A WILD CANTOR SET IN THE HILBERT CUBE

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Let $E^n$ be the Euclidean $n$-space. A Cantor set $C$ is a set homeomorphic with the Cantor middle-third set. Antoine and Blankinship have shown that there exists a "wild" Cantor set in any $E^n$ for $n \geq 3$, where "wild" means that $E^n - C$ is not simply connected. However it is also known that no "wild" Cantor set (in fact, compact set) can exist in many infinite dimensional spaces, such as $s$ (the countably infinite product of lines) or the Hilbert space $l_2$. A result of this paper provides a positive answer for a generalization of Blankinship's result in the Hilbert cube.

If $X$ is a space, we denote by $X^n$ the space $\Pi_{i=1}^n X_i$ and $X^\infty$ the space $\Pi_{i=1}^\infty X_i$ with $X_i = X$. Let $\tau_n$ denote the projecting function of $X^\infty$ onto $X^n$ and $\pi_n$ the projecting function of $X^\infty$ onto $X_n$. Let $J, J^\infty$ denote intervals $[-1,1], (-1,1)$ respectively. The Hilbert cube is the space $J^\infty$ under the metric $\rho(x, y) = \sum_{i \geq 1}(|x_i - y_i|)/2^i$. Hilbert space, $l_2$, is the space of all square summable sequences of real numbers with metric $d((x_i), (y_i)) = \sqrt{\sum_{i=1}^\infty (x_i - y_i)^2}$. The space $J^\infty$ is also denoted by $s$. Let $E^n = \Pi_{i=1}^n E_i$ be the Euclidean $n$-space.

A Cantor set is a set homeomorphic with the Cantor middle-third set. The existence of a Cantor set $C$ in $E^n$ ($n \geq 3$) such that $E^n - C$ is not simply connected was first demonstrated by Antoine [4] in 1921 and constructed by W. A. Blankinship [5] in 1951. It is known that every Cantor set is $s$ (or in $l_2$) must be tame, in the sense that its complement in $s$ (or in $l_2$) is topologically as nice as the space itself. In fact it has been proved (by V. Klee in the case of $l_2$ [9] and by R. D. Anderson [1] in the case of $s$, using Klee's method) that if $K$ is a compact set in $X$ (for $X = s$ or $l_2$), then $X - K \approx X$. The question as to whether a finite dimensional closed set can leave the Hilbert cube multiply connected (in particular, whether a Cantor set can have this property) was then raised in [5] by Blankinship and was also later mentioned in [7] by Klee. In this paper we shall give such a question a positive answer by constructing a Cantor set $C$ in the Hilbert cube $J^\infty$ such that $J^\infty - C$ is not homotopically trivial. In fact, we shall apply the result of Blankinship [5] to show that $J^\infty - C$ has nontrivial 1st-Homotopy group. We remark that such a set $C$ cannot be constructed as a subset of $J^\infty$. Note that Anderson [1] (by using Klee's method) proved that any Cantor set $C$ (in fact, any compact set) in $J^\infty$ can be carried into an end-face, say $K_1 = \{x \in J^\infty | \pi_1(x) = 1\}$, by a homeomorphism on $J^\infty$. It is quite clear that the complement of any Cantor subset (in fact, any compact subset)
of \( K \) in \( J^\infty \) is homotopically trivial, therefore, if the complement of \( C \) in \( J^\infty \) is to be homotopically nontrivial, \( C \) must, in a sense, join various end-faces of \( J^\infty \).

2. Some notation and lemma. All homeomorphisms concerned are assumed to be geometric homeomorphisms, and when a homeomorphism has domain in \( E^n \), it is assumed to be linear. Two subsets of \( E^n \) are similar if they are homeomorphic under some homeomorphism. Let \( \Delta \) denote the boundary of the unit square in \( E^2 \). A \(*\)-circle is a set homeomorphic to \( A \). An \( n \)-tube, \( n \geq 3 \), is a set homeomorphic to the product of a circular 2-cell with \((n - 2) \)*-circles.

We shall choose a fixed set of positive real numbers \( r_1, r_2, \ldots \) with the properties that (1) \( r_1 > 1 \) and (2) \( r_{n+1} > 2(\sum_{i=1}^{n} r_i) \). Let \( L_i = [r_i - 1, r_i + 1] \subset E_i \) and \( L^n = \prod_{i=1}^{n} L_i \times (r_{n+1}, r_{n+2}, \ldots) \). We shall regard \( E^n \) as a subset of \( E^{n+1} \) by considering \( E^n \) as \( E^0 \).

**Lemma 1.** If \( X \) is a Hausdorff space and \( A_1, A_2, \ldots \) is a decreasing sequence of compact subsets of \( X \) such that each \( A_i \) is dense in itself, then \( \bigcup_{i=1}^{\infty} A_i \) is dense in itself.

**Proof.** If \( x \) is an isolated point of \( \bigcap_{i=1}^{\infty} A_i \), then for some \( i, x \) is an isolated point of \( A_i \), contrary to the hypothesis.

3. Brief outline of the construction. The construction is an inductive modification of the construction by Antoine [4] and by Blankinship [5]. The Cantor set \( C \) will be the intersection of a decreasing sequence of compact subsets \( K_1, K_2, \ldots \) of the Hilbert cube \( L^\infty = \prod_{i=1}^{\infty} L_i \). For each \( n \geq 3 \), \( K_n \) will be the product of a compact subset \( K'_n \) of \( L^n \) with \( \prod_{i=n+1}^{\infty} L_i \). \( K'_n \) is the intersection of a simple chain of linking 3-tubes of \( E^3 \) with \( L^3 \). \( K'_n \) will be contained in \( K'_n \times L_4 \) and is the intersection of a simple chain of linking 4-tubes of \( E^4 \) with \( L^4 \) and so on.

4. Construction of \( K_3 \).

**Definition.** Let \( r, s \) be positive integers and \( d_r \) an arbitrary real number. Let \( S \) be a compact subset of \( E^\infty (= \prod_{i=1}^{\infty} E_i) \) such that \( \pi_r(S) = d_r \). We say \( \tilde{S} \) is the set generated by rotating \( S \) about the hyperplane \( x_r = d_r \) and \( x_s = 0 \) if

\[
\tilde{S} = \left\{ x \in E^\infty : \exists y \in S \exists (x_r, x_s) \in \text{Bd}([d_r - y_s, d_r + y_s] \times [-y_s, y_s]) \right\}
\]

and \( x_i = y_i \) for \( i \neq r, s \) if

\[
\text{where } [d_r - y_s, d_r + y_s] \subset E_r, [-y_s, y_s] \subset E_s.
\]
The following Lemma is evident:

**Lemma 2.** Suppose $S$ is the set defined above and $\pi_s(S) > 0$, then $\tilde{S}$ is homeomorphic to the product of $S$ with a $\ast$-circle.

**Definition.** Let

$$T^2 = \{ x \in E^\infty : (x_1 - r_1)^2 + (x_2 - r_2)^2 \leq \left(\frac{1}{4}\right)^2 \text{ and } x_i = r_i \text{ for } i \geq 3 \}$$

$$A_0 = \{ x \in E^\infty : (x_1 - r_1)^2 + (x_2 - r_2)^2 = \left(\frac{1}{2}\right)^2 \text{ and } x_i = r_i \text{ for } i \geq 3 \}.$$

For $n \geq 3$, define $T^n$ inductively to be the set generated by rotating $T^{n-1}$ about the hyperplane $x_{n-1} = 0$, $x_n = r_n$.

**Lemma 3.** For $n \geq 2$, $\min \pi_n(T^n) \geq 1$.

**Proof.** It is clear for $n = 2$. For $n \geq 3$, it follows from the fact $\min \pi_n(T^n) = r_n - (1/4 + r_2 + \cdots + r_{n-1})$ and from the hypothesis of $r_i$.

**Lemma 4.** For $n \geq 3$, $T^n$ is an $n$-tube in $E^n$.

**Proof.** $\pi_2(T^n) > 0$ by Lemma 3. Then by Lemma 2, $T^3$ is a 3-tube. Inductively, $T^n$ is an $n$-tube.

**Lemma 5.** For $n \geq 3$, $T^n \cap L^n = \tau^3(T^3) \times \prod_{i=3}^n L_i \times (r_{n+1}, r_{n+2}, \cdots)$.

**Proof.** This is a consequence of Lemma 3.

Let $\{t_i^3\}_{i=1}^l$ be a chain of cyclically linked disjoint 3-tubes contained in the interior of $T^3$ and looping once around the axis of $T^3$. We assume (1) they are all similar to $T^3$, (2) $l \equiv 0 \pmod{4}$ and $l$ is large enough so that each $t_i^3$ can be regarded as the set generated by rotating a small circular 2-cell $t_i^2$ along a small $\ast$-circle $A_i$, (3) $\text{diam}(t_i^3) < 1/3(\text{diam } T^3)$ for all $i$, and (4) Only two members of $\{t_i^3\}_{i=1}^l$ intersect $\text{Bd}(L^2)$ (one in each side) and the intersection of each such $t_i^3$ with $\text{Bd}(K)^3$ is exactly two disjoint 2-cells. Let $A_3 = \bigcup_{i=1}^l t_i^3$, $K' = A_3 \cap L^2$ and $K_3 = K' \times \prod_{i=4}^\infty L_i$.

5. Construction of $K_4$, $K_5$, \ldots. For the purpose of simplicity, we shall give only the construction of $K_4$ and assert that for $n \geq 5$, $K_n$ can be inductively constructed.

**Step 1.** For each $i$, let $h_i$ be a (linear) homeomorphism of $T^3$
onto \( t^3_i \). Hence \( \{ t^3_{ij} = h_i(t^3_j) \}_{j=1}^{n} \) is a similar chain of cyclically linked disjoint 3-tubes in \( t^3_i \). We require that each \( h_i \) is so chosen that (1) if \( t^3_i \) is a member that intersects \( \text{Bd}(L^3) \), then only two members of \( \{ t^3_{ij} \}_{j=1}^{n} \) intersect \( \text{Bd}(L^3) \) and the intersection of each such member with \( \text{Bd}(L^3) \) is exactly two disjoint 2-cells and (2) \( \text{diam}(t^3_{ij}) < (1/3^n)\text{diam}(T^3) \) for all \( ij \).

**Step 2.** For each \( i,j \), let \( t^4_{ij} \) be the 4-tube in \( T^n \) generated by rotating \( t^3_i \) about planes \( x_3 = 0, x_4 = r_i \). We now regard each \( t^4_{ij} \) as the set generated by rotating a small 2-cell \( t^3_{ij} \) along a small *-circle. We assume further that \( t^4_{ij} \) is contained in \( L^3 \) whenever \( t^3_i \) intersects \( \mathbb{R}^3 \). Let \( \widehat{t}^3_{ij} \) be the set generated by rotating \( t^3_{ij} \) about planes \( x_2 = 0, \alpha = \pi \). Then \( t^4_{ij} \) can be regarded as the geometric product of \( \widehat{t}^3_{ij} \) with \( \Delta_{ij} \). Let \( h_{ij} \) be a linear homeomorphism of \( T^n \) onto \( \widehat{t}^3_{ij} \). Let \( t^4_{ijk} = h_{ij}(t_k) \), \( k = 1,2,\ldots,l \). We require each \( h_{ij} \) is so chosen that (1) if \( t^3_{ij} \subset \mathbb{R}^3 \), then only two members of \( \{ t^4_{ijk} \}_{k=1}^{l} \) intersect \( \mathbb{R}^3 \times \text{Bd}(L) \) (one in each side) and the intersection of each such member with \( \mathbb{R}^3 \times \text{Bd}(L) \) is exactly two disjoint 2-cells and (2) \( \text{diam}(t^4_{ijk}) < (1/3^n)\text{diam}(T^3) \). Let \( t^4_{ijk} \) denote the geometric product of \( t^3_{ijk} \) with \( \Delta_{ij} \).

Let \( A_4 = \bigcup_{i,j,k=1}^{n} t^4_{ijk}, K'_4 = A_4 \cap L^4 \) and \( K_4 = K'_4 \times \prod_{i=0}^{n} L_i \).

6. **Theorem 1.** Let \( C = \bigcap_{i=3}^{n} K_i \). Then \( C \) is a Cantor set in \( \mathbb{R}^n \).

**Proof.** It follows from the construction that \( K_3, K_4, \ldots \) is a decreasing sequence of compact subset of \( \mathbb{R}^n \) and each \( K_i \) is dense in itself. Hence \( C \) is dense in itself by Lemma 1. Furthermore, each \( K_i \) is a finite union of disjoint compact subsets whose diameters are uniformly small and tend to zero as \( i \to \infty \). We conclude then that \( C \) is a compact zero-dimensional space which is dense in itself, hence is a Cantor set.

**Theorem 2.** If \( F \) is a mapping of \( \Delta_0 \times I \) into \( \mathbb{R}^n \) (\( n \geq 3 \)) such that \( F \mid_{\Delta_0 \times 0} = \text{identity on} \ \Delta_0 \) and \( F(\Delta_0 \times 1) \) is a point, then \( F(\Delta_0 \times I) \cap K_n' \neq \emptyset \).

**Proof.** The proof is due to [5]. Basically Blankinship had constructed a Cantor set \( C' \) in \( A_n \) such that \( C' \) links \( \Delta_0 \) in \( E^n \), hence \( A_n \) also links \( \Delta_0 \) in \( E^n \). As a consequence, \( K_n' = A_n \cap \mathbb{R}^n \) links \( \Delta_0 \) in \( \mathbb{R}^n \).

**Theorem 3.** \( \mathbb{R}^n - C \) has nontrivial 1st-Homotopy group.

**Proof.** Let \( F \) be a mapping of \( \Delta_0 \times I \) into \( \mathbb{R}^n \) such that \( F \mid_{\Delta_0 \times 0} = \emptyset \).
identity on $Δ_o$ and $F(Δ_o \times 1)$ is a point. For each $n \geq 3$, $τ_n(F')$ is a mapping of $Δ_o \times I$ into $L^n$ satisfying $(τ_n F')_{Δ_o \times 0} = \text{identity on } Δ_o$ and $(τ_n F')(Δ_o \times 1)$ is a point. Hence by Theorem 2, $(τ_n F')(Δ_o \times I) \cap K'_n \neq \phi$. This implies $F(Δ_o \times I) \cap K_n \neq \phi$, hence $F(Δ_o \times I) \cap C \neq \phi$.

**Theorem 4.** There exist two Cantor sets in the Hilbert cube such that no homeomorphism of one onto the other can be extended to a homeomorphism on the whole Hilbert cube.

Let $\hat{L}_i = \text{Int}(L_i)$ and let $(\hat{L})^\infty = \prod_{i=1}^\infty \hat{L}_i$. Let $V'_n = K'_n \cap \text{Int}(L^n)$ and $V_n = V'_n \times \prod_{i=n+1}^\infty \hat{L}_i$. Then each $V_n$ is a closed subset of $(\hat{L})^\infty$ and hence $C_0 = \bigcap_{n=3}^\infty V_n$ is both zero-dimensional and closed in $(\hat{L})^\infty$. By similar reasoning $C_0$ links $Δ_o$ in $(\hat{L})^\infty$. Finally, using the fact $s \simeq (\hat{L})^\infty$ and $l_z \simeq s$ [2], we conclude:

**Theorem 5.** $s$ and $l_z$ contain zero-dimensional closed sets whose complements are not simply-connected.

**References**


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Louisiana State University, Baton Rouge
University of California, Los Angeles
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