A WILD CANTOR SET IN THE HILBERT CUBE

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Let $E^n$ be the Euclidean $n$-space. A Cantor set $C$ is a set homeomorphic with the Cantor middle-third set. Antoine and Blankinship have shown that there exists a “wild” Cantor set in any $E^n$ for $n \geq 3$, where “wild” means that $E^n - C$ is not simply connected. However it is also known that no “wild” Cantor set (in fact, compact set) can exist in many infinite dimensional spaces, such as $s$ (the countably infinite product of lines) or the Hilbert space $l_2$. A result of this paper provides a positive answer for a generalization of Blankinship’s result in the Hilbert cube.

If $X$ is a space, we denote by $X^n$ the space $\prod_{i=1}^{n} X_i$ and $X^\infty$ the space $\prod_{i=1}^{\infty} X_i$ with $X_i = X$. Let $\tau_n$ denote the projecting function of $X^\infty$ onto $X^n$ and $\pi_n$ the projecting function of $X^\infty$ onto $X_n$. Let $J, \tilde{J}$ denote intervals $[-1, 1], (-1, 1)$ respectively. The Hilbert cube is the space $J^\infty$ under the metric $\rho(x, y) = \sum_{i=1}^{\infty} (|x_i - y_i|)/2^i$. Hilbert space, $l_2$, is the space of all square summable sequences of real numbers with metric $d((x_i), (y_i)) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$. The space $\tilde{J}^\infty$ is also denoted by $s$. Let $E^n = \prod_{i=1}^{n} E_i$ be the Euclidean $n$-space.

A Cantor set is a set homeomorphic with the Cantor middle-third set. The existence of a Cantor set $C$ in $E^n$ ($n \geq 3$) such that $E^n - C$ is not simply connected was first demonstrated by Antoine [4] in 1921 and constructed by W. A. Blankinship [5] in 1951. It is known that every Cantor set is (or in $l_2$) must be tame, in the sense that its complement in $s$ (or in $l_2$) is topologically as nice as the space itself. In fact it has been proved (by V. Klee in the case of $l_2$ [9] and by R. D. Anderson [1] in the case of $s$, using Klee’s method) that if $K$ is a compact set in $X$ (for $X = s$ or $l_2$), then $X - K \approx X$. The question as to whether a finite dimensional closed set can leave the Hilbert cube multiply connected (in particular, whether a Cantor set can have this property) was then raised in [5] by Blankinship and was also later mentioned in [7] by Klee. In this paper we shall give such a question a positive answer by constructing a Cantor set $C$ in the Hilbert cube $J^\infty$ such that $J^\infty - C$ is not homotopically trivial. In fact, we shall apply the result of Blankinship [5] to show that $J^\infty - C$ has nontrivial 1st-Homotopy group. We remark that such a set $C$ cannot be constructed as a subset of $\tilde{J}^\infty$. Note that Anderson [1] (by using Klee’s method) proved that any Cantor set $C$ (in fact, any compact set) in $\tilde{J}^\infty$ can be carried into an end-face, say $K_i = \{x \in J^\infty \mid \pi_i(x) = 1\}$, by a homeomorphism on $J^\infty$. It is quite clear that the complement of any Cantor subset (in fact, any compact subset)
of $K_1$ in $J^\omega$ is homotopically trivial, therefore, if the complement of $C$ in $J^\omega$ is to be homotopically nontrivial, $C$ must, in a sense, join various end-faces of $J^\omega$.

2. Some notation and lemma. All homeomorphisms concerned are assumed to be geometric homeomorphisms, and when a homeomorphism has domain in $E^n$, it is assumed to be linear. Two subsets of $E^n$ are similar if they are homeomorphic under some homeomorphism. Let $\Delta$ denote the boundary of the unit square in $E^2$. A $*-circle$ is a set homeomorphic to $\Delta$. An $n$-tube, $n \geq 3$, is a set homeomorphic to the product of a circular 2-cell with $(n-2)$ $*-circles$.

We shall choose a fixed set of positive real numbers $r_1, r_2, \ldots$ with the properties that (1) $r_i > 1$ and (2) $r_{i+1} > 2(\sum_{j=1}^{i} r_j)$. Let $L_i = [r_i = 1, r_i + 1] \subset E_i$ and $L^n = \prod_{i=1}^{n} L_i \times (r_{n+1}, r_{n+2}, \ldots)$. We shall regard $E^n$ as a subset of $E^{n+1}$ by considering $E^n$ as $E x^0$.

**Lemma 1.** If $X$ is a Hausdorff space and $A_1, A_2, \ldots$ is a decreasing sequence of compact subsets of $X$ such that each $A_i$ is dense in itself, then $\bigcap_{i=1}^{\infty} A_i$ is dense in itself.

**Proof.** If $x$ is an isolated point of $\bigcap_{i=1}^{\infty} A_i$, then for some $i$, $x$ is an isolated point of $A_i$, contrary to the hypothesis.

3. Brief outline of the construction. The construction is an inductive modification of the construction by Antoine [4] and by Blankinship [5]. The Cantor set $C$ will be the intersection of a decreasing sequence of compact subsets $K_1, K_2, \cdots$ of the Hilbert cube $L^\omega = \prod_{i=1}^{\omega} L_i$. For each $n \geq 3$, $K_n$ will be the product of a compact subset $K_n'$ of $L^n$ with $\prod_{i=n+1}^{\omega} L_i$. $K_n'$ is the intersection of a simple chain of linking 3-tubes of $E^3$ with $L^3$. $K_n'$ will be contained in $K_n' \times L_i$ and is the intersection of a simple chain of linking 4-tubes of $E^4$ with $L^4$ and so on.

4. Construction of $K^\omega$.

**Definition.** Let $r, s$ be positive integers and $d_r$ an arbitrary real number. Let $S$ be a compact subset of $E^\omega(= \prod_{i=1}^{\omega} E_i)$ such that $\pi_r(S) = d_r$. We say $\tilde{S}$ is the set generated by rotating $S$ about the hyperplane $x_r = d_r$ and $x_s = 0$ if

$$\tilde{S} = \left\{ x \in E^\omega : \exists y \in S \exists (x_r, x_s) \in \text{Bd}([d_r - y_s, d_r + y_s] \times [-y_s, y_s]) \quad \text{and} \quad x_i = y_i \text{ for } i \neq r, s \right\}$$

where $[d_r - y_s, d_r + y_s] \subset E_r$, $[-y_s, y_s] \subset E_s$. 

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The following Lemma is evident:

**LEMMA 2.** Suppose $S$ is the set defined above and $\pi_1(S) > 0$, then $\tilde{S}$ is homeomorphic to the product of $S$ with a $^*$-circle.

**DEFINITION.** Let

$$T^2 = \{x \in E^\infty : (x_i - r_i)^2 + (x_2 - r_2)^2 \leq \left(\frac{1}{4}\right)^2 \text{ and } x_i = r_i \text{ for } i \geq 3\}$$

$$A_i = \{x \in E^\infty : (x_i - r_i)^2 + (x_2 - r_2)^2 = \left(\frac{1}{2}\right)^2 \text{ and } x_i = r_i \text{ for } i \geq 3\}.$$

For $n \geq 3$, define $T^n$ inductively to be the set generated by rotating $T^{n-1}$ about the hyperplane $x_{n-1} = 0$, $x_n = r_n$.

**LEMMA 3.** For $n \geq 2$, $\min \pi_2(T^n) \geq 1$.

**Proof.** It is clear for $n = 2$. For $n \geq 3$, it follows from the fact $\min \pi_2(T^n) = r_n - (1/4 + r_2 + \cdots + r_{n-1})$ and from the hypothesis of $r_i$.

**LEMMA 4.** For $n \geq 3$, $T^n$ is an $n$-tube in $E^n$.

**Proof.** $\pi_2(T^n) > 0$ by Lemma 3. Then by Lemma 2, $T^3$ is a 3-tube. Inductively, $T^n$ is an $n$-tube.

**LEMMA 5.** For $n \geq 3$, $T^n \cap L^n = \tau_2(T^n) \times \prod_{i=3}^n L_i \times (r_{n+1}, r_{n+2}, \cdots)$.

**Proof.** This is a consequence of Lemma 3.

Let $\{t_i^2\}_{i=1}^l$ be a chain of cyclically linked disjoint 3-tubes contained in the interior of $T^3$ and looping once around the axis of $T^3$. We assume (1) they are all similar to $T^3$, (2) $l \equiv 0 \pmod{4}$ and $l$ is large enough so that each $t_i^2$ can be regarded as the set generated by rotating a small circular 2-cell $t_i^2$ along a small $^*$-circle $A_i$, (3) $\text{diam}(t_i^2) < 1/3(\text{diam } T^3)$ for all $i$, and (4) Only two members of $\{t_i^2\}_{i=1}^l$ intersect $\text{Bd}(L^3)$(one in each side) and the intersection of each such $t_i^2$ with $\text{Bd}(K^3)$ is exactly two disjoint 2-cells. Let $A_3 = \bigcup_{i=1}^l t_i^2$, $K_3' = A_3 \cap L^3$ and $K_3 = K_3' \times \prod_{i=4}^\infty L_i$.

5. **Construction of $K_4, K_5, \cdots$.** For the purpose of simplicity, we shall give only the construction of $K_4$ and assert that for $n \geq 5$, $K_n$ can be inductively constructed.

**Step 1.** For each $i$, let $h_i$ be a (linear) homeomorphism of $T^3$
onto $t_i^a$. Hence $\{t_{ij}^a = h_i(t_{ij}^a)\}_{j=1}^n$ is a similar chain of cyclically linked disjoint 3-tubes in $t_i^a$. We require that each $h_i$ is so chosen that (1) if $t_i^a$ is a member that intersects $\text{Bd}(L^3)$, then only two members of $\{t_{ij}^a\}_{j=1}^n$ intersect $\text{Bd}(L^3)$ and the intersection of each such member with $\text{Bd}(L^3)$ is exactly two disjoint 2-cells and (2) $\text{diam}(t_{ij}^a) < (1/3\delta)\text{diam}(T^n)$ for all $ij$.

**Step 2.** For each $i,j$, let $t_{ij}^4$ be the 4-tube in $T^4$ generated by rotating $t_{ij}^3$ about planes $x_3 = 0$, $x_4 = r_i$. We now regard each $t_{ij}^3$ as the set generated by rotating a small 2-cell $t_{ij}^3$ along a small $\star$-circle. We assume further that $t_{ij}^3$ is contained in $L^3$ whenever $t_{ij}^3$ intersects $L^4$. Let $\tilde{t}_{ij}^3$ be the set generated by rotating $t_{ij}^3$ about planes $x_2 = 0$, $x_4 = r_i$. Then $t_{ij}^3$ can be regarded as the geometric product of $\tilde{t}_{ij}^3$ with $A_{ij}$. Let $h_{ij}^3$ be a linear homeomorphism of $T^3$ onto $\tilde{t}_{ij}^3$. Let $t_{ij}^4 = h_{ij}(t_{ij}^3)$, $k = 1, 2, \ldots, l$. We require each $h_{ij}^3$ is so chosen that (1) if $t_{ij}^3 \subset L^3$, then only two members of $\{t_{ijk}^3\}_{k=1}^l$ intersect $L^3 \times \text{Bd}(L^4)$ (one in each side) and the intersection of each such member with $L^3 \times \text{Bd}(L^4)$ is exactly two disjoint 2-cells and (2) $\text{diam}(t_{ijk}^3) < (1/3\delta)\text{diam}(T^n)$. Let $t_{ijk}^3$ denote the geometric product of $t_{ijk}^3$ with $A_{ij}$. Let $A_i = \bigcup_{i,j,k=1}^n t_{ijk}^4$, $K_i = A_i \cap L^4$ and $K_i = K_i \times \prod_{i=5}^n L_i$.

6. **Theorem 1.** Let $C = \bigcap_{i=3}^n K_i$. Then $C$ is a Cantor set in $L^n$.

*Proof.* It follows from the construction that $K_a, K_i, \ldots$ is a decreasing sequence of compact subset of $L^n$ and each $K_i$ is dense in itself. Hence $C$ is dense in itself by Lemma 1. Furthermore, each $K_i$ is a finite union of disjoint compact subsets whose diameters are uniformly small and tend to zero as $i \to \infty$. We conclude then that $C$ is a compact zero-dimensional space which is dense in itself, hence is a Cantor set.

**Theorem 2.** If $F$ is a mapping of $A_0 \times I$ into $L^n$ ($n \geq 3$) such that $F|_{A_0 \times 0} = \text{identity on } A_0$ and $F(A_0 \times 1)$ is a point, then $F(A_0 \times I) \cap K_i' \neq \emptyset$.

*Proof.* The proof is due to [5]. Basically Blankinship had constructed a Cantor set $C'$ in $A_n$ such that $C'$ links $A_0$ in $E^n$, hence $A_n$ also links $A_0$ in $E^n$. As a consequence, $K_n' = A_n \cap L^n$ links $A_0$ in $L^n$.

**Theorem 3.** $L_0 \rightarrow C$ has nontrivial 1st-Homotopy group.

*Proof.* Let $F$ be a mapping of $A_0 \times I$ into $L^n$ such that $F|_{A_0 \times 0} = 

identity on $D_0$ and $F(D_0 \times 1)$ is a point. For each $n \geq 3$, $\tau_n(F')$ is a mapping of $D_n \times I$ into $L^n$ satisfying $(\tau_n F)_{D_0 \times 0} = \text{identity on } D_0$ and $(\tau_n F)(D_0 \times 1)$ is a point. Hence by Theorem 2, $(\tau_n F)(D_0 \times I) \cap K_n' \neq \emptyset$. This implies $F(D_0 \times I) \cap K_n = \emptyset$, hence $F(D_0 \times I) \cap C \neq \emptyset$.

**Theorem 4.** There exist two Cantor sets in the Hilbert cube such that no homeomorphism of one onto the other can be extended to a homeomorphism on the whole Hilbert cube.

Let $\hat{L}_i = \text{Int}(L_i)$ and let $(\hat{L})^\omega = \prod_{i=1}^{\omega} \hat{L}_i$. Let $V'_n = K'_n \cap \text{Int}(L^n)$ and $V_n = V'_n \times \prod_{i=n+1}^{\omega} \hat{L}_i$. Then each $V_n$ is a closed subset of $(\hat{L})^\omega$ and hence $C_0 = \bigcap_{n=3}^{\omega} V'_n$ is both zero-dimensional and closed in $(\hat{L})^\omega$. By similar reasoning $C_0$ links $D_0$ in $(\hat{L})^\omega$. Finally, using the fact $s \simeq (\hat{L})^\omega$ and $l_2 \cong s [2]$, we conclude:

**Theorem 5.** $s$ and $l_2$ contain zero-dimensional closed sets whose complements are not simply-connected.

**References**


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