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The deviation of the polynomial $p_0(x) \equiv c$ from the given function $f(x) \equiv |x|^{1/\alpha} sg\, x$, $p + \alpha > 2$, $w(x)$ nonnegative, bounded, and integrable but not a null function, is defined as $\delta(c) \equiv \int_{-1}^{1} w(x)|c - f(x)|^p dx$, whence $\delta''(0) < 0$. Thus the error function $c - f(x)$ has a strong oscillation in the interval $[-1, 1]$, yet $\delta(c)$ has a local *maximum* at $c = 0$ provided $\delta'(0) = 0$; this is true for every (allowable) choice of $w(x)$. For suitably chosen $w(x)$, the deviation $\delta(c)$ has a global maximum at $c = 0$, $|c| \leqq 1$.

Least p^{th} power approximating polynomials of degree n on an interval are known to require $(n + 1)$-fold strong oscillation of the error function (if the latter is not identically zero) in the case $p > 1$, and to require either $(n + 1)$-fold strong oscillation of the error function or its vanishing on a set of positive measure in the case $p = 1$; see Jackson [2, 3], Hoel [1], Walsh and Motzkin [5]. Conversely, if a polynomial with those characteristics is given, there exists a positive continuous weight function such that the polynomial is a least p^{th} power approximator [5]. The facts [1, 6] are quite different in the case $0 < p < 1$, and the object of the present note is to exhibit in that case an approximating polynomial $p_0(x) \equiv c$ of degree zero where strong oscillation occurs yet so also does a local *maximum* of the deviation (as a function of c), for a large class of weight functions. In § 5 we show that global maxima exist, in § 6 we give some special but illuminating examples, and present this contrasting behavior for various values of p in § 7 below.

1. Results.

We proceed to prove

THEOREM 1. *Suppose $f(x) \equiv |x|^{1/\alpha} sg\, x$, $0 < p < 1$, $p + \alpha > 2$, $p_0(x) \equiv c$, $\eta > 0$, $w(x)$ nonnegative bounded and integrable, but not a null function, and define the deviation as*

$$\delta(c) \equiv \int_{-\eta^\alpha}^{\eta^\alpha} w(x)|c - f(x)|^p dx, \eta > 0 . \tag{1}$$

Then we have for $-\eta < c < \eta$

$$\delta'(c) = p\int_{-\eta^\alpha}^{\eta^\alpha} w(x)|c - f(x)|^{p-1} sg\,[c - f(x)]dx , \tag{2}$$

$$\delta''(0) = p(p-1)\int_{-\eta^\alpha}^{\eta^\alpha} w(x)|\, x\,|^{(p-2)/\alpha} dx\ . \tag{3}$$

THEOREM 2. *With the hypothesis of Theorem* 1, *although the error function* $f(x) - c$, $-\eta < c < \eta$ *has a strong oscillation in the interval* $[-\eta^\alpha, \eta^\alpha]$, *the deviation* $\delta(c)$ *has a local* MAXIMUM *at* $c = 0$ *provided* $\delta'(0) = 0$; *this is true for every (allowable) choice of* $w(x)$.

2. First derivative of deviation. The detailed study of $\delta(c)$ and its derivatives involves improper integrals, which need to be treated with care. The transformation $z = x^{1/\alpha}$, $x = z^\alpha$, $dx = \alpha z^{\alpha-1}dz$, gives $(c > 0)$

$$\begin{aligned}\delta(c)/\alpha \equiv &\int_0^\eta w(-z^\alpha)(c+z)^p z^{\alpha-1}dz\\ &+\int_0^c w(z^\alpha)(c-z)^p z^{\alpha-1}dz\\ &+\int_c^\eta w(z^\alpha)(z-c)^p z^{\alpha-1}dz\ ,\end{aligned}$$

so by Leibnitz's rule and elementary inequalities, which the reader can supply by methods used below,

$$\begin{aligned}\delta'(c)/(p\alpha) = &\int_0^\eta w(-z^\alpha)(c+z)^{p-1} z^{\alpha-1}dz\\ &+\int_0^c w(z^\alpha)(c-z)^{p-1} z^{\alpha-1}dz\\ &-\int_c^\eta w(z^\alpha)(z-c)^{p-1} z^{\alpha-1}dz\ ,\end{aligned} \tag{4}$$

from which (2) follows.

The relation

$$\delta'(0^+)/(p\alpha) = \int_0^\eta w(-z^\alpha) z^{p+\alpha-2}dz - \int_0^\eta w(z^\alpha) z^{p+\alpha-2}dz$$

can be similarly proved, and indeed follows from (4), so we have $\delta'(0^-) = \delta'(0^+) = \delta'(0)$.

3. Second derivative. We proceed to compute $\delta''(0)$ from (4), and denote by $J_k(c)$ the k^{th} integral in the second member of (4), $c > 0$. We have

$$\frac{J_2(c) - J_2(0)}{c} = \frac{1}{c}\int_0^c w(z^\alpha)(c-z)^{p-1}dz\ . \tag{5}$$

Here we make the substitution $y = z/c$, $z = cy$, $dz = cdy$. The second member of (5) becomes

$$c^{p+\alpha-2}\int_0^1 w(c^\alpha y^\alpha)(1-y)^{p-1}y^{\alpha-1}dy ,$$

which approaches zero with c, whence

$$J_2'(0^+) = 0 . \tag{6}$$

We now consider for $c \downarrow 0$

$$\frac{J_1(c)-J_1(0)}{c} = \int_0^\eta w(-z^\alpha)\frac{(z+c)^{p-1}-z^{p-1}}{c}z^{\alpha-1}dz . \tag{7}$$

The second factor in the integrand can be expressed $(0 < z \leqq \eta)$

$$\frac{(z+c)^{p-1}-z^{p-1}}{c} = (p-1)(z+\theta c)^{p-2} ,$$

so the integral in (7) lies between the two integrals

$$\begin{aligned}&(p-1)\int_0^\eta w(-z^\alpha)(z+c)^{p-2}z^{\alpha-1}dz ,\\ &(p-1)\int_0^\eta w(-z^\alpha)z^{\alpha+p-3}dz ;\end{aligned} \tag{8}$$

the first integrand in (8) increases monotonically as $c \downarrow 0$ and approaches the second integrand uniformly except in the neighborhood of the point $z = 0$. The second integral converges and

$$\int_0^{c_0} w(-z^\alpha)z^{\alpha+p-3}dz$$

can be made as small as desired merely by choosing c_0 sufficiently small, $0 < c_0 < \eta$. Thus the first integral in (8) also converges, and

$$\int_0^{c_0} w(-z^\alpha)(z+c)^{p-2}z^{\alpha-1}dz$$

is less than the corresponding integral with $c = 0$. The first integral in (8) with the lower limit of integration replaced by c_0 approaches the second integral in (8) with the lower limit replaced by c_0, so we have

$$J_1'(0^+) = (p-1)\int_0^\eta w(-z^\alpha)z^{\alpha+p-3}dz . \tag{9}$$

It remains to study $J_3(c)$ as $c \downarrow 0$:

$$\begin{aligned}\frac{J_3(c)-J_3(0)}{c} &= \int_c^\eta w(z^\alpha)\frac{(z-c)^{p-1}-z^{p-1}}{c}z^{\alpha-1}dz\\ &\quad - \frac{1}{c}\int_0^c w(z^\alpha)z^{p+\alpha-2}dz .\end{aligned} \tag{10}$$

The second term on the right can be compared to a constant multiple of

$$-\frac{1}{c}\int_0^c z^{p+\alpha-2}dz = -\frac{c^{p+\alpha-2}}{p+\alpha-1}\,,$$

which approaches zero with c. The first integral in (10) can be treated somewhat like the integral in (7); we choose c_0 fixed but as yet undetermined, $0 < c < c_0 < \eta$, and notice that

$$\int_{c_0}^{\eta} w(z^\alpha)\left[\frac{(z-c)^{p-1}-z^{p-1}}{c} + (p-1)z^{p-2}\right]z^{\alpha-1}dz \tag{11}$$

approaches zero with c, since by the law of the mean the integrand approaches zero uniformly in $[c_0, \eta]$. We isolate the integral

$$(p-1)\int_c^{c_0} w(z^\alpha)z^{p+\alpha-3}dz\,, \tag{12}$$

which can be made as small as desired by suitable choice of c_0, uniformly in c (in particular we may choose $c = 0$). It remains to treat

$$\int_c^{c_0} w(z^\alpha)\left[\frac{(z-c)^{p-1}-z^{p-1}}{c} + (p-1)z^{p-2}\right]z^{\alpha-1}dz\,. \tag{13}$$

The contribution of the last term in brackets to (13) is (12), so that term can be ignored. In continuing the study of (13) we suppress the factor $w(z^\alpha)$, which is not important in proving that (13) can be made as small as desired by suitable choice of c_0 and then of c.

In the modified (13) we set $y = z/c$, $z = cy$, $dz = cdy$, and obtain

$$c^{p+\alpha-2}\int_1^{c_0/c} y^{\alpha-1}[(y-1)^{p-1} - y^{p-1}]dy\,, \tag{14}$$

which for sufficiently small c equals $c^{p+\alpha-2}$ times the corresponding integral over the interval $[2, 3]$ (which approaches zero with c) plus

$$\begin{aligned}
&c^{p+\alpha-2}\int_2^{c_0/c} y^{p+\alpha-2}[(1-1/y)^{p-1} - 1]dy \\
&= c^{p+\alpha-2}\int_2^{c_0/c} y^{p+\alpha-2}\left[-\frac{p-1}{y} + \frac{(p-1)(p-2)}{2y^2} - \cdots\right]dy \\
&= c^{p+\alpha-2}\left[-(p-1)\frac{(c_0/c)^{p+\alpha-2} - 2^{p+\alpha-2}}{p+\alpha-2}\right. \\
&\qquad \left. + \frac{(p-1)(p-2)}{2}\,\frac{(c_0/c)^{p+\alpha-3} - 2^{p+\alpha-3}}{p+\alpha-3} - \cdots\right].
\end{aligned}$$

This last expression (which requires slight modification if $p + \alpha$ is an integer) can be written (K_0 is a numerical constant)

$$-\frac{(p-1)c_0^{p+\alpha-2}}{p+\alpha-2}+\frac{(p-1)(p-2)}{2(p+\alpha-3)}cc_0^{p+\alpha-3}-\cdots+c^{p+\alpha-2}K_0\,,$$

and can be made numerically as small as desired by choosing c_0 so that the first term is, say, less than a given $\varepsilon(>0)$, then choosing c so small that the entire expression is numerically less than ε.

Consequently (14), (13), (12), and (11) can each be made as small as desired by suitable choice of c_0 and then of c, so we have

$$J_3'(0^+) = -(p-1)\int_0^\eta w(z^\alpha)z^{p+\alpha-3}dz\,. \tag{15}$$

Combining (4), (6), (9), and (15) now yields

$$\begin{aligned}\delta''(0^+) &= p(p-1)\alpha\int_0^\eta w(-z^\alpha)z^{\alpha+p-3}dz\\ &\quad + p(p-1)\alpha\int_0^\eta w(z^\alpha)z^{\alpha+p-3}dz\,,\end{aligned}$$

which is equivalent to (3).

To be sure, we have computed merely $\delta''(0^+)$, but by the symmetry of $f(x)$ and of the notation, the value of $\delta''(0^-)$ is the same, so $\delta''(0^-)=\delta''(0^+)=\delta''(0)$.

4. Proof of Theorem 2. It is clear that $\delta(c)$ can have neither a maximum nor a minimum at $c=0$ unless $\delta'(0)=0$. If $\delta'(0)=0$ it follows from (3) that $\delta'(c)<0$ for small positive c, and $\delta'(c)>0$ for small negative c. By the law of the mean we conclude that $\delta(c)$ can never have a minimum at $c=0$, and that whenever $\delta'(0)=0$, $\delta(c)$ has a strong local maximum at $c=0$, whatever may be the bounded integrable weight function $w(x)(\not\equiv 0)$. This conclusion is obtained despite the strong oscillation of the error function $f(x)-c\equiv f(x)$.

In particular the condition $\delta'(0)=0$ is satisfied here whenever the weight function $w(x)$ is an even function; $\delta(c)$ has a strong local maximum at $c=0$.

5. Global maxima. Theorems 1 and 2 illustrate the existence of local maxima of $\delta(c)$ at $c=0$ but do not show the possibility of a global maximum. We shall prove

THEOREM 3. *For every $p, 0<p<1$, and for every α with $p+\alpha>2$, there exists an even function $w(x)$ positive at every point of $[-1,1]$, integrable and bounded there, such that the deviation*

$$\delta(c)\equiv\int_{-1}^1 w(x)|\,c-f(x)\,|^p dx,\ f(x)\equiv|\,x\,|^{1/\alpha}sg\ x\,, \tag{16}$$

has a proper global maximum $\delta(0)$ *in the interval* $[-1, 1]$

As a preliminary remark, we note the inequality $(0 < p < 1)$

$$|1 - X|^p + |1 + X|^p < 2|X|^p, \text{ for } |X| \geqq 1 . \tag{17}$$

This inequality expresses the fact that for the concave curve $y = x^p$, $x \geqq 0$, the chord joining the points whose abscissas are $|X| + 1$ and $|X| - 1$ passes below the point of the curve whose abscissa is $|X|$. Since the strong inequality is valid for $|X| = 1$, it is also valid for all X such that $|X| \geqq x_1$, where x_1 is suitably chosen, $0 < x_1 < 1$.

If $c \neq 0$ we can now write for $|X| \geqq |c|x_1$

$$\begin{aligned} |c - X|^p + |c + X|^p &= |c|^p\left[\left|1 - \frac{X}{c}\right|^p + \left|1 + \frac{X}{c}\right|^p\right] \\ &< 2|c|^p\left|\frac{X}{c}\right|^p = 2|X|^p . \end{aligned} \tag{18}$$

The validity of (18) is assured if we have

$$x_1 \leqq |X| \leqq 1, 0 < |c| \leqq 1 ,$$

for these inequalities imply $|X| \geqq x_1 \geqq |c|x_1$; if $c = 0$, but for no other value of $c, |c| \leqq 1$, the inequality (18) between the extreme members becomes an equality.

We choose now the weight function $w_1(x)$ as any nonnegative, integrable, bounded, even, nonnull function on the intervals $x_1^\alpha \leqq |x| \leqq 1$, and zero elsewhere on $-1 \leqq x \leqq 1$. For the function $f(x)$ in (16) the corresponding deviation $\delta_1(c)$ is

$$\delta_1(c) \equiv \int_{x_1^\alpha}^1 w_1(x)[|c - x^{1/\alpha}|^p + |c + x^{1/\alpha}|^p]dx ,$$

whence

$$\delta_1(c) - \delta_1(0) \equiv \int_{x_1^\alpha}^1 w_1(x)[|c - x^{1/\alpha}|^p + |c + x^{1/\alpha}|^p - 2x^{p/\alpha}]dx . \tag{20}$$

We identify the first member of (18) minus the last member with the bracket in (20), where $X = x^{1/\alpha}$, and note that for $x_1^\alpha \leqq x \leqq 1$ the bracket is negative for $0 < |c| \leqq 1$. Thus $\delta_1(c)$ has a global maximum at $c = 0$. However, the weight function $w_1(x)$ is not positive at every point of $-1 \leqq x \leqq 1$.

We continue to envisage $f(x)$ as in (16), but now with the weight function $w_2(x) \equiv 1$ in $-1 \leqq x \leqq 1, p + \alpha > 2$, and with the deviation denoted by $\delta_2(c)$. It is shown in [4] under these conditions that $\delta_2(c)$ has at $c = 0$ a local maximum, and $\delta_2(c) - \delta_2(0) \sim A|c|^{p+\alpha}$ as $|c| \to 0$, $A > 0$. On the other hand, for $x \geqq x_1^\alpha$ and for $c \downarrow 0$, by the binomial theorem we find uniformly in $x_1^\alpha \leqq x \leqq 1$

$$(x^{1/\alpha} - c)^p + (x^{1/\alpha} + c)^p - 2x^{p/\alpha} \sim p(p-1)c^2 x^{(p-2)/\alpha} ,$$

whence $\delta_1(c) - \delta_1(0) \sim Bc^2, B < 0$.

We now define the weight function $w(x) \equiv w_1(x) + \varepsilon w_2(x)$, where $\varepsilon(>0)$ is to be determined, and denote the corresponding deviation by $\delta(c) = \delta_1(c) + \varepsilon\delta_2(c)$. For $c \downarrow 0$ there follows $\delta(c) - \delta(0) \sim Bc^2 + \varepsilon A|c|^{p+\alpha}$, so for sufficiently small ε we have $\delta(c) - \delta(0) < 0$ throughout some deleted neighborhood $0 < |c| \leqq \beta, \beta > 0$; it will be noted that a change to a smaller ε allows β to be increased if desired. Choose ε also less than

$$\min\left[\frac{-[\delta_1(c) - \delta_1(0)]}{\delta_2(c) - \delta_2(0)}, c \text{ on } E_0\right],$$

where E_0 is the subset of $\beta \leqq |c| \leqq 1$ on which $\delta_2(c) - \delta_2(0) > 0$, provided E_0 is not empty; such a (positive) minimum exists by the continuity of $\delta_1(c)$ and $\delta_2(c)$ in $|c| \leqq 1$. Thus for c on E_0

$$\varepsilon < \frac{-[\delta_1(c) - \delta_1(0)]}{\delta_2(c) - \delta_2(0)} ,$$

$$\delta_1(c) - \delta_1(0) + \varepsilon[\delta_2(c) - \delta_2(0)] < 0, \delta(c) - \delta(0) < 0 .$$

However, on the complement of E_0 with respect to $\beta \leqq |c| \leqq 1$, we have $\delta_2(c) - \delta_2(0) \leqq 0, \delta(c) - \delta(0) < 0$, so Theorem 3 is established.

It may be noted that $w_1(x)$ can be chosen continuous in $[-1, 1]$, in which case $w(x)$ is continuous there. We also note that Theorem 3 remains valid if $p + \alpha = 2$.

6. Finite sets versus intervals, $0 < p < 1$. We add several remarks relative to hypotheses analogous to, but different from, the hypothesis of Theorem 1, still with $0 < p < 1$. If we modify the hypothesis of Theorem 1 by choosing $f(x) \equiv \lambda x, \lambda > 0$, and $w(x) \equiv 1$, we have

$$\delta(c) \equiv \int_{-\eta}^{\eta} |c - \lambda x|^p dx \equiv \frac{1}{\lambda}\int_{-\lambda\eta}^{\lambda\eta} |c - x'|^p dx' ,$$

so to study the behavior of $\delta'(c)$ it is no essential loss of generality to choose $\lambda = 1$. There follow the equations $(0 < c < \eta)$

$$\delta(c) \equiv \int_0^\eta (c + x)^p dx + \int_0^c (c - x)^p dx + \int_c^\eta (x - c)^p dx ,$$

$$(p + 1)\delta(c) \equiv (c + \eta)^{p+1} + (\eta - c)^{p+1} ,$$

$$\delta'(c) \equiv (c + \eta)^p - (\eta - c)^p ,$$

which approaches zero with c,

$$\delta''(c)/p \equiv (c+\eta)^{p-1} + (\eta - c)^{p-1};$$

we have $\delta(0^+) = \delta(0^-) = 0$, $\delta''(0^+) = \delta''(0^-) = \delta''(0) > 0$, so $\delta(c)$ has a strong *minimum* at $c = 0$, in great contrast to the situation of Theorems 1 and 2. Indeed, it can be shown [4] that $\delta(c)$ has a minimum at $c = 0$ for approximation on $[-\eta, \eta]$ to $\lambda|x|^{1/\alpha}$ for every $\alpha \leqq 1, \lambda > 0$.

It is illuminating to compare Theorems 1 and 2 with least p^{th} power approximation $(0 < p < 1)$ to $f(x) \equiv x$ not on an interval but on the finite set $S: \{-1, 1\}$ by a polynomial $p_0(x) \equiv c$ of degree 0, $-1 \leqq c \leqq 1$, with weights w_1 and w_2. The deviation is

$$\delta(c) \equiv w_2(1-c)^p + w_1(c+1)^p,$$

which has a *maximum* for $c = 0$ if $\delta'(c) = 0$, as in Theorems 1 and 2; the graph of $\delta(c)$ is concave downward in $-1 \leqq c \leqq 1$.

Likewise for least p^{th} power approximation $(0 < p < 1)$ to the discontinuous function $f(x) \equiv sg\, x$ on the interval $-1 \leqq x \leqq 1$ by a polynomial $p_0(x) \equiv c$ of degree zero, $-1 \leqq c \leqq 1$, the deviation is

$$\delta(c) \equiv \int_{-1}^{0} (c+1)^p dx + \int_0^1 (1-c)^p dx \equiv (1-c)^p + (c+1)^p$$

as before; $\delta(c)$ has again a maximum for $c = 0$ and its graph is concave downward in $-1 \leqq c \leqq 1$. The minimum of $\delta(c)$ occurs for $c = \pm 1$.

In sum, for approximation on a finite set $S, 0 < p < 1$, strong oscillation of the function $f(x) - c$ may lead to a local maximum of $\delta(c)$ when $\delta'(c) = 0$, as in the example above; but the function

$$\delta(c) \equiv \sum w_k |c - f(x_k)|^p,\ w_k > 0,$$

is continuous and piecewise concave downward, so its local and global minima must occur in values of c equal to some $f(x_k)$; such a minimum involves weak oscillation and is independent of strong oscillation. On the other hand, for approximation on an interval E, strong oscillation of $f(x) - c$ with $\delta'(c) = 0$ may lead to a local maximum of $\delta(c)$ as in Theorems 1 and 2, and [4] weak oscillation as with $f(x) \equiv |x|^{1/\alpha}, \alpha \leqq 1$, on $-1 \leqq x \leqq 1$ may lead to a global minimum; it is no accident that the cases $\alpha > 1$ and $\alpha < 1$ are respectively characterized by vertical and horizontal tangents of $f(x)$ at $x = 0$, corresponding with $\delta'(0) = 0$ to maxima and minima of $S(c)$.

7. Summary of results, arbitrary p. We summarize some of the known results on approximation for various values of p, on a real finite point set S or on a closed interval E, for comparison with each other and with Theorems 1 and 2. In each case we approximate by a polynomial $p_n(x)$ of degree n, either to a continuous function $f(x)$ on E, or to a function on a finite set S: $\{x_k\}$ consisting of more than n points. We compare oscillation of the error $f(x) - p_n(x)$ on the one hand to the existence of maxima and minima of the deviation

$$\delta[p_n(x)] = \int_E w(x)|f(x) - p_n(x)|^p dx \text{ or}$$
$$\delta[p_n(x)] = \sum_k w_k|f(x_k) - p_n(x_n)|^p ,$$

where $w(x)$ is nonnegative and not a null function, and we assume $\delta[p_n(x)]$ to be different from zero for all $p_n(x)$.

For $p > 1$, $\delta[p_n(x)]$ is never a local maximum; every local minimum is also a strong global minimum, and the error $f(x) - p_n(x)$ has at least $n + 1$ strong oscillations. Conversely, if the error has $n + 1$ strong oscillations, then there exists a $w(x)$ (continuous for approximation on E) such that $\delta[p_n(x)]$ is a strong global minimum.

For $p = 1$, $\delta[p_n(x)]$ has never a strong local maximum; every local minimum (which can be a weak minimum for approximation on S) is also a global minimum. For approximation on E and every minimum of δ, the error has either at least $n + 1$ strong oscillations or vanishes identically on a subset of E of positive measure; conversely, if the error $f(x) - p_n(x)$ has either $n + 1$ strong oscillations or vanishes on a subset of E of positive measure, $\delta[p_n(x)]$ has a local minimum for suitable continuous weight. For approximation on S, the error has at least $n + 1$ weak oscillations on S if the error has a local minimum; conversely, if the error has at least $n + 1$ weak oscillations, the deviation has a local minimum for suitable weights.

For $0 < p < 1$ and approximation on S, if $\delta[p_n(x)]$ is minimum, then $p_n(x)$ coincides with $f(x)$ in at least $n + 1$ points of S; conversely, if $p_n(x)$ coincides with $f(x)$ in at least $n + 1$ points of $f(x)$, $\delta[p_n(x)]$ is a minimum for suitable weights. For $0 < p < 1$ and approximation on E, coincidence of $p_n(x)$ with $f(x)$ in $n + 1$ points of E is neither necessary nor sufficient that $\delta[p_n(x)]$ be a minimum, and even strong oscillation is neither necessary nor sufficient. Indeed, with strong oscillation and $n = 0$ it may occur (Theorem 2) that $\delta[p_n(x)]$ has a strong *maximum*.

It is clear that the deviation $\delta[p_n(x)]$ varies both with changes in $p_n(x)$ and the weight, and the deviation may also have a maximum or minimum which varies with those changes. In particular, Theorem 2 indicates stability of a maximum of $\delta(c)$ with respect to changes in $w(x)$ that preserve the relation $\delta'(0) = 0$. The writers plan to discuss stability in more detail on another occasion.

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