

Pacific Journal of Mathematics

GALOIS COHOMOLOGY OF ABELIAN GROUPS

DALTON TARWATER

GALOIS COHOMOLOGY OF ABELIAN GROUPS

DALTON TARWATER

Normal and separable algebraic extensions of abelian groups have been defined in a manner similar to that of the field theory. In this paper it is shown that if N is a normal algebraic extension of the torsion group $K = \sum K_p$, where the p -components K_p of K are cyclic or divisible, and if G is the group of K -automorphisms of N , then there is a family $\{G_B\}_{B \in X}$ of subgroups of G such that $\{G, \{G_B\}_{B \in X}, N\}$ is a field formation.

All groups mentioned are abelian. If K is a subgroup of E , then $A_K(E)$ denotes the group of K -automorphisms of E . If S is a subgroup of the automorphism group $A(E)$ of E , then E^S is the subgroup of E fixed by S . E is an algebraic extension of K if every $e \in E$ satisfies an equation $ne = k \neq 0, k \in K$. E is a normal extension of K in an algebraic closure D of (minimal divisible group containing) K if every K -automorphism of D induces an automorphism of E and E is a separable extension of K if for every $e \in E, e \notin K$, there is a $\sigma \in A_K(D)$ such that $e \neq \sigma(e) \in E$. A formation is a field formation [1] if it satisfies:

AXIOM I. For each Galois extension F/E ,

$$H^1(F/E) = H^1(G_E/G_F, F) = 0 .$$

The following are proved in [6]:

I (THEOREM 8). *Let N be a normal and separable extension of K in D and let $E (\neq N)$ be an extension of K in N . E is a normal extension of K if and only if $A_E(N)$ is a normal subgroup of $A_K(N)$ and then*

$$A_K(E) \cong A_K(N)/A_E(N) .$$

II (THEOREM 11). *If G' is a closed subgroup of G (in the topology defined below) and $E = N^{G'}$, then $G' = A_E(N)$.*

We now state the

III THEOREM. *Let $K = \sum K_p$ be a torsion group such that K_2 is divisible or trivial and for a prime $p \geq 3, K_p$ is divisible or cyclic. If N is a normal extension of K in an algebraic closure D of K , if $G = A_K(N)$, and if X is the class of groups E such that $K \subseteq E \subseteq N$*

and $G_E = A_E(N)$ is of finite index in G , then $\{G, \{G_E\}_{E \in X}, N\}$ is a field formation.

Proof. Since K is a torsion group, it follows (page 54 of [6]) that N is a separable extension of K . G is the complete direct product of the groups $A_{K_p}(N_p)$ which are abelian, being cyclic if N_p is cyclic or being isomorphic to a subgroup of the multiplicative group of p -adic units of $N_p = D_p \cong Z(p^\infty)$ and K_p is cyclic.

Let \mathcal{L} be the class of groups L such that $K \subseteq L \subseteq N$ and if K_p is cyclic while $N_p = D_p$ then L_p is cyclic. Topologize G by taking as a filter base for the neighborhoods of 0 all groups $G_L = A_L(N)$ with $L \in \mathcal{L}$.

Every member of X is in \mathcal{L} . For if $E \in X$, then by I, $G/G_E \cong A_K(E) \cong \pi A_{K_p}(E_p)$ is a finite group. So $E_p = K_p$ for almost all primes p and if $E_p \neq K_p$ then E_p is cyclic (otherwise $A_{K_p}(E_p)$ is of the power of the continuum). Hence $E \in \mathcal{L}$.

We have

A. If E and E' are in X , then $G_E \cap G_{E'} = G_{E+E'}$ and $E + E'$ is in X .

B. If $E \in X$ and $G_E \subseteq G' \subseteq G$, then $G' = G_{E'}$, where $E' = N^{G'} \in X$.

Proof of B. G' is of finite index and is closed in the topology on G . An application of II completes the proof.

C. For $E \in X$, every conjugate of G_E equals G_E .

D. For each $x \in N$, $\Gamma(x) = \{\gamma(x) \mid \gamma \in G\}$ is one of the G_E with $E \in X$.

Proof of D. $\{K, x\}$, the group generated by K and x , is in X . For if $\gamma' \in \gamma G_{\{K, x\}}$ then $\gamma'(x) = \gamma(x)$; but there are only finitely many members of $\Gamma(x)$ since there are only finitely many elements of N which are not in K and have the same order as x . So $G_{\{K, x\}}$ is of finite index. Also, $G_{\{K, x\}} \subseteq \Gamma(x) \subseteq G$. So by B, $\Gamma(x)$ is one of the G_E with $E \in X$.

Statements A thru D establish that $\{G, \{G_E\}_{E \in X}, N\}$ is a formation [5]. It remains to be proved that if $G_F \subseteq G_E$ for E and F in X , then $H^1(F/E) = H^1(A_E(F), F) = 0$. The proof will be established first for cyclic p -groups ($p \neq 2$). The following lemma will facilitate this proof. The proof of the lemma will be found below.

LEMMA. *If p is an odd prime and $M = \sum(1 + p^m)^i, i = 0, 1, \dots, p^{n-m} - 1$, where $n > m \geq 1$, then p^{n-m} is an exact divisor of M .*

Now let F_p be cyclic of order p^n and algebraic over its subgroup E_p of order $p^m, m \geq 1$. If $t \in A_{E_p}(F_p)$ is defined by $t(x) = (1 + p^m)x$, then t generates $A_{E_p}(F_p)$. By Theorem 7.1 of [4],

$$H^1(A_{E_p}(F_p), F_p) \cong \{f \in F_p \mid Mf = 0\} / \{(t - 1)f \mid f \in F_p\},$$

where $Mf = \sum(1 + p^m)^i f, i = 0, 1, \dots, p^{n-m} - 1$, and $(t - 1)f = p^m f$. From the lemma, $Mf = 0$ implies $f = p^m f'$ for some $f' \in F_p$. Thus $H^1(A_{E_p}(F_p), F_p) = 0$, concluding the primary cyclic case.

To complete the proof of the theorem, let E and F be in X such that $G_F \subseteq G_E$, i.e., F/E is a Galois extension. Then by Theorem 10.1 of [2]

$$H^1(F/E)_p = H^1(A_{E_p}(F_p), F_p) = 0$$

for each prime p and therefore $H^1(F/E) = 0$. $\{G, \{G_E\}_{E \in X}, N\}$ is a field formation.

Proof of lemma (suggested by A. A. Gioia). The series defining M is geometric so $p^m M = (1 + p^m)^{p^{n-m}} - 1$. By Theorem 4-5 of [3], p^n divides the right hand side of this equation. If p^{n+1} also divides $p^m M$, then Theorem 4-5 of [3]—which requires $p \neq 2$ —can be applied again to yield:

$$1 + p^m \equiv 1 \pmod{p^{n+1-(n-m)}}$$

which is false. The lemma is proved.

REFERENCES

1. E. Artin and J. Tate, *Class Field Theory*, Harvard, 1961.
2. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956.
3. W. J. LeVeque, *Topics in Number Theory*, Vol. I, Addison-Wesley, Reading, 1956.
4. S. MacLane, *Homology*, Springer-Verlag, Berlin, 1963.
5. J. P. Serre, *Corps Locaux*, Hermann, Paris, 1962.
6. D. Tarwater, *Galois theory of abelian groups*, Math. Zeit. **95** (1967), 50-59.

Received February 20, 1967. Partly supported by NFS Grant GP-2214.

WESTERN MICHIGAN UNIVERSITY
 KALAMAZOO, MICHIGAN
 AND
 NORTH TEXAS STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN

Stanford University
Stanford, California

J. P. JANS

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

Department of Mathematics
Rice University
Houston, Texas 77001

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL ORDNANCE TEST STATION

Harry P. Allen, <i>Lie algebras of type D_4 over algebraic number fields</i>	1
Charles Ballantine, <i>Products of positive definite matrices. II</i>	7
David W. Boyd, <i>The spectral radius of averaging operators</i>	19
William Howard Caldwell, <i>Hypercyclic rings</i>	29
Francis William Carroll, <i>Some properties of sequences, with an application to noncontinuable power series</i>	45
David Fleming Dawson, <i>Matrix summability over certain classes of sequences ordered with respect to rate of convergence</i>	51
D. W. Dubois, <i>Second note on David Harrison's theory of preprimes</i>	57
Edgar Earle Enochs, <i>A note on quasi-Frobenius rings</i>	69
Ronald J. Ensey, <i>Isomorphism invariants for Abelian groups modulo bounded groups</i>	71
Ronald Owen Fulp, <i>Generalized semigroup kernels</i>	93
Bernard Robert Kripke and Richard Bruce Holmes, <i>Interposition and approximation</i>	103
Jack W. Macki and James Sai-Wing Wong, <i>Oscillation of solutions to second-order nonlinear differential equations</i>	111
Lothrop Mittenthal, <i>Operator valued analytic functions and generalizations of spectral theory</i>	119
T. S. Motzkin and J. L. Walsh, <i>A persistent local maximum of the pth power deviation on an interval, $p < 1$</i>	133
Jerome L. Paul, <i>Sequences of homeomorphisms which converge to homeomorphisms</i>	143
Maxwell Alexander Rosenlicht, <i>Liouville's theorem on functions with elementary integrals</i>	153
Joseph Goeffrey Rosenstein, <i>Initial segments of degrees</i>	163
H. Subramanian, <i>Ideal neighbourhoods in a ring</i>	173
Dalton Tarwater, <i>Galois cohomology of abelian groups</i>	177
James Patrick Williams, <i>Schwarz norms for operators</i>	181
Raymond Y. T. Wong, <i>A wild Cantor set in the Hilbert cube</i>	189