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Normal and separable algebraic extensions of abelian groups have been defined in a manner similar to that of the field theory. In this paper it is shown that if N is a normal algebraic extension of the torsion group $K = \sum K_p$, where the p-components K_p of K are cyclic or divisible, and if G is the group of K-automorphisms of N, then there is a family $\{G_E\}_{E \in X}$ of subgroups of G such that $\{G, \{G_E\}_{E \in X}, N\}$ is a field formation.

All groups mentioned are abelian. If K is a subgroup of E, then $A_K(E)$ denotes the group of K-automorphisms of E. If S is a subgroup of the automorphism group $A(E)$ of E, then E^S is the subgroup of E fixed by S. E is an algebraic extension of K if every $e \in E$ satisfies an equation $ne = k \neq 0, k \in K$. E is a normal extension of K in an algebraic closure D of (minimal divisible group containing) K if every K-automorphism of D induces an automorphism of E and E is a separable extension of K if for every $e \in E$, $e \notin K$, there is a $\sigma \in A_K(D)$ such that $e \neq \sigma(e) \in E$. A formation is a field formation [1] if it satisfies:

AXIOM I. For each Galois extension F/E,

$$H^1(F/E) = H^1(G_E/G_F, F) = 0 .$$

The following are proved in [6]:

I (THEOREM 8). *Let N be a normal and separable extension of K in D and let $E(\neq N)$ be an extension of K in N. E is a normal extension of K if and only if $A_E(N)$ is a normal subgroup of $A_K(N)$ and then*

$$A_K(E) \cong A_K(N)/A_E(N) .$$

II (THEOREM 11). *If G' is a closed subgroup of G (in the topology defined below) and $E = N^{G'}$, then $G' = A_E(N)$.*

We now state the

III THEOREM. *Let $K = \sum K_p$ be a torsion group such that K_2 is divisible or trivial and for a prime $p \geqq 3$, K_p is divisible or cyclic. If N is a normal extension of K in an algebraic closure D of K, if $G = A_K(N)$, and if X is the class of groups E such that $K \subseteqq E \subseteqq N$*

and $G_E = A_E(N)$ *is of finite index in* G, *then* $\{G, \{G_E\}_{E\in X}, N\}$ *is a field formation.*

Proof. Since K is a torsion group, it follows (page 54 of [6]) that N is a separable extension of K. G is the complete direct product of the groups $A_{K_p}(N_p)$ which are abelian, being cyclic if N_p is cyclic or being isomorphic to a subgroup of the multiplicative group of p-adic units of $N_p = D_p \cong Z(p^\infty)$ and K_p is cyclic.

Let $\mathscr{L}$ be the class of groups L such that $K \subseteqq L \subseteqq N$ and if K_p is cyclic while $N_p = D_p$ then L_p is cyclic. Topologize G by taking as a filter base for the neighborhoods of 0 all groups $G_L = A_L(N)$ with $L \in \mathscr{L}$.

Every member of X is in $\mathscr{L}$. For if $E \in X$, then by I, $G/G_E \cong A_K(E) \cong \pi A_{K_p}(E_p)$ is a finite group. So $E_p = K_p$ for almost all primes p and if $E_p \neq K_p$ then E_p is cyclic (otherwise $A_{K_p}(E_p)$ is of the power of the continuum). Hence $E \in \mathscr{L}$.

We have

A. If E and E' are in X, then $G_E \cap G_{E'} = G_{E+E'}$ and $E + E'$ is in X.

B. If $E \in X$ and $G_E \subseteqq G' \subseteqq G$, then $G' = G_{E'}$, where $E' = N^{G'} \in X$.

Proof of B. G' is of finite index and is closed in the topology on G. An application of II completes the proof.

C. For $E \in X$, every conjugate of G_E equals G_E.

D. For each $x \in N$, $\Gamma(x) = \{\gamma(x) \mid \gamma \in G\}$ is one of the G_E with $E \in X$.

Proof of D. $\{K, x\}$, the group generated by K and x, is in X. For if $\gamma' \in \gamma G_{\{K,x\}}$ then $\gamma'(x) = \gamma(x)$; but there are only finitely many members of $\Gamma(x)$ since there are only finitely many elements of N which are not in K and have the same order as x. So $G_{\{K,x\}}$ is of finite index. Also, $G_{\{K,x\}} \subseteqq \Gamma(x) \subseteqq G$. So by B, $\Gamma(x)$ is one of the G_E with $E \in X$.

Statements A thru D establish that $\{G, \{G_E\}_{E\in X}, N\}$ is a formation [5]. It remains to be proved that if $G_F \subseteqq G_E$ for E and F in X, then $H^1(F/E) = H^1(A_E(F), F) = 0$. The proof will be established first for cyclic p-groups ($p \neq 2$). The following lemma will facilitate this proof. The proof of the lemma will be found below.

LEMMA. *If p is an odd prime and $M = \sum(1+p^m)^i$, $i = 0, 1, \cdots, p^{n-m} - 1$, where $n > m \geqq 1$, then p^{n-m} is an exact divisor of M.*

Now let F_p by cyclic of order p^n and algebraic over its subgroup E_p of order p^m, $m \geqq 1$. If $t \in A_{E_p}(F_p)$ is defined by $t(x) = (1+p^m)x$, then t generates $A_{E_p}(F_p)$. By Theorem 7.1 of [4],

$$H^1(A_{E_p}(F_p), F_p) \cong \{f \in F_p \mid Mf = 0\}/\{(t-1)f \mid f \in F_p\} ,$$

where $Mf = \sum(1+p^m)^i f$, $i = 0, 1, \cdots, p^{n-m} - 1$, and $(t-1)f = p^m f$. From the lemma, $Mf = 0$ implies $f = p^m f'$ for some $f' \in F_p$. Thus $H^1(A_{E_p}(F_p), F_p) = 0$, concluding the primary cyclic case.

To complete the proof of the theorem, let E and F be in X such that $G_F \subseteqq G_E$, i.e., F/E is a Galois extension. Then by Theorem 10.1 of [2]

$$H^1(F/E)_p = H^1(A_{E_p}(F_p), F_p) = 0$$

for each prime p and therefore $H^1(F/E) = 0$. $\{G, \{G_E\}_{E \in X}, N\}$ is a field formation.

Proof of lemma (*suggested by A. A. Gioia*). The series defining M is geometric so $p^m M = (1+p^m)^{p^{n-m}} - 1$. By Theorem 4-5 of [3], p^n divides the right hand side of this equation. If p^{n+1} also divides $p^m M$, then Theorem 4-5 of [3]—which requires $p \neq 2$—can be applied again to yield:

$$1 + p^m \equiv 1 \bmod p^{n+1-(n-m)}$$

which is false. The lemma is proved.

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