

# Pacific Journal of Mathematics

**GALOIS COHOMOLOGY OF ABELIAN GROUPS**

DALTON TARWATER

## GALOIS COHOMOLOGY OF ABELIAN GROUPS

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**Normal and separable algebraic extensions of abelian groups have been defined in a manner similar to that of the field theory. In this paper it is shown that if  $N$  is a normal algebraic extension of the torsion group  $K = \sum K_p$ , where the  $p$ -components  $K_p$  of  $K$  are cyclic or divisible, and if  $G$  is the group of  $K$ -automorphisms of  $N$ , then there is a family  $\{G_B\}_{B \in X}$  of subgroups of  $G$  such that  $\{G, \{G_B\}_{B \in X}, N\}$  is a field formation.**

All groups mentioned are abelian. If  $K$  is a subgroup of  $E$ , then  $A_K(E)$  denotes the group of  $K$ -automorphisms of  $E$ . If  $S$  is a subgroup of the automorphism group  $A(E)$  of  $E$ , then  $E^S$  is the subgroup of  $E$  fixed by  $S$ .  $E$  is an algebraic extension of  $K$  if every  $e \in E$  satisfies an equation  $ne = k \neq 0, k \in K$ .  $E$  is a normal extension of  $K$  in an algebraic closure  $D$  of (minimal divisible group containing)  $K$  if every  $K$ -automorphism of  $D$  induces an automorphism of  $E$  and  $E$  is a separable extension of  $K$  if for every  $e \in E, e \notin K$ , there is a  $\sigma \in A_K(D)$  such that  $e \neq \sigma(e) \in E$ . A formation is a field formation [1] if it satisfies:

AXIOM I. For each Galois extension  $F/E$ ,

$$H^1(F/E) = H^1(G_E/G_F, F) = 0 .$$

The following are proved in [6]:

I (THEOREM 8). *Let  $N$  be a normal and separable extension of  $K$  in  $D$  and let  $E (\neq N)$  be an extension of  $K$  in  $N$ .  $E$  is a normal extension of  $K$  if and only if  $A_E(N)$  is a normal subgroup of  $A_K(N)$  and then*

$$A_K(E) \cong A_K(N)/A_E(N) .$$

II (THEOREM 11). *If  $G'$  is a closed subgroup of  $G$  (in the topology defined below) and  $E = N^{G'}$ , then  $G' = A_E(N)$ .*

We now state the

III THEOREM. *Let  $K = \sum K_p$  be a torsion group such that  $K_2$  is divisible or trivial and for a prime  $p \geq 3, K_p$  is divisible or cyclic. If  $N$  is a normal extension of  $K$  in an algebraic closure  $D$  of  $K$ , if  $G = A_K(N)$ , and if  $X$  is the class of groups  $E$  such that  $K \subseteq E \subseteq N$*

and  $G_E = A_E(N)$  is of finite index in  $G$ , then  $\{G, \{G_E\}_{E \in X}, N\}$  is a field formation.

*Proof.* Since  $K$  is a torsion group, it follows (page 54 of [6]) that  $N$  is a separable extension of  $K$ .  $G$  is the complete direct product of the groups  $A_{K_p}(N_p)$  which are abelian, being cyclic if  $N_p$  is cyclic or being isomorphic to a subgroup of the multiplicative group of  $p$ -adic units of  $N_p = D_p \cong Z(p^\infty)$  and  $K_p$  is cyclic.

Let  $\mathcal{L}$  be the class of groups  $L$  such that  $K \subseteq L \subseteq N$  and if  $K_p$  is cyclic while  $N_p = D_p$  then  $L_p$  is cyclic. Topologize  $G$  by taking as a filter base for the neighborhoods of 0 all groups  $G_L = A_L(N)$  with  $L \in \mathcal{L}$ .

Every member of  $X$  is in  $\mathcal{L}$ . For if  $E \in X$ , then by I,  $G/G_E \cong A_K(E) \cong \pi A_{K_p}(E_p)$  is a finite group. So  $E_p = K_p$  for almost all primes  $p$  and if  $E_p \neq K_p$  then  $E_p$  is cyclic (otherwise  $A_{K_p}(E_p)$  is of the power of the continuum). Hence  $E \in \mathcal{L}$ .

We have

A. If  $E$  and  $E'$  are in  $X$ , then  $G_E \cap G_{E'} = G_{E+E'}$  and  $E + E'$  is in  $X$ .

B. If  $E \in X$  and  $G_E \subseteq G' \subseteq G$ , then  $G' = G_{E'}$ , where  $E' = N^{G'} \in X$ .

*Proof of B.*  $G'$  is of finite index and is closed in the topology on  $G$ . An application of II completes the proof.

C. For  $E \in X$ , every conjugate of  $G_E$  equals  $G_E$ .

D. For each  $x \in N$ ,  $\Gamma(x) = \{\gamma(x) \mid \gamma \in G\}$  is one of the  $G_E$  with  $E \in X$ .

*Proof of D.*  $\{K, x\}$ , the group generated by  $K$  and  $x$ , is in  $X$ . For if  $\gamma' \in \gamma G_{\{K, x\}}$  then  $\gamma'(x) = \gamma(x)$ ; but there are only finitely many members of  $\Gamma(x)$  since there are only finitely many elements of  $N$  which are not in  $K$  and have the same order as  $x$ . So  $G_{\{K, x\}}$  is of finite index. Also,  $G_{\{K, x\}} \subseteq \Gamma(x) \subseteq G$ . So by B,  $\Gamma(x)$  is one of the  $G_E$  with  $E \in X$ .

Statements A thru D establish that  $\{G, \{G_E\}_{E \in X}, N\}$  is a formation [5]. It remains to be proved that if  $G_F \subseteq G_E$  for  $E$  and  $F$  in  $X$ , then  $H^1(F/E) = H^1(A_E(F), F) = 0$ . The proof will be established first for cyclic  $p$ -groups ( $p \neq 2$ ). The following lemma will facilitate this proof. The proof of the lemma will be found below.

LEMMA. *If  $p$  is an odd prime and  $M = \sum(1 + p^m)^i, i = 0, 1, \dots, p^{n-m} - 1$ , where  $n > m \geq 1$ , then  $p^{n-m}$  is an exact divisor of  $M$ .*

Now let  $F_p$  be cyclic of order  $p^n$  and algebraic over its subgroup  $E_p$  of order  $p^m, m \geq 1$ . If  $t \in A_{E_p}(F_p)$  is defined by  $t(x) = (1 + p^m)x$ , then  $t$  generates  $A_{E_p}(F_p)$ . By Theorem 7.1 of [4],

$$H^1(A_{E_p}(F_p), F_p) \cong \{f \in F_p \mid Mf = 0\} / \{(t - 1)f \mid f \in F_p\},$$

where  $Mf = \sum(1 + p^m)^i f, i = 0, 1, \dots, p^{n-m} - 1$ , and  $(t - 1)f = p^m f$ . From the lemma,  $Mf = 0$  implies  $f = p^m f'$  for some  $f' \in F_p$ . Thus  $H^1(A_{E_p}(F_p), F_p) = 0$ , concluding the primary cyclic case.

To complete the proof of the theorem, let  $E$  and  $F$  be in  $X$  such that  $G_F \subseteq G_E$ , i.e.,  $F/E$  is a Galois extension. Then by Theorem 10.1 of [2]

$$H^1(F/E)_p = H^1(A_{E_p}(F_p), F_p) = 0$$

for each prime  $p$  and therefore  $H^1(F/E) = 0$ .  $\{G, \{G_E\}_{E \in X}, N\}$  is a field formation.

*Proof of lemma (suggested by A. A. Gioia).* The series defining  $M$  is geometric so  $p^m M = (1 + p^m)^{p^{n-m}} - 1$ . By Theorem 4-5 of [3],  $p^n$  divides the right hand side of this equation. If  $p^{n+1}$  also divides  $p^m M$ , then Theorem 4-5 of [3]—which requires  $p \neq 2$ —can be applied again to yield:

$$1 + p^m \equiv 1 \pmod{p^{n+1-(n-m)}}$$

which is false. The lemma is proved.

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