Pacific Journal of Mathematics

SEMIGROUP ALGEBRAS THAT ARE GROUP ALGEBRAS

Donald Brooks Coleman

Vol. 24, No. 2 June 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 2, 1968

SEMIGROUP ALGEBRAS THAT ARE GROUP ALGEBRAS

D. B. COLEMAN

If S is a finite semigroup, and if K is a field, under what conditions is there a group G such that the semigroup algebra KS is isomorphic to the group algebra KG?

The following theorems are proved:

1. Let S have odd order n, and let K be either a real number field or $GF(q)$, where q is a prime less than any prime divisor of n. If $KS \cong KG$ for a group G, then S is a group.

2. Let K be a cyclotomic field over the rationals, and let G be an abelian group. Then $KG \cong KS$ for a semigroup S that is not a group if and only if for some prime p and some positive integer k, K contains all p^kth roots of unity and the cyclic group of order p^k is a direct factor of G.

3. Let S be a commutative semigroup of order n, and let $K = GF(p)$, where p is a prime not exceeding the smallest prime dividing n. If $KS \cong KG$ for a group G, then S is a group.

The semigroup ring of a semilattice is also considered.

1. Preliminary remarks.

The basic definitions and concepts involving semigroups that are used here can be found in [2].

For related literature, see [5], [6], [7], [9], [10], and § 5.2 in [2].

Let S be a finite semigroup and let K be a field. The *semigroup algebra* KS is the free algebra on S; that is S forms a K-basis for KS and multiplication in KS is induced by that in S.

If S has a zero element z, let K_0S denote the *contracted semigroup algebra* of S. We see that K_0S is an algebra that has the nonzero members of S as a basis, with multiplication $\circ$ determined by

$$s \circ t = st \text{ if } st \neq z \quad \text{and} \quad s \circ t = 0 \text{ if } st = z;\ t \in S \backslash \{z\}\ .$$

If J is an ideal in S, let S/J denote the Rees quotient semigroup of S modulo J.

It is easy to verify that if J is an ideal in S, then the factor algebra KS/KJ is isomorphic to the contracted algebra of S/J. Also, if S has a zero, then $K_0S/K_0J \cong K_0(S/J)$. [2, p. 160].

If A is an algebra over K, we denote by A_k the algebra of $k \times k$ matrices over A, where k is a positive integer.

By a *nongroup* we mean a semigroup that is not a group.

$GF(q)$ denotes the Galois field with q elements.

2. Odd order semigroups.

Let S be a finite semigroup, and let $\varnothing \subset J_1 \subset J_2 \subset \cdots \subset J_k = S$ be a principal series for S. Suppose that K is a field such that KS is semisimple. Then by [2, pp. 161-162], each J_i/J_{i-1} is 0-simple, $i = 2, \cdots, k$, and

$$KS \cong KJ_1 \oplus K_0(J_2/J_1) \oplus \cdots \oplus K_0(J_k/J_{k-1}) .$$

According to M. Teissier (see [2, p. 165]), J_1 is a group. Also, for each $i = 2, \cdots, k$, there is a group H_i such that $K_0(J_i/J_{i-1}) \cong (KH_i)_{k_i}$, the algebra of $k_i \times k_i$ matrices over KH_i, for some positive integer k_i. This is due to W. D. Munn; see [2, p. 162]. Each KH_i, being semisimple, has K as a direct summand. It follows that each $K_0(J_i/J_{i-1})$ has K_{k_i} as a simple direct summand. It is well known that the group algebra KG is semisimple if and only if the characteristic of K does not divide the order of G. Thus we have

THEOREM 2.1. *Let G be a finite group of order n, and let K be a field whose characteristic does not divide n. Suppose that $KG \cong K \oplus \sum_{i=1}^{t} (D_i)_{k_i}$, where each D_i is a division algebra properly containing K. If S is a semigroup such that $KS \cong KG$, then S is a group.*

If n is odd, and if K contains no n-th roots of unity except 1, then it follows from [1] that the hypothesis of the theorem holds. Hence we have the following special case.

COROLLARY 2.2. *Let K be a field of real numbers, and let S be a semigroup of odd order. If $KS \cong KG$ for some group G, then S is itself a group.*

COROLLARY 2.3. *Let S be a semigroup of order n, and let $K = GF(p^m)$, where p is a prime such that no prime divisor of n divides $p(p^m - 1)$. If $KS \cong KG$ for some group G, then S is a group.*

A CONSTRUCTION 2.4. Suppose that A is an algebra over K such that $A = A_0 \oplus A_1 \oplus \cdots \oplus A_t$, for ideals A_i. Suppose further that $A_0 = KS_0$ for a semigroup S_0, and that for each $i = 1, \cdots, t$, A_i is either KS_i or K_0S_i' for a semigroup S_i or a semigroup $S_i' = S_i \cup 0$ with zero, respectively.

Let $S = S_0 \cup \{x + e_0 : x \in \bigcup_{i=1}^{t} S_i\}$, where e_0 is an idempotent in S_0. Since $A_iA_j = (0)$ for $i \neq j$, we see that S is a semigroup. Since $S_0 \cup S_1 \cup \cdots \cup S_t$ is a basis for A, we have that $A = KS$. Since S_0 is an ideal in S, S is not a group.

This construction follows that in the proof of Theorem 5.30 in [2]. In that case $S_0 = \{e_0\}$ and A_i is a full matrix algebra, for $i > 0$.

We now see that the hypothesis that n is odd is needed in 2.2. For let D denote the dihedral group of order 8, and let K be a field of characteristic $\neq 2$. Then $KD \cong K \oplus K \oplus K \oplus K \oplus K_2$. By 2.4 there is a nongroup S such that $KS \cong KD$. If K has characteristic 2, there is no such S. In fact, if G is a p-group, if K is a field of characteristic p, and if $KS \cong KG$, then S is a group. For in this case KG has no idempotents except 0 and 1; thus $KG \cong KS$ forces S to have exactly one idempotent which must be an identity. (Notice that a zero element in S is not the zero of KS). Thus the finite semigroup S is a group.

Another example is of interest here. Let $G = S_3$, the symmetric group on 3 letters, and let K have characteristic $\neq 3$. Then $KG \cong KC \oplus K_2$, where C is the group of order 2. Thus, as before, $KG \cong KS$ for some nongroup S.

In examining examples we use the fact that the matrix algebra K_m is a contracted semigroup algebra. This raises the question: What are the semigroups S such that $K_0S \cong K_m$? From Theorem 5.19 and Corollary 3.12 in [2] we get the following answer.

Let P be a nonsingular $m \times m$ matrix over K all of whose entries are either 0 or 1. Let $\{E_{ij}\}$ be the usual m^2 matrix units; $E_{ij}E_{kr} = \delta_{jk}E_{ir}$. Let $U(P)$ denote the multiplicative semigroup of matrices consisting of the zero matrix and all matrices of the form PE_{ij}; $1 \leqq i, j \leqq m$. If S is a semigroup with zero, then $K_0S \cong K_m$ if and only if $S \cong U(P)$ for some such nonsingular P. Moreover, $U(P) \cong U(P')$ if and only if P and P' have the same number of entries equal to one. We see that there are exactly $m^2 - 2m + 2$ nonisomorphic semigroups $U(P)$. Note also that $U(P) \cong U(P')$ if and only if there is a nonsingular matrix T such that $T^{-1}U(P)T = U(P')$.

3. Commutative semigroup algebras.

Let G be an abelian group of order n, and let K be a field whose characteristic does not divide n. Then according to [8], we have

$$KG \cong \bigoplus \sum a_d K(\zeta_d) \; ; \qquad (1)$$

summation is over divisors of n, ζ_d is a primitive d-th root of unity, and $a_dK(\zeta_d)$ indicates $K(\zeta_d)$ as a direct summand a_d times. Further $a_d = n_d/v_d$, where n_d is the number of elements of order d in G and $v_d = \deg(K(\zeta_d)/K)$.

If there are groups $G_1, \cdots, G_m$, with $m > 1$, such that $KG \cong KG_1 \oplus \cdots \oplus KG_m$, then by 2.4 there is a nongroup S such that $KS \cong KG$. By Theorem 5.21 in [2], we see that the converse holds.

Thus given the abelian group G, the semigroups S such that $KS \cong KG$ are precisely those commutative semigroups S such that

(i) S is the disjoint union of groups, $G_1, \cdots, G_s$; and

(ii) $KG \cong KG_1 \oplus \cdots \oplus KG_s$.

By Theorem 4.11 in [2] all semigroups satisfying (i) can be determined. Also, since all finite groups of order less than n, and their corresponding numbers n_d, can be determined, we can use formula (1) to check condition (ii).

Note that if K contains a primitive p^k-th root of unity, and if the cyclic group $C(p^k)$ of order p^k is a direct factor of G, then condition (ii) holds. For in this case $KC(p^k) \cong p^k K$, so that

$$K(C(p^k) \times H) \cong KC(p^k) \otimes KH \cong p^k K(H) .$$

In the following case the converse holds.

Let Q denote the rational field. To avoid trivialities, when we write $K(\zeta_d)$ we assume that d is either odd or divisible by 4.

THEOREM 3.1. *Let $K = Q(\zeta)$, where ζ is a primitive m-th root of unity, and let G be an abelian group. There is a nongroup S such that $KS \cong KG$ if and only if there is a prime p and a positive integer k such that K contains all the p^k-th roots of unity and $C(p^k)$ is a direct factor of G.*

Proof. We just observed the sufficiency of the condition.

Suppose conversely that

$$(2) \qquad KG \cong KG_1 \oplus \cdots \oplus KG_s , \qquad s > 1 .$$

Assume that each group algebra KG_i is indecomposable as a direct sum of group algebras. Then for each i, either $G_i = 1$, or KG_i is the direct sum of fields $K(\zeta_d)$, not all equal to K.

Suppose that q is a prime dividing the order of G_i; then q divides the order n of G. For there is some power q^a of q such that $K < K(\zeta_{q^a}) = K(\zeta_d)$ for a divisor d of n. (Otherwise, using the remarks preceding the theorem, KG_i would be decomposable.) Thus $K < Q(\zeta_t) = K(\zeta_{q^a}) = K(\zeta_d)$, where $t = [m, q^a] = [m, d]$, the least common multiple. Since q^a does not divide m, we have that q divides d.

Suppose now that our condition fails, and let $p_1, \cdots, p_r$ be the distinct prime divisors of n. Then for each i, there is a positive integer n_i such that $p_i^{n_i}$ does not divide m and $C(p_i^{n_i})$ is a subgroup of every nontrivial cyclic direct factor of the p_i-Sylow subgroup of G. Choose each n_i to be the smallest such integer. We may assume without loss of generality that $p_i^{n_i - 1}$ divides m.

In (2), think of KG and each KG_i being expressed as in (1). Now delete all fields $K(\zeta_d)$ for which $([m, d], n)$ exceeds $p_1^{n_1} \cdots p_r^{n_r}$. On the left of (2) we have left the group algebra of a subgroup of G whose p_i-Sylow subgroup is of type $(p_i^{n_i}, \cdots, p_i^{n_i})$. On the right, after possibly some further decomposition, we have a like situation. We may thus assume that for each i, the p_i-Sylow subgroup P_i of G is of type $(p_i^{n_i}, \cdots, p_i^{n_i})$, with say k_i factors; and for each G_j, the p_i-Sylow subgroup P_i^j of G_j is either trivial or of type $(p_i^{n_i}, \cdots, p_i^{n_i})$, with say k_{ij} factors. Take $k_{ij} = 0$ in case $P_i^j = 1$.

Using (1), we have that

$$KP_i \cong a_i K \oplus b_i K(\zeta_d)\ , \tag{3}$$

where $d = p_i^{n_i}$, $a_i = p_i^{(n_i-1)k_i}$, and

$$b_i = (p_i^{n_i k_i} - p_i^{(n_i-1)k_i})/\delta_i\ ; \qquad \delta_i = \deg(K(\zeta_d)/K)\ .$$

Similarly

$$KP_i^j \cong a_{ij} K \oplus b_{ij} K(\zeta_d)\ , \tag{4}$$

where $a_{ij} = p_i^{(n_i-1)k_{ij}}$ and

$$b_{ij} = (p_i^{n_i k_{ij}} - p_i^{(n_i-1)k_{ij}})/\delta_i\ .$$

For some pair α, β we have $k_\alpha > k_{\alpha\beta}$. Otherwise some G_j would be isomorphic to G.

Now use formulas (3) and (4) and the fact that $K(A \times B) \cong KA \otimes KB$ to count the number of summands on each side of (2) that are isomorphic to K. We obtain

$$\prod_{i=1}^{r} p_i^{(n_i-1)k_i} = \sum_{j=1}^{s} \prod_{i=1}^{r} p_i^{(n_i-1)k_{ij}}\ . \tag{5}$$

Let $f = p_\alpha^{n_\alpha}$; use (3) and (4) to count all summands on each side of (2) isomorphic to $K(\zeta_f)$. Then add the terms in (5) to each side of the resulting equation, getting

$$\prod_{i\neq\alpha} p_i^{(n_i-1)k_i} \cdot p_\alpha^{n_\alpha k_\alpha} = \sum_{j=1}^{s} \prod_{i\neq\alpha} p_i^{(n_i-1)k_{ij}} \cdot p_\alpha^{n_\alpha k_{\alpha j}}\ . \tag{6}$$

Multiplying (5) by $\prod_{i=1}^{r} p_i^{k_i}$, we have

$$\prod_{i=1}^{r} p_i^{n_i k_i} = \sum_{j=1}^{s} \prod_{i=1}^{r} p_i^{n_i k_{ij} + k_i - k_{ij}}\ . \tag{7}$$

Multiplying (6) by $\prod_{i\neq\alpha} p_i^{k_i}$, we have

$$\prod_{i=1}^{r} p_i^{n_i k_i} = \sum_{j=1}^{s} \prod_{i\neq\alpha} p_i^{n_i k_{ij}} \cdot p_\alpha^{n_\alpha k_{\alpha j}}\ . \tag{8}$$

But $k_\alpha > k_{\alpha\beta}$, so that (7) and (8) cannot both hold. This contradiction completes the proof.

COROLLARY 3.2. *Let G be a finite abelian group such that $QG \cong QS$ for a nongroup S. Then $C(2)$ is a direct factor of G.*

REMARK 3.3. Let S be a commutative semigroup of order $2m$, where m is odd. If $QS \cong QG$ for a group G, then either $S \cong G$ or S is the disjoint union of two copies of the group H, where $G = C(2) \times H$.

Proof. Suppose that $QS \cong QG$. Let $G = C(2) \times H$, where H has order m. According to [8], QG completely determines G. Hence if S is a group, then $S \cong G$.

QG has two simple direct summands isomorphic to Q. Thus if S is not a group, $QS \cong QG_1 \oplus QG_2$ for groups G_1 and G_2. It is clear that the orders of G_1 and G_2 have the same prime divisors, and those are the prime divisors of m. Let p be one of these primes, and let P, P_1 and P_2 be the p-Sylow subgroups of H, G_1 and G_2, respectively. Then we have

$$Q(C(2) \times P) \cong QP_1 \oplus QP_2 . \tag{9}$$

This leads to an equation $2p^a = p^b + p^c$, which implies $b = c = a$. Thus P, P_1 and P_2 all have the same order p^a. By induction on the exponent p^e of P we see that $P \cong P_1 \cong P_2$. If $e = 1$, then P, P_1 and P_2 are all elementary abelian of the same order, hence isomorphic. Suppose $e > 1$. Deleting direct summands $Q(\zeta_{p^e})$ from both sides of (9) we have

$$Q(C(2) \times P') \cong QP_1' \oplus QP_2' ,$$

where $P' = \{x \in P: x^{p^e} = 1\}$. As before, P', P_1' and P_2' have the same order; and by induction $P' \cong P_1' \cong P_2'$. From (9), and the fact that P, P_1 and P_2 have the same order, the three groups have the same number of elements of order p^e. Thus $P \cong P_1 \cong P_2$.

Theorem 3.1 fails for arbitrary finite extensions of Q. For let $K = Q(\sqrt{3})$, and let $G = C(12)$. Notice that

$$K(\zeta_3) = K(\zeta_4) = K(\zeta_6) = K(\zeta_{12}) = Q(\sqrt{3}, i) .$$

Using this we see that

$$KG \cong KG_1 \oplus KG_2 ,$$

where $G_1 = C(3)$ and $G_2 = C(3) \times C(3)$.

Theorem 3.1 also fails for the prime fields $GF(p), p$ a prime. To see this, let $K = GF(5)$. Then $KC(8) \cong KC(2) \oplus KC(6)$. Here $K(\zeta_4) = K$ and $K(\zeta_3) = K(\zeta_6) = K(\zeta_8) \cong GF(25)$.

THEOREM 3.4. *Let K be a field of characteristic $p \neq 0$; let*

$G = P \times H$, where P is a p-group and H is an abelian group of order prime to p. Then $KG \cong KS$ for a nongroup S if and only if $KH \cong KT$ for a nongroup T.

Proof. If $KH \cong KT$, and if T is a nongroup, then $S = P \times T$ is a nongroup, and $KG \cong KS$.

Conversely, suppose that S is a nongroup and that $R = KS \cong KG$. Let $KH \cong K_1 \oplus \cdots \oplus K_r$ for fields K_i. Then $R = R_1 \oplus \cdots \oplus R_k$, where $R_i \cong K_i \otimes KP$. The R_i are the indecomposable components of R. As a ring, R_i is isomorphic with K_iP. Thus every element in R_i is either nilpotent or a unit. Let $\pi_1, \cdots, \pi_k$ be the projections of R onto the R_i. Let $X = \{1, \cdots, k\}$.

Let $X_1 = \{i \in X: \pi_i(s)$ is a unit in R_i for all $s \in S\}$. Then $X_1 \neq \emptyset$; otherwise the element $s_1 \cdot s_2 \cdot \cdots \cdot s_n$, the product of all members of S, would be the zero element of R. Let $G_1 = \{s \in S: \pi_j(s) = 0$ for $j \notin X_1\}$. Then G_1 is a group, KG_1 is an ideal in R, and $KG_1 = \sum R_i$ $(i \in X_1)$. Also $R = KG_1 \oplus K_0U$, where $U = \{\rho_1(s): s \in S, s \notin G_1\}$; $\rho_1 = \sum \pi_j$ $(j \notin X_1)$. Fix $j \notin X_1$, and choose $t \in S$ such that $\pi_j(t)$ is a unit in R_j. There is such an element; for if not, R_j would be nilpotent. Let $X_2 = \{i \in X: i \notin X_1$ and $\pi_i(t)$ is a unit in $R_i\}$. Suppose $X \neq X_1 \cup X_2$. Let $\eta = \sum \pi_i$ $(i \in X_2)$ and $\rho_2 = \sum \pi_j$ $(j \notin X_1 \cup X_2)$, and let $G_2' = \{\eta(s): s \notin G_1$ and $\rho_2(s) = 0\}$ and $G_3' = \{\rho_2(s): s \notin G_1$ and $\rho_2(s) \neq 0\} \cup \{0\}$. Note that $0 \in G_2'$.

We have $R = KG_1 \oplus K_0G_2' \oplus K_0G_3'$; $KG_1 = \sum R_i$ $(i \in X_1)$, $K_0G_2' = \sum R_i$ $(i \in X_2)$, and $K_0G_3' = \sum R_i$ $(i \notin X_1 \cup X_2)$.

We continue this procedure until we have

$$R = KG_1 \oplus K_0G_2' \oplus \cdots \oplus K_0G_m' ,$$

with $m > 1$, where the set X is partitioned into disjoint subsets $X_1, \cdots, X_m$; $K_0G_q' = \sum R_j$ $(j \in X_q)$ and for each $q \geqq 1$, either $G_q' = G_q \cup 0$ for a group G_q, or $K_0G_q' \cong R_j$ for some j, and G_q' is not a group with zero. Suppose that the former holds for $q = 1, \cdots, w$, and that X_q is a singleton for $q > w$. Let N be the radical of R, and for each q, let N_q be the radical of K_0G_q'. If $q > w$, then $K_0G_q'/N_q \cong K$. For since $K_0G_q' \cong R_j$ has no nontrivial idempotents, it follows that G_q' has at most two idempotents. If G_q' has only one idempotent, then R_j is nilpotent. This is not the case. Thus G_q' has exactly two idempotents, the 0 and 1 in R_j. Thus G_q' is the disjoint union of a nilpotent semigroup Z and a group V. Clearly $K_0Z \subset N_q$. Thus there is a homomorphism μ of $KV \cong K_0G_q'/K_0Z$ onto $K_j \cong R_j/\mathrm{Rad}\, R_j$. The normalized units of finite order in $K_j \otimes KP$ have order a power of p. Thus V is a p-group (perhaps trivial). Thus the kernel of μ is the radical of KV and $K_j \cong K$.

According to Deskins [4], $R/N \cong KH$ and $KG_q/N_q \cong KH_q$ for $q \leqq w$, where H_q is the p-complement of G_q. Thus

$$KH \cong KH_1 \oplus \cdots \oplus KH_w \oplus K \oplus \cdots \oplus K .$$

This completes the proof.

COROLLARY 3.5. *Let S be a commutative semigroup of order n, and let $K = GF(p)$, where p is the smallest prime dividing n. If $KS \cong KG$ for a group G, then S is a group.*

COROLLARY 3.6. *Let $K = GF(2)$. If S is a commutative semigroup, and if $KS \cong KG$ for some group G, then S is a group.*

Note that $GF(2)$ and transcendental extensions of $GF(2)$ are the only fields K for which Corollary 3.6 will hold. For if K contains $GF(2^t)$, and if G is the cyclic group of order $2^t - 1$, then $KG \cong \sum K$. If K has characteristic $\neq 2$, then $KC(2) \cong K \oplus K$.

THEOREM 3.7. *Let K be the real number field, and let S be a commutative nongroup of order n. Then there is a group G such that $KS \cong KG$ if and only if the following conditions hold:*

(i) *n is even;*

(ii) *S is the disjoint union of group $G_1, \cdots, G_m$;*

(iii) *If 2^{e_i} is the number of elements x in G_i such that $x^2 = 1$, then $\sum_{i=1}^{m} 2^{e_i}$ is a power of 2 dividing n.*

Proof. The necessity of the conditions follows from the fact that if G is an abelian group, then $GK \cong aK \oplus bL$, where $a - 1$ is the number of elements of G of order 2, and L is the complex field.

Conversely, suppose the conditions hold, and let $\sum_{i=1}^{m} 2^{e_i} = 2^e$. Let $n = 2^e \cdot 2^f \cdot m$, with m odd; let $G = C(2) \times \cdots \times C(2) \times C(2^{f+1}) \times H$, where there are $e - 1$ factors $C(2)$ and H is any abelian group of order m. Then clearly $KS \cong KG$.

4. **Semilattices.** A semigroup in which every element is idempotent is called a *band*. A commutative band is a (lower) semilattice under the ordering: $e \leqq f$ if $e = ef$. Conversely, any semilattice is a commutative band under the operation $e \cdot f = e \wedge f$.

If S is a semilattice, and if R is a commutative ring with identity, then the semigroup ring RS has an identity. ([6, Th. 7.5]). Corresponding to Theorem 5.27 in [5] we have

THEOREM 4.1. *Let S be a semilattice of order n. Then RS is* *the direct sum of n copies of R and R_0S is the direct sum of $n-1$ copies of R.*

Proof. The theorem is trivial for $n = 1$. If $n = 2$, and $S = \{z, e\}$, with $ez = ze = z$, then $R_0S = Re$ and $RS = Rz \oplus R(e - z)$, so the theorem holds.

Suppose that $n > 2$ and proceed inductively. Choose $f \in S$ such that f is neither the zero of S nor the identity of S, in case there is one. Let $J = Sf$. Then $RS = (RS)f \oplus RS(1 - f) \cong RJ \oplus R_0(S/J)$. Since both J and S/J are semilattices of order less than n, we have by induction that RJ and $R_0(S/J)$ are direct sums of copies of R, and hence so is RS.

Similarly $R_0S \cong R_0J \oplus R_0(S/J)$ and induction gives R_0S as a sum of copies of R.

As a partial converse we have

THEOREM 4.2. *Let S be a semigroup of order n, and let R be an integral domain such that no prime $p \leqq n$ is a unit in R. If RS is the direct sum of copies of R, then S is a semilattice.*

Proof. Let $RS \cong R \oplus \cdots \oplus R$, and let K be the quotient field of R. Then $KS \cong K \oplus \cdots \oplus K$, so that KS is semisimple. Hence by [2, Cor. 5.15] S is a semisimple commutative semigroup. Thus S has a principal series $\phi < S_1 < S_2 < \cdots < S_k = S$ such that the kernel $S_1 = G_1$ is a group and S_i/S_{i-1} is a group with zero $G_i \cup 0$ for $i = 2, \cdots, k$. Thus $RS \cong RG_1 \oplus \cdots \oplus RG_k$. By [3] each RG_i is indecomposable; but by hypothesis each is the direct sum of copies of R. Thus each G_i is trivial, so that S is a semilattice.

Using Theorem 4.2 and the results of § 3, we have

PROPOSITION 4.3. Let S be a semilattice, let T be a commutative semigroup of the same order, and let K be a field of characteristic 0. Then $KS \cong KT$ if and only if T is the disjoint union of groups $G_1 \cup \cdots \cup G_k$ such that if G_i has exponent m_i, then K contains the m_i-th roots of unity.

Using Theorem 4.2 and the fact that for a band S, KS is semisimple if and only if S is commutative [2, p. 169], we see

PROPOSITION 4.4. Let S be a band, and let G be a group of the same order n. Let K be a field whose characteristic does not divide n. Then $KS \cong KG$ if and only if S and G are commutative and F contains the m-th roots of unity, where m is the exponent of G.

Let $R = GF(2)$. Using the fact that $RS \cong R \oplus \cdots \oplus R$ for a finite semilattice S, we may derive the following well known result:

Every semilattice S of order n can be embedded in the lattice 2^n (subsets of the set $\{1, 2, \cdots, n\}$. In fact, S can be considered as linearly independent subset of 2^n, where 2^n is viewed as $R \times \cdots \times$

The author thanks W. E. Deskins for suggesting this problem.

REFERENCES

1. S. D. Berman, *On the theory of representations of finite groups*, Doklady Aka Nauk SSSR (N.S.) **86** (1952), 885-888. (Russian)
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. Mathematical Surveys, No. 7, Amer. Math. Soc., 1961.
3. D. B. Coleman, *Idempotents in group rings*, Proc. Amer. Math. Soc. **17** (196 962.
4. W. E. Deskins, *Finite abelian groups with isomorphic group algebras*, Duke Mat J. **23** (1956), 35-40.
5. E. Hewitt and H. S. Zuckerman, *Finite dimensional convolution algebras*, Act Math. **93** (1955), 67-119.
6. ———, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. **8** (1956), 70-97.
7. W. D. Munn, *On semigroup algebras*, Proc. Cambridge Phil. Soc. **51** (1955), 1-1
8. S. Perlis and G. L. Walker, *Abelian group algebras of finite order*, Trans. Ame Math. Soc. **68** (1950), 420-426.
9. Marianne Teissier, *Sur l'algebre d'un demi-groupe fini simple*, C. R. Acad. Sc Paris **234** (1952), 2413-2414.
10. ———, *Sur l'algebre d'un demi-groupe fini simple*, II, Cas general, C. R. Aca Sci. Paris **234** (1952), 2511-2513.

Received November 8, 1966.

UNIVERSITY OF KENTUCKY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 24, No. 2 June, 1968