GROUPS ADMITTING A FIXED-POINT-FREE AUTOMORPHISM OF ORDER $2^n$

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Let $G$ be a finite solvable group which admits a fixed-point-free automorphism of order $2^n$. The main result of this paper is that the nilpotent length of $G$ is at most $2n - 2$ for $n \geq 2$. This is an improvement on earlier results in that no assumptions are made regarding the Sylow subgroups of $G$.

Suppose $G$ is a finite solvable group which admits a fixed-point-free automorphism of order $p^n$ where $p$ is a prime. Then it is known that the nilpotent length of $G$ is at most $n$ provided that $p \neq 2$ ([8], [10], [6]). This result also holds for $p = 2$ if the Sylow $q$-subgroups of $G$ are abelian for all Mersenne primes $q$ ([8], [10]). The purpose of the present paper is to obtain an upper bound on the nilpotent length in the case $p = 2$ without imposing any restrictions on the Sylow subgroups of $G$. Our result is

**Theorem 1.1.** If $G$ is a finite group admitting a fixed-point-free automorphism of order $2^n$, then $G$ is solvable and has nilpotent length at most $\max\{2n - 2, n\}$.

Here it should be noted that if $G$ admits a 2-group as a fixed-point-free operator group then $G$ must have odd order and thus must be solvable from [2].

The usual methods employed to prove results about solvable groups admitting a fixed-point-free automorphism of order $p^n$ are so similar to the methods used by Hall and Higman [7] to find upper bounds on the $p$-length that it seems natural to ask whether both types of results might follow from some general theorem about linear groups. If $p = 2$ this can be done and the theorem is the following:

**Theorem 1.2.** Let $G$ be a finite solvable linear group over a field $K$ such that the order of $F'(G)$ is divisible by neither 2 nor the characteristic of $K$. Assume that $g$ is an element of order $2^n$ in $G$ such that the minimal polynomial of $g$ has degree $< 2^n$. Then $g^{2^n-1}$ must belong to $F_2(G)$.

Here $F'(G)$ is the greatest normal nilpotent subgroup of $G$ and $F_2(G) = F'(G \mod F'(G))$. In addition to implying Theorem 1.1, Theorem 1.2 also immediately implies Theorem B of [4] which in turn implies that $l_2(G) \leq \max\{2e_2(G) - 2, e_3(G)\}$ for any solvable group $G$ ([4], [5]).
Preliminary results. For the rest of this paper we adopt the convention that all groups referred to are assumed to be finite. If $G$ is a linear group operating on $V$ and $U$ is a $G$-invariant subspace, then $[G | U]$ denotes the restriction of $G$ to $U$. If $g$ is an element of a linear group such that the minimal polynomial of $g$ has degree less than the order of $g$, then $g$ is said to be exceptional. The rest of the notation used agrees with that of [2].

Before proceeding to the proof of Theorem 1.2, some preliminary results are needed.

**Lemma 2.1.** Let $Q$ be an extra-special $q$-group which is operated upon by an automorphism $g$ of order $p^n$ where $p$ is a prime distinct from $q$. Assume that $[Q', g] = 1$ and let $K$ be an algebraically closed field of characteristic different from $q$. Then, if $M$ is any irreducible $K - Q(g)$ module which represents $Q$ faithfully, it follows that $M$ is an irreducible $K - Q$ module.

This follows from either [1, Th. 1.30] or [7, Lemma 2.2.3] depending on whether the characteristic of $K$ differs from or is equal to $p$, respectively. Next we need a generalization of Theorem 2.5.4 of [7].

**Theorem 2.2.** Suppose that

(i) $Q$ is an extra-special $q$-group which admits an automorphism $g$ of order $p^n$ where $p$ is a prime distinct from $q$.

(ii) $[Q', g] = 1$.

(iii) $K$ is a field of characteristic different from $q$.

(iv) $M$ is a faithful, irreducible $K - Q(g)$ module.

(v) $g$ is exceptional on $M$.

Then the following must hold:

(a) $p^n - 1 = q^d$.

(b) If $Q/Q'$ is a subgroup of $Q/Q'$ that is transformed faithfully and irreducibly by $\langle g \rangle$, then $|Q/Q'| = q^d$ and $[Q, g] \leq Q_i$.

(c) The minimal polynomial of $g$ on $M$ has degree $p^n - 1$.

**Proof.** First we show that $K$ may be taken to be algebraically closed. Let $L$ be an algebraically closed extension of $K$ and let $N$ be an irreducible $L - Q(g)$ submodule of $M \otimes_K L$. Now if $c$ generates $Q'$, then, since $c \in Z(Q(g))$, $c$ has no nonzero fixed vectors in $M$. It immediately follows from this that $c$ is not the identity on $N$. Since any nontrivial normal subgroup of $Q(g)$ must contain $c$, this implies that $N$ is a faithful $L - Q(g)$ module.

Thus in proving the theorem we may as well assume that $K$ is algebraically closed. The lemma now implies that $M$ is an irreducible $K - Q$ module. If char $(K) = p$, then the theorem follows from
Theorems 2.5.1. and 2.5.4 of [7]. Hence we now suppose that char \((K) \neq p\).

\(Q/Q'\) is the direct product of groups transformed irreducibly by \(g\). Thus there is a subgroup \(Q_i/Q'\) such that \(g\) transforms \(Q_i/Q'\) irreducibly according to some automorphism of order \(p^n\). Now if \(Q_i\) were abelian, then, since \(g^{p^n-1}\) does not centralize \(Q_i\) and \(M\) is a completely reducible \(K-Q_i\) module, it would follow easily that the minimal polynomial of \(g\) would have degree \(p^n\). Hence \(Q_i\) is not abelian and so must be extra-special. This implies that \(|Q_i| = q^{d+1}\) for some \(d\).

Now if \(N\) is an irreducible \(K-Q_i\langle g \rangle\) submodule of \(M\), \(N\) must faithfully represent \(Q_i\) since \(c\) is represented by a scalar matrix. Hence \(N\) is an irreducible \(K-Q_i\) module.

Since \(g\) is exceptional, there is at least one \(p^n\)-th root of unity in \(K\) which is not an eigenvalue of \(g\). The argument given in [10, pp. 706-707] now implies that \(p^n - 1 = q^d\) and exactly \((p^n - 1)\) \(p^n\)-th roots of unity occur as eigenvalues of \(g\). Thus it only remains to show that \([Q, g] \leq Q_1\) to complete the proof of the theorem. If \(Q_1 = Q_2\), this is trivial. Therefore assume that \(Q = Q_1\). Then if \(Q_2 = C_q(Q_1)\) we find that \(Q_2\) admits \(g\) and \(Q\) is the central product of \(Q_1\) and \(Q_2\).

We now use the construction given in [7, p. 21] to construct linear groups \(H_1, H_2\) where \(H_i = Q_i\langle g_i \rangle\) and \(g_i\) is a \(p\)-element which transforms \(Q_i\) in the same way as \(g\). In the Kronecker product of \(H_1\) and \(H_2\), the product of \(Q_1\) and \(Q_2\) becomes identified with \(Q\). Since \(M\) is an irreducible \(K-Q\) module, it follows that \(g_1 \otimes g_2\) differs from \(g\) only by a scalar factor. Since \(g\) is of order \(p^n\), we find that

\[g = \alpha(g_1 \otimes g_2)\]

where \(\alpha^{p^n} = 1\). Now if \([Q_2, g] \neq 1\), then \(g_2\) has at least two distinct eigenvalues \(\beta, \gamma\). But \(g_1\) has \(p^n - 1\) distinct eigenvalues. Thus if \(\lambda\) is any \(p^n\)-th root of unity then at least one of \(\lambda/\alpha\beta\) and \(\lambda/\alpha\gamma\) must be an eigenvalue of \(g_1\). But this would imply that \(\lambda\) would be an eigenvalue of \(g\). Since \(g\) is exceptional, we must have that \([Q_2, g] = 1\).

**Corollary 2.3.** Under the hypothesis of the theorem let \(V\) be \(Q/Q'\) written additively and consider \(V\) as a \(GF(q) - \langle g \rangle\) module. Then the minimal polynomial of \(g\) on \(V\) is of degree at most \(2d + 1\).

**Proof.** This follows immediately from (b).

**Theorem 2.4.** Let \(G = PQ\) be a linear group over a field \(K\) where \(Q\) is a \(q\)-group normal in \(G\) (\(q \neq 2\)) and \(P\) is cyclic of order \(2^n > 2\) generated by an element \(g\) such that \([Q, g^{p^n-1}] \neq 1\). Assume that \(\text{char}(K) \neq q\) and that the minimal polynomial of \(g\) is of degree at most 3. Then we must have \(q = 3\) and \(n = 2\).
Proof. Extending $K$ affects neither hypothesis nor conclusion so we may as well assume that $K$ is algebraically closed. Now let $S$ be a subgroup of $Q$ which is minimal with respect to being normalized by $g$ but not centralized by $h$ where $h = g^{2n-1}$. Then $S$ is a special $q$-group.

If $V$ is the space on which $G$ operates, then $V = V_1 \oplus V_2 \oplus \cdots$ where the $V_i$ are the homogeneous $K - S$ submodules of $V$. Without loss of generality we may assume that $[S, h]$ is not the identity on $V_i$. But if $g^{2m}$ is the first power of $g$ fixing $V_i$, then the minimal polynomial of $g$ has degree at least $2^m$ times the degree of the minimal polynomial of $\{g^{2m} | V_i\}$. This implies that $g$ must fix $V_i$.

Now let $U$ be an irreducible $K - PS$ submodule of $V_i$. $[S, h]$ is not the identity on $U$ but $Z(S \setminus U)$ is cyclic generated by a scalar matrix. Thus we conclude that $\{S \setminus U\}$ is an extra special $q$-group whose center is centralized by $\{g \setminus U\}$. From Theorem 2.2 we now obtain that $2^m = q^d + 1$ and the minimal polynomial of $\{g \setminus U\}$ has degree $2^m - 1$. This implies that $n = 2$ and $q = 3$.

3. Proof of Theorem 1.2. Neither the hypothesis nor the conclusion of the theorem is affected by extending the field $K$. Thus we may assume without loss of generality that $K$ is algebraically closed. Now if $n = 1$, then, since $g$ is exceptional, $g$ would have to be a scalar matrix which would imply that $g \in Z(G)$. Hence we assume that $n > 1$ and let $h = g^{2n-2}$.

If $Q$ is any normal nilpotent subgroup of $G$, then char $(K) \nmid |Q|$ and so $V$, the space on which $G$ operates, is a completely reducible $K - Q$ module. Therefore $V = V_1 \oplus V_2 \oplus \cdots$ where the $V_i$ are the homogeneous $K - Q$ submodules. $G$ must permute the $V_i$ since $Q \leq G$. Now if $h^2$ did not fix each $V_i$, then it would follow that the minimal polynomial of $g$ would be of degree $2^n$ which is a contradiction. Let $H$ be the set of all elements in $G$ which fix each minimal characteristic $K - Q$ submodule of $V$ for each normal nilpotent subgroup $Q$ in $G$. Clearly $H \leq G$. Hence $F_i(H) \leq F_i(G)$ for $i = 1, 2$. Also we have shown that $h^2 \in H$.

It follows from [4, Lemmas 3.2 and 3.3] that $[Q, H] = 1$ if $Q$ is any normal abelian subgroup of $G$ and that $F_i(H)$ is of class 2. $F_i(H) = Q_1 \times Q_2 \times \cdots$ where $Q_i$ is the Sylow $q_i$-subgroup of $F_i(H)$ and $q_i$ is an odd prime. Since $Q_i$ is of class at most 2, $Q_i$ is a regular $q_i$-group. Then the elements of order at most $q_i$ form a subgroup $R_i$ in $Q_i$. If $R = R_1 \times R_2 \times \cdots$, then $C_H(R) \leq F_i(H)$ [9, Hilfssatz 1.5].

The proof now divides into two parts. First we will show that $h^2$ induces the identity automorphism on any $2'$-subgroup of $F_2(H)/F_1(H)$. In the second part we consider how $h^2$ operates on a 2-subgroup of $F_2(H)/F_1(H)$. 
**Part I.** Suppose that \( p \) is an odd prime which divides \( |F_2(H)/F_1(H)| \).

It is easy to show that there is a Sylow \( p \)-subgroup \( P \) of \( F_2(H) \) which is normalized by \( g \). We now proceed to prove that

\[
[P, h^2] \leq F_1(H).
\]

To do this we first note that, since \( P \leq F_1(H) \), \( C_P(O_P, (F_1(H))) = P \cap F_1(H) \). Now let \( N = P \cap F_1(H) \) and suppose that \([P, h^2] \leq N\).

Since \( C_P(O_P, (F_1(H))) = N \), there is a \( q_i \neq p \) such that \([h^2, P, R_i] \neq 1\). Now let \( U \) be a minimal characteristic \( K - R_i \) submodule of \( V \) on which \([h^2, P, R_i] \) is not the identity. Let \( q = q_i, \ S = \{P \mid U\}, \) and \( Q = \{R_j \mid U\} \). \( h^2 \) must fix \( U \) but cannot be a scalar matrix on \( U \) since \([h^2, P, R_i] \mid U \neq 1\). Let \( g^{2^{n-m}} \) be the first power of \( g \) to fix \( U \) and let \( g_i \) be the restriction of \( g^{2^{n-m}} \) to \( U \). But if \( g_i \) were not exceptional then \( g_i \) could not be exceptional. Hence \( g_i \) is exceptional and so \( m \) must be \( > 1 \). Now let \( h_i = g_i^{2^{n-2}} \).

Then \([h_i^2, S, Q] \neq 1\). Since \( U \) is the sum of isomorphic, irreducible \( K - Q \) modules, \( Z(Q) \) must be cyclic generated by a scalar matrix. Therefore \([Z(Q), S \langle g_i \rangle] = 1\) and, since \( Q \) is a homomorphic image of a class 2 group of exponent \( q \), \( Q \) must be an extra-special \( q \)-group.

Next let \( U_i \) be an irreducible \( K - Q \langle g_i \rangle \) submodule of \( U \). Lemma 2.1 implies that \( U_i \) is an irreducible \( K - Q \) module and so \( U \) is the sum of \( K - Q \) modules isomorphic to \( U_i \). From Theorem 2.2 we obtain that \( 2^m - 1 = q^d \) and \([Q : C_Q(g_i)] = q^{2d}\). Then \( q \) must be a Mersenne prime and \( d = 1 \).

Now let \( W \) be \( Q/Q' \) written additively and consider \( W \) as a \( GF(q) - S \langle g_i \rangle \) module. The minimal polynomial of \( g_i \) on \( W \) has degree at most 3 from Corollary 2.3. Since \([h_i^2, S] \) is not the identity on \( W \), Theorem 2.4 now implies that \( m = 2 \) and \( p = 3 \) which contradicts \( p \neq q = 2^m - 1 \).

Thus we have shown that \( h^2 \) induces the identity automorphism on any \( 2' \)-subgroup of \( F_2(H)/F_1(H) \).

**Part II.** The \( 2 \)-subgroups of \( F_2(H)/F_1(H) \) have to be handled differently and we apply the method of [4, pp. 1224–1228]. Accordingly, let \( V = V_1 \oplus V_2 \oplus \cdots \) where the \( V_{ij} \) are the homogeneous \( K - R_i \) submodules of \( V \). For each \( i \) and \( j \), let

\[
C_{ij} = \{x \mid x \in H \text{ and } ([R_i x] \mid V_{ij}) = 1\}.
\]

Next let \( H_i \) be the intersection of all the \( C_{ij} \) which contain \( h^2 \). If \( h^2 \) belongs to no \( C_{ij} \) then set \( H_i \) equal to \( H \). In any event \( H_i \triangleleft H \).
$h^2 \in H_1$, and $g$ normalizes $H_1$.

Now choose $P$ to be a Sylow 2-subgroup of $F_2(H_1)$ such that $P\langle g \rangle$ is a 2-group. If $x \in P$, we now assert that $[h^2, x] = [h, x]^2$. The proof of this is identical with the proof of Lemma 3.4 in [4] and, for this reason, is omitted.

Now from the above we see that $[h^2, P] \leq D(P)$. This combined with our result proved in Part I implies that $[h^2, F_2(H_1)] \leq D(F_2(H_1)) \mod F_1(H_1)$. But this implies that $h^2 \in F_2(H_1)$. Since $F_2(H_1) \leq F_2(H)$ and $F_2(H) \leq F_2(G)$, this completes the proof of the theorem.

4. Proof of Theorem 1.1. Let $\sigma$ denote the fixed-point-free automorphism of order $2^n$. If $n \leq 2$, then the result is a known one [3]. Consequently, we assume that $n \geq 3$ and proceed by induction on the order of $G$.

Now if $G$ has two distinct minimal $\sigma$-admissible normal subgroups $H_1$ and $H_2$, then by induction, $(G/H_1) \times (G/H_2)$ has nilpotent length at most $2n - 2$. Since $G$ is isomorphic to a subgroup of $(G/H_1) \times (G/H_2)$, the theorem would follow immediately.

Therefore we may assume that $G$ has a unique minimal $\sigma$-admissible normal subgroup. This implies that $F_4(G)$ is a $p$-group for some $p$. Then we may consider $H = \langle \sigma \rangle G / F_4(G)$ as a linear group operating on $V$ where $V$ is $F_4(G) / D(F_4(G))$ written additively. Now $p$ cannot be 2 and $(\sigma - 1)$ must be nonsingular on $V$. Thus $\sigma$ must be exceptional and we obtain from Theorem 1.2 that $\sigma^{2n-1} \in F_2(H)$.

This implies that $\sigma^{2n-1}$ centralizes $F_4(G) / F_2(G)$ which in turn implies that $\sigma^{2n-1}$ centralizes $G / F_4(G)$ [8, Lemma 4]. Thus, by induction, the nilpotent length of $G / F_4(G)$ is at most $\max \{2n - 4, n - 1\}$. Since we are assuming that $n \geq 3$, this implies that $G$ has nilpotent length at most $2n - 2$.

REFERENCES

5. ———, The 2-length of groups whose Sylow 2-groups are of exponent 4, J. Algebra 2 (1965), 312-314.

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