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THE RATIONAL HOMOTOPY OF A WEDGE

ALLAN CLARK AND LARRY SMITH

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The rational homotopy of a wedge $X \vee Y$ is given in terms of the rational homotopy of X and Y.

Let X be a pathwise connected and simply connected space with base point e_X , which is a neighborhood deformation retract in X. (See [5].) We shall say that X is a nicely pointed space. The rational homotopy of X is the connected graded Lie algebra over Q, $\mathcal{L}_*(X)$, defined by setting $\mathcal{L}_n(X) = \pi_{n+1}(X, e_X) \otimes Q$, with the Lie product induced by the Whitehead product on homotopy groups.

The purpose of this note is to show that the functor \mathcal{L}_* preserves coproducts. More precisely we show:

THEOREM 1. Let X and Y be nicely pointed spaces which are pathwise connected and simply connected and whose rational homotopy has finite type. Then there is a natural isomorphism of graded Lie algebras

$$\phi(X, Y)$$
: $\mathcal{L}_*(X \vee Y) \approx \mathcal{L}_*(X) \perp \mathcal{L}_*(Y)$

where \perp denotes the coproduct in the category of connected graded Lie algebras over Q (defined below).

The result follows easily from the natural isomorphism of $\mathcal{L}_*(X)$ with $H_*(\Omega X; Q)$, the Lie algebra of primitive elements of the Hopf algebra $H_*(\Omega X; Q)$. This isomorphism was discovered by Cartan and Serre; a revised statement [4, page 263] is due to John Moore to whom we are indebted for many useful conversations. Due to this isomorphism we may view \mathcal{L}_* as the composition of four functors: $\mathcal{L}_* = \mathcal{P}H\mathcal{F}\mathscr{C}$ where

- 1. \mathscr{C} is the functor which assigns to a pathwise and simply connected space the connected differential graded Q-coalgebra formed by its simply connected singular chain complex over Q;
 - 2. F is the cobar construction;
 - 3. H is the homology functor;
- 4. \mathscr{T} is the functor which assigns to a connected graded Hopf algebra over Q the associated connected graded Lie algebra of primitive elements.

The idea of the proof is to show that each of the required categories has coproducts preserved by the four functors involved.

This result has long been a part of the folk literature, but to

the best of our knowledge no proof appears in print. This result extends and compliments results of Hilton, and Porter on the integral homotopy of a wedge.

1. Coproducts. A category \mathscr{C} has coproducts if to every pair of objects A and B of \mathscr{C} , there is assigned a diagram in \mathscr{C}

$$A \xrightarrow{i_A} A \perp B \xleftarrow{i_B} B$$

with the property that for any morphisms $/: A \to C$ and $g: B \to C$ of \mathscr{C} , there is a *unique* morphism $/\perp g: A \perp B \to C$ such that

$$A \xrightarrow{i_A} A \perp B \xleftarrow{i_B} B$$

$$C$$

is a diagram in \mathscr{C} . ("Diagram in \mathscr{C} " means a commutative diagram of objects and morphisms of \mathscr{C} .)

If \bot is a coproduct on $\mathscr C$, then as an immediate consequence of the definition, there are natural $\mathscr C$ -isomorphisms $A \bot B \approx B \bot A$ and $A \bot (B \bot C) \approx (A \bot B) \bot C$.

EXAMPLE 1. The wedge \vee or one point union is a coproduct on the category of pointed spaces \mathscr{T}_* .

In the remaining examples K will be a commutative ring with unit.

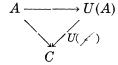
EXAMPLE 2. $\mathscr{C} = the \ category \ of \ connected \ graded \ K-modules.$ For each object A of, we have $A_0 \approx K$. The coproduct is defined by $(A \perp B)_n = A_n \oplus B_n$ for n > 0.

EXAMPLE 3. $\mathscr{C}=$ the category of connected graded K-algebras. For $A\in\mathscr{C}$ we define a graded K-module \overline{A} by $\overline{A}_n=A_n$ for n>0 and $A_0=0$. Then $T(\overline{A})=K\oplus\sum_{n=1}^{\infty}(\overline{A}\otimes\cdots(n)\cdots\otimes\overline{A})$, the tensor algebra of \overline{A} , is an object of \mathscr{C} and there is a canonical homomorphism $T(\overline{A})\to A$ in \mathscr{C} , the kernel of which we denote I(A). A coproduct is defined by $A\perp B=T(\overline{A}\oplus\overline{B})/(I(A),I(B))$ where the denominator denotes the ideal of $T(A\oplus\overline{B})$ generated by I(A) and I(B). It is routine to verify that \bot is indeed a coproduct. A simple diagram chase shows that $T(\overline{A})\perp T(\overline{B})=T(\overline{A}\oplus\overline{B})$.

EXAMPLE 4. $\mathscr{C} = the \ category \ of \ connected \ graded \ Lie \ algebras \ over \ Q.$ Each $A \in \mathscr{C}$ is a graded K-module with $A_0 = 0$. We set

¹ The word 'algebra' means 'associative algebra with unit'.

U(A) = T(A)/J where J is the ideal generated by all elements $x \otimes y - (-1)^{pq}y \otimes x - [x,y]$ with $x \in A_p$, $y \in A_q$. Then U(A) is a connected graded Q-algebra, called the *universal enveloping algebra of A*. There is a canonical morphism $A \to U(A)$ such that if $f : A \to C$ is a map of A into a connected graded Q-algebra, such that f[x,y] = [f(x,f(y))], then there is a unique map of algebras $U(f): U(A) \to C$ such that the following diagram is commutative:



To form the coproduct \perp in $\mathscr C$ we begin by forming $U(A) \perp U(B)$ as in Example 2. We define $A \perp B$ to be the sub Lie algebra of (the associated Lie algebra of) $U(A) \perp U(B)$ generated by the images of A and B. Thus we have a diagram

It is routine to check the universal property. We note that uniqueness implies $U(A \perp B) \approx U(A) \perp U(B)$ as graded Q-algebras.

EXAMPLE 5. $\mathscr{C} = the \ category \ of \ connected \ graded \ Hopf \ algebras$ over K. Since each object of \mathscr{C} is a graded connected K-algebra, we may form the coproduct as in Example 2. Then we need to check that $A \perp B$ is still a Hopf algebra. In the category of graded connected algebras we have the diagram:

$$A \xrightarrow{i_{A}} A \perp B \xleftarrow{i_{B}} B$$

$$\downarrow^{A_{A} \downarrow} \qquad \downarrow^{A_{A \perp B}} \qquad \downarrow^{A_{B}} \downarrow^{A_{B}}$$

$$A \bigotimes_{K} A \xrightarrow{i_{A} \otimes i_{A}} (A \perp B) \bigotimes_{K} (A \perp B) \xleftarrow{i_{B} \otimes i_{B}} B \bigotimes_{K} B$$

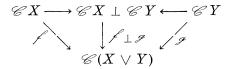
In other words $\Delta_{A\perp B}=(i_A\otimes i_A)\Delta_A\perp (i_B\otimes i_B)\Delta_B$ is a morphism of graded connected algebras and $A\perp B$ is a Hopf algebra. The required universal property is easily verified.

EXAMPLE 6. $\mathscr{C} = the \ category \ of \ connected \ differential \ graded \ K-coalgebras.$ The coproduct here is defined as in Example 2 it is only necessary to check that the differential and comultiplication behave well.

EXAMPLE 7. $\mathcal{C} = the \ category \ of \ connected \ differential \ graded \ K-algebras.$ The coproduct is defined as in Example 3, and the differential extends naturally.

EXAMPLE 8. $\mathscr{C} = the \ category \ of \ connected \ differential \ graded$ Hopf algebras over K. The coproduct is defined as in Example 5 and the differential extends naturally.

2. Functors which preserve coproducts. Let \mathscr{T}_*^1 denote the category of nicely pointed 1-connected spaces. Let $\mathscr{C}X$ for $X \in \mathscr{T}_*^1$ denote the normalized singular chains of X with all edges at the base point e_X . In other words $\mathscr{C}X = C_N(E_2(X, e_X))$, the normalized chain complex of $E_2(X, e_X)$, the second Eilenberg subcomplex. [3; p. 430.] Then \mathscr{C} is a functor with range the category of 1-connected differential graded coalgebras over Z, which we denote C^1DGCO . \mathscr{C} does not preserve coproducts. However there is a diagram in C^1DGCO :



where f and g are induced by the inclusions into $X \vee Y$. An elementary argument shows that $f \perp g$ induces a homology isomorphism of coalgebras.

The cobar construction \mathscr{F} is a functor with domain C^1DGCO and range C^0DGAl , the category of connected differential graded algebras. We want to show that

2.1. Proposition. F preserves coproducts.

Proof. Let C_1 and C_2 belong to C^1DGCO . Then \mathscr{F} induces maps $\mathscr{F}(C_i) \to \mathscr{F}(C_1 \perp C_2)$. Consequently we have in C^0DGAl :

$$\mathscr{F}(C_1) \longrightarrow \mathscr{F}(C_1) \perp \mathscr{F}(C_2) \longleftarrow \mathscr{F}(C_2)$$

$$\downarrow^{\phi} \qquad \qquad \mathscr{F}(C_1 \perp C_2)$$

Let \sharp denote the functors which forget the differentials in various categories. Then $\mathscr{F}(C)_{\sharp} = T(C_{\sharp})$ so that

$$\phi_{\sharp} \colon T(\bar{C}_{1\sharp}) \ \bot \ T(\bar{C}_{2\sharp}) \longrightarrow T(\bar{C}_{1\sharp} \oplus \bar{C}_{2\sharp})$$

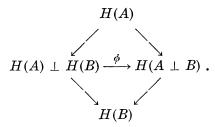
is an isomorphism. Since \sharp is faithful, ϕ is an isomorphism.

Next we restrict our attention to algebras over the rational field

Q and consider the homology functor $H_*: C^{\circ}DGAl/Q \to C^{\circ}GAl/Q$, the category of connected graded Q-algebras.

2.2. Proposition. $H_*(A \perp B) \approx H_*(A) \perp H_*(B)$.

Proof. We can readily construct a diagram in $C^{\circ}DGAl/Q$



The additive isomorphisms $A \perp B = (\bar{A} \oplus \bar{B}) \oplus ((\bar{A} \otimes \bar{B}) \oplus (\bar{B} \otimes \bar{A})) + \cdots$ and

$$H(A) \perp H(B)$$

$$= (\bar{H}(\bar{A}) \oplus \bar{H}(\bar{B})) \oplus ((\bar{H}(\bar{A}) \otimes \bar{H}(\bar{B})) \oplus (\bar{H}(\bar{B}) \otimes H(A))) + \cdots$$

together with the Kunneth Theorem implies that ϕ is an isomorphism.

3. Proof of Theorem 1. In the notation above we have isomorphisms of graded Q-algebras

$$\begin{split} H_*(\mathscr{F}\mathscr{C}X) \perp H_*(\mathscr{F}\mathscr{C}Y) &\stackrel{\approx}{\longrightarrow} H_*(\mathscr{F}\mathscr{C}X \perp \mathscr{F}\mathscr{C}Y) \\ &\stackrel{\approx}{\longrightarrow} H_*(\mathscr{F}(\mathscr{C}X \perp \mathscr{C}Y)) \stackrel{\approx}{\longrightarrow} H_*(\mathscr{F}\mathscr{C}(X \vee Y)) \;. \end{split}$$

By a theorem of Adams, for any pathwise and simply connected space Z, there is a natural isomorphism of algebras, $H_*(\Omega Z; Q) \to H_*(\mathcal{F} \mathcal{C} Z)$. Consequently the morphism of Hopf algebras

$$H_*(\Omega X; Q) \perp H_*(\Omega Y; Q) \longrightarrow H_*(\Omega(X \vee Y); Q)$$

is an isomorphism of algebras, and hence of Hopf algebras. Moore's statement says $H_*(\varOmega X;Q)=U(\mathscr{L}_*(X))$ so we have

$$U(\mathscr{L}_*(X) \perp \mathscr{L}_*(Y)) \approx U(\mathscr{L}_*(X)) \perp U(\mathscr{L}_*(Y)) \approx U(\mathscr{L}_*(X \vee Y))$$
 and since PU is the identity, $\mathscr{L}_*(X) \perp \mathscr{L}_*(Y) \approx \mathscr{L}_*(X \vee Y)$.

REMARK. It is apparent from the above argument and the theorem of Adams that

$$H_*(\Omega X; k) \perp H_*(\Omega Y; k) \longrightarrow H_*(\Omega(X \vee Y); k)$$

is an isomorphism of Hopf algebras for any field k.

This has been proved by Berstein in [2] by slightly differer methods.

REMARK. The calculation of the Poincaré Series of the coproduct of two Lie algebras is a difficult number theoretic problem involvin Witt numbers.

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Pacific Journal of Mathematics

Vol. 24, No. 2

June, 1968

John Suemper Alin and Spencer Ernest Dickson, Goldie's torsion theory	
and its derived functor	195
Steve Armentrout, Lloyd Lesley Lininger and Donald Vern Meyer,	205
Equivalent decomposition of R^3	205
James Harvey Carruth, A note on partially ordered compacta	229
Charles E. Clark and Carl Eberhart, A characterization of compact	
connected planar lattices	233
Allan Clark and Larry Smith, The rational homotopy of a wedge	241
Donald Brooks Coleman, Semigroup algebras that are group algebras	247
John Eric Gilbert, Convolution operators on $L^p(G)$ and properties of	
locally compact groups	257
Fletcher Gross, Groups admitting a fixed-point-free automorphism of order	
2^n	269
Jack Hardy and Howard E. Lacey, Extensions of regular Borel measures	277
R. G. Huffstutler and Frederick Max Stein, <i>The approximation solution of</i>	
$y' = F(x, y) \dots $	283
Michael Joseph Kascic, Jr., Polynomials in linear relations	291
Alan G. Konheim and Benjamin Weiss, A note on functions which	
operate	297
Warren Simms Loud, Self-adjoint multi-point boundary value problems	303
Kenneth Derwood Magill, Jr., Topological spaces determined by left ideals	
of semigroups	319
Morris Marden, On the derivative of canonical products	331
J. L. Nelson, A stability theorem for a third order nonlinear differential	
equation	341
Raymond Moos Redheffer, Functions with real poles and zeros	345
Donald Zane Spicer, Group algebras of vector-valued functions	379
Myles Tierney, Some applications of a property of the functor Ef	401
Jan 1977 - Property of the factor of the fac	