K_\alpha$ and $A = \sum_{\alpha<\tau} L_\alpha^\alpha[p^{n_\alpha}]$. As above $G/A \simeq K \simeq H$. This completes the proof.

LEMMA 4.21. *Let G be as in 4.20. For each $\alpha < \tau$, let $G_\alpha = S_\alpha \oplus T_\alpha$. Let H be a countable reduced p-group with Ulm factors $p^nS_\alpha \oplus T_\alpha$, $n \geqq 0$. Then $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. In fact, there is an epimorphism $f\colon G \to p^nH$ such that* Ker f *is bounded by p^{3n}.*

Proof. Let $I_1 = \{\alpha < \tau \mid S_\alpha$ and T_α are both unbounded$\}$, $I_2 = \{\alpha < \tau \mid S_\alpha$ is unbounded and T_α is bounded$\}$, $I_3 = \{\alpha < \tau \mid S_\alpha$ is bounded and T_α is unbounded, and $I_4 = \{\tau - 1\}$ if $\tau - 1$ exists and $G_{\tau-1}$ is bounded, otherwise $I_4 = \varnothing$. If $\alpha \in I_2$, let $S_\alpha = S'_\alpha \oplus S''_\alpha$ where both S'_α and S''_α are unbounded. If $\alpha \in I_3$, let $T_\alpha = T'_\alpha \oplus T''_\alpha$ where both T'_α and T''_α are unbounded. Let K have Ulm factors as follows:

$$p^n S_\alpha \oplus T_\alpha \text{ if } \alpha \in I_1 \cap I_4 ,$$
$$p^n S'_\alpha \oplus S''_\alpha \oplus T_\alpha = P^n S_\alpha \oplus T_\alpha \text{ if } \alpha \in I_2 ,$$
$$p^n S_\alpha \oplus p^n T'_\alpha \oplus T''_\alpha \text{ if } \alpha \in I_3 .$$

By 4.20, $G/A \simeq K$ for some $A \subseteqq G[p^n]$. Let L have Ulm factors as follows:

$$p^n S_\alpha \oplus T_\alpha \text{ if } \alpha \in I_1 \cap I_4 ,$$
$$p^n S'_\alpha \oplus p^n S''_\alpha \oplus T_\alpha = p^n S_\alpha \oplus T_\alpha \text{ if } \alpha \in I_2 ,$$
$$p^n S_\alpha \oplus p^n T'_\alpha \oplus T''_\alpha \text{ if } \alpha \in I_3 .$$

By 4.20, $K/B \simeq L$ for some $B \subseteqq G[p^n]$. Let H have Ulm factors as follows:

$$p^n S_\alpha \oplus T_\alpha \text{ if } \alpha \in I_1 \cap I_2 \cap I_4 ,$$
$$p^n S_\alpha \oplus T'_\alpha \oplus T''_\alpha = p^n S_\alpha \oplus T_\alpha \text{ if } \alpha \in I_3 .$$

By 4.20, $H/C \simeq L$ for some $C \subseteqq H[p^n]$. Hence $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Let f be the composition $G \to G/A \simeq K \to K/B \simeq L \simeq H/C \to H/H[p^n] \simeq p^n H$. The lemma follows.

LEMMA 4.22. *Let G be a direct sum of countable reduced p-groups, $n \geqq 0$, and suppose G has length τ. Let G have Ulm factors G_α. For each $\alpha < \tau$, let $G_\alpha = S_\alpha \oplus T_\alpha$. Then there is a p-group H such that*

(i) *H is a direct sum of countable reduced groups,*
(ii) *H has Ulm factors $p^n S_\alpha \oplus T_\alpha$,*
(iii) *$G \simeq H$ in $\mathscr{A}/\mathscr{B}$.*

Proof. Let $G = \sum_{\lambda \in \Lambda} X_\lambda$ where $|X_\lambda| \leqq \aleph_0$ for each $\lambda \in \Lambda$. Let τ_λ be the length of X_λ and $X_{\lambda\alpha}$ its αth Ulm factor. For $\tau_\lambda \leqq \alpha < \tau$, $X_{\lambda\alpha} = 0$. Now $G_\alpha = \sum_{\lambda \in \Lambda} X_{\lambda\alpha}$ for each $\alpha < \tau$. Let $X^B_{\lambda\alpha}$, S^B_α, and T^B_α be bases of $X_{\lambda\alpha}$, S_α, and T_α, respectively. For each $\alpha < \tau$, there is an equivalence $\pi_\alpha \colon \bigcup_{\lambda \in \Lambda} X^B_{\lambda\alpha} \to S^B_\alpha \cup T^B_\alpha$ which preserves exponents. For $\lambda \in \Lambda$ and $\alpha < \tau_\lambda$, define

$$S_{\lambda\alpha} = \sum \{\langle a \rangle \mid a \in X^B_{\lambda\alpha}, \pi_\alpha(a) \in S^B_\alpha\} ,$$
$$T_{\lambda\alpha} = \sum \{\langle a \rangle \mid a \in X^B_{\lambda\alpha}, \pi_\alpha(a) \in T^B_\alpha\} .$$

If $\tau_\lambda \leqq \alpha < \tau$, let $S_{\lambda\alpha} = T_{\lambda\alpha} = 0$. Then $X_{\lambda\alpha} = S_{\lambda\alpha} \oplus T_{\lambda\alpha}$, $\sum_{\lambda\in\Lambda} S_{\lambda\alpha} \simeq S_\alpha$, and $\sum_{\lambda\in\Lambda} T_{\lambda\alpha} \simeq T_\alpha$. Let $Y_{\lambda\alpha} = p^n S_{\lambda\alpha} \oplus T_{\lambda\alpha}$. Then $Y_{\lambda\alpha}$ is countable and if $\alpha < \tau_\lambda$, $Y_{\lambda\alpha}$ is unbounded with the possible exception of $Y_{\lambda,\tau_\lambda - 1}$ if $\tau_\lambda - 1$ exists. Let Y_λ be a countable reduced p-group of length τ_λ and with Ulm factors $Y_{\lambda\alpha}$. By 4.20, there is an epimorphism $f_\lambda: X_\lambda \to p^n Y_\lambda$ such that $p^{3n} \operatorname{Ker} f = 0$. Let $H = \sum_{\lambda\in\Lambda} Y_\lambda$. Then there is an epimorphism $f: G \to p^n H$ such that $p^{3n} \operatorname{Ker} f = 0$. Hence $G \simeq H$ in $\mathscr{A}/\mathscr{B}$. Clearly H is a direct sum of countable groups with Ulm factors $p^n S_\alpha \oplus T_\alpha$.

LEMMA 4.23. *Let G be a direct sum of countable reduced p-groups. Let G have length τ and Ulm factors G_α. Then there is a reduced p-group H such that*

(i) *H is a direct sum of countable groups,*

(ii) *H has Ulm factors $G_\alpha \oplus \sum_{|G_\alpha|\aleph_0} C(p)$, $\alpha < \tau$,*

(iii) *$G \simeq H$ in $\mathscr{A}/\mathscr{B}$.*

Proof. Write $G = \sum_{\lambda\in\Lambda} X_\lambda$ where $|X_\lambda| \leqq \aleph_0$ for each $\lambda \in \Lambda$. Let τ_λ be the length of X_λ and $X_{\lambda\alpha}$ its αth Ulm factor. Then $G_\alpha = \sum_{\lambda\in\Lambda} X_{\lambda\alpha}$. For $\lambda \in \Lambda$ and $\alpha < \tau_\lambda$, let $Y_{\lambda\alpha} \simeq X_{\lambda\alpha} \oplus \sum_{\aleph_0} C(p)$. If $\tau_\lambda \leqq \alpha < \tau$, let $Y_{\lambda\alpha} = 0$. Let Y_λ be the countable reduced p-group of length τ_λ and with Ulm factors $Y_{\lambda\alpha}$, $\alpha < \tau_\lambda$. For $\alpha < \tau$, let $\Lambda_\alpha = \{\lambda \in \Lambda \mid X_{\lambda\alpha} \neq 0\}$. Then $|G_\alpha| \aleph_0 = |\Lambda_\alpha| \aleph_0$ for $\alpha < \tau$. Let $H = \sum_{\lambda\in\Lambda} Y_\lambda$. Then H is a direct sum of countable reduced p-groups whose αth Ulm factor is

$$\begin{aligned} H_\alpha &= \textstyle\sum_{\lambda\in\Lambda} Y_{\lambda\alpha} \simeq \sum_{\lambda\in\Lambda} (X_{\lambda\alpha} \oplus \sum_{\aleph_0} C(p)) \\ &= \textstyle\sum_{\lambda\in\Lambda} X_{\lambda\alpha} \oplus \sum_{\lambda\in\Lambda} \sum_{\aleph_0} C(p) = G_\alpha \oplus \sum_{|G_\alpha|} \sum_{\aleph_0} C(p)\ . \end{aligned}$$

By 4.22, H is isomorphic in $\mathscr{A}/\mathscr{B}$ to the direct sum of countable groups whose Ulm factors are $G_\alpha \oplus p(\sum_{|G_\alpha|} \sum_{\aleph_0} C(p)) = G_\alpha$; namely, $H \simeq G$ in $\mathscr{A}/\mathscr{B}$.

LEMMA 4.24. *Let G and H be direct sums of countable reduced p-groups whose corresponding Ulm factors are uniformly quasi-isomorphic. Let G and H have the same length τ and Ulm factors G_α and H_α, respectively. Suppose $f_{G_\alpha}(0) = |G_\alpha| \geqq \aleph_0$ and $f_{H_\alpha}(0) = |H_\alpha| \geqq \aleph_0$ for each $\alpha < \tau$. Then there exists direct sums of countable reduced p-groups K and L with Ulm factors K_α and L_α, respectively, such that*

(i) *$f_{K_\alpha}(0) = |K_\alpha|\,|L_\alpha|\,\aleph_0 = f_{L_\alpha}(0)$ for each $\alpha < \tau$,*

(ii) *$G \simeq K$ in $\mathscr{A}/\mathscr{B}$,*

(iii) *$H \simeq L$ in $\mathscr{A}/\mathscr{B}$.*

Proof. Let $I_1 = \{\alpha < \tau \mid |G_\alpha| = |H_\alpha|\}$, $I_2 = \{\alpha < \tau \mid |G_\alpha| < |H_\alpha|\}$, and $I_3 = \{\alpha < \tau \mid |H_\alpha| < |G_\alpha|\}$. There is an integer $k \geqq 0$ and for each $\alpha < \tau$, subgroups $S_\alpha \subseteqq G_\alpha$ and $T_\alpha \subseteqq H_\alpha$ such that $p^k G_\alpha \subseteqq S_\alpha$, $p^k H_\alpha \subseteqq T_\alpha$, and $S_\alpha \simeq T_\alpha$. Thus $| H_\alpha < | G_\alpha | \geqq | p^{2k} H_\alpha |$ for $\alpha \in I_2$ and $| G_\alpha | > | H_\alpha | \geqq | p^{2k} G_\alpha |$ for $\alpha \in I_3$. If $\alpha \in I_2$, write $H_\alpha = H'_\alpha \oplus \sum_{|G_\alpha|} C(p)$. If $\alpha \in I_3$, write $G_\alpha = G'_\alpha \oplus \sum_{|H_\alpha|} C(p)$. By 4.22, there is a direct sum of countable reduced p-groups K whose Ulm factors K_α are G_α for $\alpha \in I_1 \cup I_2$ and $p^{2k} G'_\alpha \oplus \sum_{|H_\alpha|} C(p)$ for $\alpha \in I_3$ and such that $G \simeq K$ in $\mathscr{A}/\mathscr{B}$. Similarly, there is a direct sum of countable reduced p-groups L whose Ulm factors L_α are H_α for $\alpha \in I_1 \cup I_3$ and $p^{2k} H'_\alpha \oplus \sum_{|G_\alpha|} C(p)$ for $\alpha \in I_2$ and such that $L \simeq H$ in $\mathscr{A}/\mathscr{B}$. For $\alpha \in I_1 \cup I_2$, $f_{K_\alpha}(0) = | G_\alpha | = | K_\alpha |$. For

$$\begin{aligned} \alpha \in I_3,\ | H_\alpha | \geqq f_{K_\alpha}(0) \leqq | K_\alpha | &\leqq | p^{2k} G'_\alpha | + | H_\alpha | \\ &\leqq | G_\alpha | + | H_\alpha | \leqq | H_\alpha | + | H_\alpha | \,. \end{aligned}$$

By hypothesis, $| G_\alpha | \geqq \aleph_0$ and $| H_\alpha | \geqq \aleph_0$. Thus $| K_\alpha | = f_{K_\alpha}(0) = \min (| G_\alpha |, | H_\alpha |)$ for $\alpha < \tau$. Likewise, $| L_\alpha | = f_{L_\alpha}(0) = \min (| G_\alpha |, | H_\alpha |)$ for $\alpha < \tau$. It now follows that $f_{K_\alpha}(0) = | K_\alpha | \, | L_\alpha | \aleph_0 = f_{L_\alpha}(0)$ for $\alpha < \tau$.

THEOREM 4.25. *Let G and H be direct sums of countable reduced p-groups. Then $G \simeq H$ in $\mathscr{A}/\mathscr{B}$ if and only if their corresponding Ulm factors are uniformly quasi-isomorphic.*

Proof. Assume that the corresponding Ulm factors of G and H are uniformly quasi-isomorphic. Let G_α and H_α be the αth Ulm factors of G and H, respectively. If G has length τ and H has length $\tau + 1$, then H_τ is bounded, H/H_τ is a direct sum of countable groups, and $G \simeq H$ in $\mathscr{A}/\mathscr{B}$ if and only if $G \simeq H/H_\tau$ in $\mathscr{A}/\mathscr{B}$. Moreover, the corresponding Ulm factors of G and H/H_τ are uniformly quasi-isomorphic. Hence we may as well assume that G and H have the same length τ. By 4.23, there exist direct sums of countable reduced p-groups K and L such that K has Ulm factors

$$G_\alpha \oplus \sum_{|G_0|} \sum_{\aleph_0} C(p) \,,$$

L has Ulm factors $H_\alpha \oplus \sum_{|H_\alpha|} \sum_{\aleph_0} C(p)$, $G \simeq K$ in $\mathscr{A}/\mathscr{B}$, and $H \simeq L$ in $\mathscr{A}/\mathscr{B}$. By 4.16, the corresponding Ulm factors of G and K (H and L) are uniformly quasi-isomorphic. In particular the corresponding Ulm factors of K and L are uniformly quasi-isomorphic. Moreover, $G \simeq H$ in $\mathscr{A}/\mathscr{B}$ if and only if $K \simeq L$ in $\mathscr{A}/\mathscr{B}$. Hence we may as well assume that $f_{G_\alpha}(0) = |G_\alpha| \aleph_0$ and $f_{H_\alpha}(0) = | H_\alpha | \aleph_0$ for each $\alpha < \tau$. By 4.24, we may likewise assume that $f_{G_\alpha}(0) = | G_\alpha | \, | H_\alpha | \aleph_0 = f_{H_\alpha}(0)$ for each $\alpha < \tau$. There is an integer $k \geqq 0$ and for each $\alpha < \tau$, subgroups $S_\alpha \subseteqq G_\alpha$ and $T_\alpha \subseteqq H_\alpha$ such that $p^k G_\alpha \subseteqq S_\alpha$, $p^k H_\alpha \subseteqq T_\alpha$ and $S_\alpha \simeq T_\alpha$. Thus for each $\alpha < \tau$ and all integers $n \geqq 0$ and $r \geqq 0$

$$\sum_{j=0}^{r} f_G(n+k+j) \leqq \sum_{j=0}^{r+2k} f_H(n+j)$$

and

$$\sum_{j=0}^{r} f_H(n+k+j) \leqq \sum_{j=0}^{r+2k} f_G(n+j) .$$

(For a proof of this fact, see [1].) Now by 4.1, we can write $G_\alpha = G_{\alpha,0} \oplus \cdots \oplus G_{\alpha,2k}$ and $H_\alpha = H_{\alpha,0} \oplus \cdots \oplus H_{\alpha,2k}$ where $p^k G_{\alpha,0} \simeq H_{\alpha,0}$, $p^{k-1} G_{\alpha,1} \simeq H_{\alpha,1}, \cdots, G_{\alpha,k} \simeq H_{\alpha,k}$, $G_{\alpha,k+1} \simeq pH_{\alpha,k+1}, \cdots, G_{\alpha,2k} \simeq p^k H_{\alpha,2k}$ for each $\alpha < \tau$. Now apply 4.22 as needed.

THEOREM 4.26. *Let G and H be direct sums of countable reduced p-groups. Then $G \simeq H$ in $\mathscr{A}/\mathscr{B}$ if and only if there is an integer $k \geqq 0$ such that for all ordinals α and all integers $r \geqq 0$*

$$\sum_{j=0}^{r} f_G(\alpha+k+j) \leqq \sum_{j=0}^{r+2k} f_H(\alpha+j)$$

and

$$\sum_{j=0}^{r} f_H(\alpha+k+j) \leqq \sum_{j=0}^{r+2k} f_G(\alpha+j) .$$

Proof. Suppose the inequalities hold. Let G_α and H_α be the αth Ulm factors of G and H, respectively. Then for each ordinal α and $r \leqq 0$

$$\begin{aligned}\sum_{j=0}^{r} f_{G_\alpha}(n+k+j) &= \sum_{j=0}^{r} f_G(\omega\alpha+n+k+j) \\ &\leqq \sum_{j=0}^{r+2k} f_H(\omega\alpha+n+j) = \sum_{j=0}^{r+2k} f_{H\alpha}(n+j)\end{aligned}$$

and similarly

$$\sum_{j=0}^{r} f_{H_\alpha}(n+k+j) \leqq \sum_{j=0}^{r+2k} f_{G_\alpha}(n+j) .$$

Now apply 4.2 to see that the corresponding Ulm factors of G and H are uniformly quasi-isomorphic. By 4.25, $G \simeq H$ in $\mathscr{A}/\mathscr{B}$.

The converse is 4.13.

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