A PERSISTENT LOCAL MAXIMUM OF THE $p$th POWER DEVIATION ON AN INTERVAL, $p < 1$

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The deviation of the polynomial $p_0(x) = c$ from the given function $f(x) = |x|^{\frac{1}{\alpha}sg x}$, $p + \alpha > 2$, $w(x)$ nonnegative, bounded, and integrable but not a null function, is defined as $\delta(c) = \int_{-1}^{1} w(x)|c - f(x)|^p dx$, whence $\delta''(0) < 0$. Thus the error function $c - f(x)$ has a strong oscillation in the interval $[-1, 1]$, yet $\delta(c)$ has a local maximum at $c = 0$ provided $\delta'(0) = 0$; this is true for every (allowable) choice of $w(x)$. For suitably chosen $w(x)$, the deviation $\delta(c)$ has a global maximum at $c = 0$, $|c| \leq 1$.

Least $p$th power approximating polynomials of degree $n$ on an interval are known to require $(n + 1)$-fold strong oscillation of the error function (if the latter is not identically zero) in the case $p > 1$, and to require either $(n + 1)$-fold strong oscillation of the error function or its vanishing on a set of positive measure in the case $p = 1$; see Jackson [2, 3], Hoel [1], Walsh and Motzkin [5]. Conversely, if a polynomial with those characteristics is given, there exists a positive continuous weight function such that the polynomial is a least $p$th power approximator [5]. The facts [1, 6] are quite different in the case $0 < p < 1$, and the object of the present note is to exhibit in that case an approximating polynomial $p_0(x) = c$ of degree zero where strong oscillation occurs yet so also does a local maximum of the deviation (as a function of $c$), for a large class of weight functions. In § 5 we show that global maxima exist, in § 6 we give some special but illuminating examples, and present this contrasting behavior for various values of $p$ in § 7 below.

1. Results. We proceed to prove

**Theorem 1.** Suppose $f(x) = |x|^{\frac{1}{\alpha}sg x}$, $0 < p < 1$, $p + \alpha > 2$, $p_0(x) = c$, $\eta > 0$, $w(x)$ nonnegative bounded and integrable, but not a null function, and define the deviation as

$$\delta(c) = \int_{-\eta}^{\eta} w(x)|c - f(x)|^p dx, \eta > 0.$$  

Then we have for $-\eta < c < \eta$

$$\delta'(c) = p\int_{-\eta}^{\eta} w(x)|c - f(x)|^{p-1}sg [c - f(x)]dx,$$
THEOREM 2. With the hypothesis of Theorem 1, although the error function \( f(x) - c, -\gamma < c < \gamma \) has a strong oscillation in the interval \([-\gamma, \gamma]\), the deviation \( \delta(c) \) has a local MAXIMUM at \( c = 0 \) provided \( \delta'(0) = 0 \); this is true for every (allowable) choice of \( w(x) \).

2. First derivative of deviation. The detailed study of \( \delta(c) \) and its derivatives involves improper integrals, which need to be treated with care. The transformation \( z = x^{1/\alpha}, x = z^{\alpha}, dx = \alpha z^{\alpha-1}dz \), gives \( c > 0 \)

\[
\frac{\delta(c)}{\alpha} = \int_0^{\gamma} w(-z^{\alpha})(c + z)^{p-1}z^{\alpha-1}dz + \int_0^{c} w(z^{\alpha})(c - z)^{p-1}z^{\alpha-1}dz + \int_c^{\gamma} w(z^{\alpha})(z - c)^{p-1}z^{\alpha-1}dz,
\]

so by Leibnitz’s rule and elementary inequalities, which the reader can supply by methods used below,

\[
\frac{\delta'(c)}{(p\alpha)} = \int_0^{\gamma} w(-z^{\alpha})(c + z)^{p-1}z^{\alpha-1}dz + \int_0^{c} w(z^{\alpha})(c - z)^{p-1}z^{\alpha-1}dz - \int_c^{\gamma} w(z^{\alpha})(z - c)^{p-1}z^{\alpha-1}dz,
\]

from which (2) follows.

The relation

\[
\frac{\delta'(0^-)}{(p\alpha)} = \int_0^{\gamma} w(-z^{\alpha})z^{p-a-2}dz - \int_0^{c} w(z^{\alpha})z^{p-a-2}dz
\]

can be similarly proved, and indeed follows from (4), so we have \( \delta'(0^-) = \delta'(0^+) = \delta'(0) \).

3. Second derivative. We proceed to compute \( \delta''(0) \) from (4), and denote by \( J_k(c) \) the \( k^{th} \) integral in the second member of (4), \( c > 0 \). We have

\[
\frac{J_k(c) - J_k(0)}{c} = \frac{1}{c} \int_0^{c} w(z^{\alpha})(c - z)^{p-1}dz.
\]

Here we make the substitution \( y = z/c, z = cy, dz = cdy \). The second member of (5) becomes
A PERSISTENT LOCAL MAXIMUM OF THE pTH POWER DEVIATION

\[
e^{p-\alpha-1} \int_0^1 w(z^\alpha)(1 - y)^{p-1} y^{a-1} dy,
\]

which approaches zero with \( c \), whence

(6) \[ J_1(0^+) = 0. \]

We now consider for \( c \downarrow 0 \)

(7) \[ \frac{J_1(c) - J_1(0)}{c} = \int_0^\eta \frac{w(-z^\alpha)(z + c)^{p-1} - z^{p-1}}{c} z^{a-1} dz. \]

The second factor in the integrand can be expressed \((0 < z \leq \eta)\)

\[
\frac{(z + c)^{p-1} - z^{p-1}}{c} = (p - 1)(z + \theta c)^{p-2},
\]

so the integral in (7) lies between the two integrals

(8) \[
(p - 1) \int_0^\eta w(-z^\alpha)(z + c)^{p-1} z^{a-1} dz,
\]

the first integrand in (8) increases monotonically as \( c \downarrow 0 \) and approaches the second integrand uniformly except in the neighborhood of the point \( z = 0 \). The second integral converges and

\[
\int_0^\eta w(-z^\alpha) z^{a+p-3} dz
\]

can be made as small as desired merely by choosing \( c_0 \) sufficiently small, \( 0 < c_0 < \eta \). Thus the first integral in (8) also converges, and

\[
\int_0^\eta w(-z^\alpha)(z + c)^{p-1} z^{a-1} dz
\]

is less than the corresponding integral with \( c = 0 \). The first integral in (8) with the lower limit of integration replaced by \( c_0 \) approaches the second integral in (8) with the lower limit replaced by \( c_0 \), so we have

(9) \[ J_1(0^+) = (p - 1) \int_0^\eta w(-z^\alpha) z^{a+p-3} dz. \]

It remains to study \( J_3(c) \) as \( c \downarrow 0 \):

(10) \[ \frac{J_3(c) - J_3(0)}{c} = \int_0^\eta \frac{w(z^\alpha)(z - c)^{p-1} - z^{p-1}}{c} z^{a-1} dz
\]
\[ - \frac{1}{c} \int_0^\xi w(z^\alpha) z^{p+a-2} dz. \]
The second term on the right can be compared to a constant multiple of

\[ -\frac{1}{c} \int_{\eta}^{\varphi} z^{p+\alpha-2} dz = -\frac{c^{p+\alpha-2}}{p + \alpha - 1} , \]

which approaches zero with \( c \). The first integral in (10) can be treated somewhat like the integral in (7); we choose \( c_0 \) fixed but as yet undetermined, \( 0 < c < c_0 < \eta \), and notice that

\[ \int_{c_0}^{\eta} w(z^a) \left( \frac{(z - c)^{p-1} - z^{p-1}}{c} + (p - 1)z^{p-2} \right)z^{\alpha-1} dz \]

approaches zero with \( c \), since by the law of the mean the integrand approaches zero uniformly in \([c_0, \eta]\). We isolate the integral

\[ (p - 1) \int_{c}^{c_0} w(z^a)z^{p+\alpha-3} dz , \]

which can be made as small as desired by suitable choice of \( c_0 \), uniformly in \( c \) (in particular we may choose \( c = 0 \)). It remains to treat

\[ \int_{c}^{c_0} w(z^a) \left( \frac{(z - c)^{p-1} - z^{p-1}}{c} + (p - 1)z^{p-1} \right)z^{\alpha-1} dz . \]

The contribution of the last term in brackets to (13) is (12), so that term can be ignored. In continuing the study of (13) we suppress the factor \( w(z^a) \), which is not important in proving that (13) can be made as small as desired by suitable choice of \( c_0 \) and then of \( c \).

In the modified (13) we set \( y = z/c, z = cy, dz = cdy \), and obtain

\[ c^{p+\alpha-1} \int_{1}^{c_0/c} y^{p-1}[(y - 1)^{p-1} - y^{p-1}] dy , \]

which for sufficiently small \( c \) equals \( c^{p+\alpha-2} \) times the corresponding integral over the interval \([2, 3]\) (which approaches zero with \( c \)) plus

\[ c^{p+\alpha-2} \int_{2}^{c_0/c} y^{p+\alpha-3}[(1 - 1/y)^{p-1} - 1] dy \]

\[ = c^{p+\alpha-2} \int_{2}^{c_0/c} y^{p+\alpha-3} \left[ \frac{p - 1}{y} + \frac{(p - 1)(p - 2)}{2y^2} - \ldots \right] dy \]

\[ = c^{p+\alpha-2} \left[ -(p - 1) \left( \frac{c_0/c}{p + \alpha - 2} \right) - \frac{2^{p+\alpha-2}}{p + \alpha - 2} \right. \]

\[ + \left. \frac{(p - 1)(p - 2)}{2} \left( \frac{c_0/c}{p + \alpha - 3} - \frac{2^{p+\alpha-3}}{p + \alpha - 3} \right) - \ldots \right] . \]

This last expression (which requires slight modification if \( p + \alpha \) is an integer) can be written (\( K_0 \) is a numerical constant)
A PERSISTENT LOCAL MAXIMUM OF THE \( p \)TH POWER DEVIATION

\[
- \frac{(p - 1)c_0^{p+a-2}}{p + \alpha - 2} + \frac{(p - 1)(p - 2)}{2(p + \alpha - 3)} \cdot c_0^{p+a-3} - \ldots + c^{p+a-2}K_0 ,
\]

and can be made numerically as small as desired by choosing \( c_0 \) so that the first term is, say, less than a given \( \epsilon (>0) \), then choosing \( c \) so small that the entire expression is numerically less than \( \epsilon \).

Consequently (14), (13), (12), and (11) can each be made as small as desired by suitable choice of \( c_0 \) and then of \( c \), so we have

\[
J_\delta(0^+) = -(p - 1) \int_0^\eta w(z^p)z^{p+a-3}dz .
\]

Combining (4), (6), (9), and (15) now yields

\[
\delta''(0^+) = p(p - 1)\alpha \int_0^\eta w(-z^p)z^{p+a-3}dz
+ p(p - 1)\alpha \int_0^\eta w(z^p)z^{p+a-3}dz ,
\]

which is equivalent to (3).

To be sure, we have computed merely \( \delta''(0^+) \), but by the symmetry of \( f(x) \) and of the notation, the value of \( \delta''(0^-) \) is the same, so \( \delta''(0^-) = \delta''(0^+) = \delta''(0) \).

4. Proof of Theorem 2. It is clear that \( \delta(c) \) can have neither a maximum nor a minimum at \( c = 0 \) unless \( \delta'(0) = 0 \). If \( \delta'(0) = 0 \) it follows from (3) that \( \delta'(c) < 0 \) for small positive \( c \), and \( \delta'(c) > 0 \) for small negative \( c \). By the law of the mean we conclude that \( \delta(c) \) can never have a minimum at \( c = 0 \), and that whenever \( \delta'(0) = 0 \), \( \delta(c) \) has a strong local maximum at \( c = 0 \), whatever may be the bounded integrable weight function \( w(x)(\neq 0) \). This conclusion is obtained despite the strong oscillation of the error function \( f(x) - c = f(x) \).

In particular the condition \( \delta'(0) = 0 \) is satisfied here whenever the weight function \( w(x) \) is an even function; \( \delta(c) \) has a strong local maximum at \( c = 0 \).

5. Global maxima. Theorems 1 and 2 illustrate the existence of local maxima of \( \delta(c) \) at \( c = 0 \) but do not show the possibility of a global maximum. We shall prove

**Theorem 3.** For every \( p, 0 < p < 1 \), and for every \( \alpha \) with \( p + \alpha > 2 \), there exists an even function \( w(x) \) positive at every point of \([-1, 1]\), integrable and bounded there, such that the deviation

\[
\delta(c) \equiv \int_{-1}^1 w(x)|c - f(x)|^pdx , \quad f(x) \equiv |x|^{\alpha}sg x ,
\]
has a proper global maximum $\delta(0)$ in the interval $[-1,1]$

As a preliminary remark, we note the inequality $(0 < p < 1)$

$$|1 - X|^p + |1 + X|^p < 2|X|^p, \text{ for } |X| \geq 1. \quad (17)$$

This inequality expresses the fact that for the concave curve $y = x^p$, $x \geq 0$, the chord joining the points whose abscissas are $|X| + 1$ and $|X| - 1$ passes below the point of the curve whose abscissa is $|X|$. Since the strong inequality is valid for $|X| = 1$, it is also valid for all $X$ such that $|X| \geq x_i$, where $x_i$ is suitably chosen, $0 < x_i < 1$.

If $c \neq 0$ we can now write for $|X| \geq |c|x_i$

$$|c - X|^p + |c + X|^p = |c|^p \left[ |1 - \frac{X}{c}|^p + |1 + \frac{X}{c}|^p \right] < 2|c|^p \left| \frac{X}{c} \right|^p = 2|X|^p. \quad (18)$$

The validity of (18) is assured if we have

$$x_i \leq |X| \leq 1, \quad 0 < |c| \leq 1,$$

for these inequalities imply $|X| \geq x_i \geq |c|x_i$; if $c = 0$, but for no other value of $c, |c| \leq 1$, the inequality (18) between the extreme members becomes an equality.

We choose now the weight function $w_1(x)$ as any nonnegative, integrable, bounded, even, nonnull function on the intervals $x_i^1 \leq |x| \leq 1$, and zero elsewhere on $-1 \leq x \leq 1$. For the function $f(x)$ in (16) the corresponding deviation $\tilde{\delta}_1(c)$ is

$$\tilde{\delta}_1(c) = \int_{x_i^1}^1 w_1(x)[|c - x^{1/\alpha}|^p + |c + x^{1/\alpha}|^p]dx,$$

whence

$$\tilde{\delta}_1(c) - \tilde{\delta}_1(0) = \int_{x_i^1}^1 w_1(x)[|c - x^{1/\alpha}|^p + |c + x^{1/\alpha}|^p - 2x^{p/\alpha}]dx. \quad (20)$$

We identify the first member of (18) minus the last member with the bracket in (20), where $X = x^{1/\alpha}$, and note that for $x_i^1 \leq x \leq 1$ the bracket is negative for $0 < |c| \leq 1$. Thus $\delta_1(c)$ has a global maximum at $c = 0$. However, the weight function $w_1(x)$ is not positive at every point of $-1 \leq x \leq 1$.

We continue to envisage $f(x)$ as in (16), but now with the weight function $w_2(x) \equiv 1$ in $-1 \leq x \leq 1, p + \alpha > 2$, and with the deviation denoted by $\delta_2(c)$. It is shown in [4] under these conditions that $\delta_2(c)$ has at $c = 0$ a local maximum, and $\delta_2(c) - \delta_2(0) \sim A|c|^{p+\alpha}$ as $|c| \to 0$, $A > 0$. On the other hand, for $x \geq x_i^1$ and for $c \neq 0$, by the binomial
A PERSISTENT LOCAL MAXIMUM OF THE \( p \)-TH POWER DEVIATION

Theorem we find uniformly in \( x^\alpha \leq x \leq 1 \)

\[
(x^\alpha - c)^p + (x^\alpha + c)^p - 2x^{\alpha/2} \sim \beta(c - x)^{p/2},
\]

whence \( \delta(c) - \delta(0) \sim \beta c^\alpha, \beta > 0. \)

We now define the weight function \( w(x) = w_1(x) + \varepsilon w_2(x) \), where \( \varepsilon(>0) \) is to be determined, and denote the corresponding deviation by \( \delta(c) = \delta_1(c) + \varepsilon \delta_2(c) \). For \( c \downarrow 0 \) there follows \( \delta(c) - \delta(0) \sim \beta c^\alpha + \varepsilon A \varepsilon \varepsilon^\alpha, \)

so for sufficiently small \( \varepsilon \) we have \( \delta(c) - \delta(0) < 0 \) throughout some deleted neighborhood \( 0 < |c| \leq \beta, \beta > 0; \) it will be noted that a change to a smaller \( \varepsilon \) allows \( \beta \) to be increased if desired. Choose \( \varepsilon \) also less than

\[
\min \left[ \frac{\delta_1(c) - \delta_1(0)}{\delta_2(c) - \delta_2(0)}, \ c \text{ on } E_0 \right],
\]

where \( E_0 \) is the subset of \( \beta \leq |c| \leq 1 \) on which \( \delta_1(c) - \delta_1(0) > 0, \) provided \( E_0 \) is not empty; such a (positive) minimum exists by the continuity of \( \delta_1(c) \) and \( \delta_2(c) \) in \( |c| \leq 1. \) Thus for \( c \) on \( E_0 \)

\[
\delta_1(c) - \delta_1(0) + \varepsilon [\delta_2(c) - \delta_2(0)] < 0, \delta(c) - \delta(0) < 0.
\]

However, on the complement of \( E_0 \) with respect to \( \beta \leq |c| \leq 1, \) we have \( \delta_1(c) - \delta_1(0) \leq 0, \delta(c) - \delta(0) < 0, \) so Theorem 3 is established.

It may be noted that \( w_1(x) \) can be chosen continuous in \([-1,1], \) in which case \( w(x) \) is continuous there. We also note that Theorem 3 remains valid if \( p + \alpha = 2. \)

6. Finite sets versus intervals, \( 0 < p < 1. \) We add several remarks relative to hypotheses analogous to, but different from, the hypothesis of Theorem 1, still with \( 0 < p < 1. \) If we modify the hypothesis of Theorem 1 by choosing \( f(x) = \lambda x, \lambda > 0, \) and \( w(x) = 1, \) we have

\[
\delta(c) \equiv \int_{-\gamma}^\gamma |c - \lambda x|^\alpha dx = \frac{1}{\lambda} \int_{-\gamma}^\gamma |c - x'|^\alpha dx',
\]

so to study the behavior of \( \delta'(c) \) it is no essential loss of generality to choose \( \lambda = 1. \) There follow the equations \( 0 < c < \gamma \)

\[
\delta(c) \equiv \int_0^\gamma (c + x)^p dx + \int_0^\gamma (x - c)^p dx + \int_0^\gamma (x - c)^p dx,
\]

\[
(p + 1)\delta(c) \equiv (c + \gamma)^{p+1} + (\gamma - c)^{p+1},
\]

\[
\delta'(c) \equiv (c + \gamma)^p - (\gamma - c)^p,
\]
which approaches zero with $c$,

$$\frac{\delta''(c)}{p} = (c + \gamma)^{p-1} + (\gamma - c)^{p-1};$$

we have $\delta(0^+) = \delta(0^-) = 0$, $\delta''(0^+) = \delta''(0^-) = \delta''(0) > 0$, so $\delta(c)$ has a strong minimum at $c = 0$, in great contrast to the situation of Theorems 1 and 2. Indeed, it can be shown [4] that $\delta(c)$ has a minimum at $c = 0$ for approximation on $[-\gamma, \gamma]$ to $\lambda |x|^{1/\alpha}$ for every $\alpha \leq 1$, $\lambda > 0$.

It is illuminating to compare Theorems 1 and 2 with least $p^{th}$ power approximation ($0 < p < 1$) to $f(x)$ not on an interval but on the finite set $S: \{-1, 1\}$ by a polynomial $p_\alpha(x) = c$ of degree 0, $-1 \leq c \leq 1$, with weights $w_1$ and $w_2$. The deviation is

$$\delta(c) = w_2 (1 - c)^p + w_1 (c + 1)^p,$$

which has a maximum for $c = 0$ if $\delta'(c) = 0$, as in Theorems 1 and 2; the graph of $\delta(c)$ is concave downward in $-1 \leq c \leq 1$.

Likewise for least $p^{th}$ power approximation ($0 < p < 1$) to the discontinuous function $f(x) = sg x$ on the interval $-1 \leq x \leq 1$ by a polynomial $p_\alpha(x) \equiv c$ of degree zero, $-1 \leq c \leq 1$, the deviation is

$$\delta(c) = \int_{-1}^{c} (c + 1)^p dx + \int_{c}^{1} (1 - c)^p dx = (1 - c)^p + (c + 1)^p$$

as before; $\delta(c)$ has again a maximum for $c = 0$ and its graph is concave downward in $-1 \leq c \leq 1$. The minimum of $\delta(c)$ occurs for $c = \pm 1$.

In sum, for approximation on a finite set $S$, $0 < p < 1$, strong oscillation of the function $f(x) - c$ may lead to a local maximum of $\delta(c)$ when $\delta'(c) = 0$, as in the example above; but the function

$$\delta(c) = \sum w_k |c - f(x_k)|^p, \quad w_k > 0,$$

is continuous and piecewise concave downward, so its local and global minima must occur in values of $c$ equal to some $f(x_k)$; such a minimum involves weak oscillation and is independent of strong oscillation. On the other hand, for approximation on an interval $E$, strong oscillation of $f(x) - c$ with $\delta'(c) = 0$ may lead to a local maximum of $\delta(c)$ as in Theorems 1 and 2, and [4] weak oscillation as with $f(x) \equiv |x|^{1/\alpha}, \alpha \leq 1$, on $-1 \leq x \leq 1$ may lead to a global minimum; it is no accident that the cases $\alpha > 1$ and $\alpha < 1$ are respectively characterized by vertical and horizontal tangents of $f(x)$ at $x = 0$, corresponding with $\delta'(0) = 0$ to maxima and minima of $S(c)$.

7. Summary of results, arbitrary $p$. We summarize some of the known results on approximation for various values of $p$, on a
real finite point set $S$ or on a closed interval $E$, for comparison with each other and with Theorems 1 and 2. In each case we approximate by a polynomial $p_n(x)$ of degree $n$, either to a continuous function $f(x)$ on $E$, or to a function on a finite set $S: \{x_k\}$ consisting of more than $n$ points. We compare oscillation of the error $f(x) - p_n(x)$ on the one hand to the existence of maxima and minima of the deviation

$$\delta[p_n(x)] = \int_E w(x)|f(x) - p_n(x)|^p \, dx \quad \text{or} \quad \delta[p_n(x)] = \sum_k w_k|f(x_k) - p_n(x_k)|^p,$$

where $w(x)$ is nonnegative and not a null function, and we assume $\delta[p_n(x)]$ to be different from zero for all $p_n(x)$.

For $p > 1$, $\delta[p_n(x)]$ is never a local maximum; every local minimum is also a strong global minimum, and the error $f(x) - p_n(x)$ has at least $n + 1$ strong oscillations. Conversely, if the error has $n + 1$ strong oscillations, then there exists a $w(x)$ (continuous for approximation on $E$) such that $\delta[p_n(x)]$ is a strong global minimum.

For $p = 1$, $\delta[p_n(x)]$ has never a strong local maximum; every local minimum (which can be a weak minimum for approximation on $S$) is also a global minimum. For approximation on $E$ and every minimum of $\delta$, the error has either at least $n + 1$ strong oscillations or vanishes identically on a subset of $E$ of positive measure; conversely, if the error $f(x) - p_n(x)$ has either $n + 1$ strong oscillations or vanishes on a subset of $E$ of positive measure, $\delta[p_n(x)]$ has a local minimum for suitable continuous weight. For approximation on $S$, the error has at least $n + 1$ weak oscillations on $S$ if the error has a local minimum; conversely, if the error has at least $n + 1$ weak oscillations, the deviation has a local minimum for suitable weights.

For $0 < p < 1$ and approximation on $S$, if $\delta[p_n(x)]$ is minimum, then $p_n(x)$ coincides with $f(x)$ in at least $n + 1$ points of $S$; conversely, if $p_n(x)$ coincides with $f(x)$ in at least $n + 1$ points of $f(x)$, $\delta[p_n(x)]$ is a minimum for suitable weights. For $0 < p < 1$ and approximation on $E$, coincidence of $p_n(x)$ with $f(x)$ in $n + 1$ points of $E$ is neither necessary nor sufficient that $\delta[p_n(x)]$ be a minimum, and even strong oscillation is neither necessary nor sufficient. Indeed, with strong oscillation and $n = 0$ it may occur (Theorem 2) that $\delta[p_n(x)]$ has a strong maximum.

It is clear that the deviation $\delta[p_n(x)]$ varies both with changes in $p_n(x)$ and the weight, and the deviation may also have a maximum or minimum which varies with those changes. In particular, Theorem 2 indicates stability of a maximum of $\delta(c)$ with respect to changes in $w(x)$ that preserve the relation $\delta'(0) = 0$. The writers plan to discuss stability in more detail on another occasion.
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AND
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Harry P. Allen, *Lie algebras of type $D_4$ over algebraic number fields* ........... 1
Charles Ballantine, *Products of positive definite matrices. II* ..................... 7
David W. Boyd, *The spectral radius of averaging operators* ......................... 19
William Howard Caldwell, *Hypercyclic rings* ........................................... 29
Francis William Carroll, *Some properties of sequences, with an application to noncontinuable power series* .......................................................... 45
David Fleming Dawson, *Matrix summability over certain classes of sequences ordered with respect to rate of convergence* ................................. 51
D. W. Dubois, *Second note on David Harrison’s theory of preprimes* ............ 57
Edgar Earle Enochs, *A note on quasi-Frobenius rings* ................................. 69
Ronald J. Ensey, *Isomorphism invariants for Abelian groups modulo bounded groups* .......................................................... 71
Ronald Owen Fulp, *Generalized semigroup kernels* ..................................... 93
Bernard Robert Kripke and Richard Bruce Holmes, *Interposition and approximation* ......................................................................................... 103
Jack W. Macki and James Sai-Wing Wong, *Oscillation of solutions to second-order nonlinear differential equations* ........................................... 111
Lothrop Mittenthal, *Operator valued analytic functions and generalizations of spectral theory* .......................................................... 119
T. S. Motzkin and J. L. Walsh, *A persistent local maximum of the $p$th power deviation on an interval, $p < 1$* ......................................................... 133
Jerome L. Paul, *Sequences of homeomorphisms which converge to homeomorphisms* .......................................................... 143
Maxwell Alexander Rosenlicht, *Liouville’s theorem on functions with elementary integrals* .......................................................... 153
Joseph Goeffrey Rosenstein, *Initial segments of degrees* .............................. 163
H. Subramanian, *Ideal neighbourhoods in a ring* ......................................... 173
Dalton Tarwater, *Galois cohomology of abelian groups* ................................ 177
James Patrick Williams, *Schwarz norms for operators* .................................. 181
Raymond Y. T. Wong, *A wild Cantor set in the Hilbert cube* ....................... 189