EXTENSIONS OF REGULAR BOREL MEASURES

Jack Hardy and Howard E. Lacey

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This paper is concerned with the extension of regular Borel measures defined on the Borel sets generated by sub-topologies of compact Hausdorff topologies.

Specifically, if $X$ is a nonempty set and $\tau$ is a topology on $X$, the Borel sets of $(X, \tau)$ are the members of the smallest $\sigma$-ring containing $\tau$. A regular Borel measure is taken to mean a finite-valued measure $\mu$ on the Borel sets of $(X, \tau)$ with property that

$$\mu(B) = \sup \{\mu(F) | F \subseteq B, F \text{ is closed}\}.$$ 

In this paper, the situation considered is the following: $\tau$ is a compact Hausdorff topology on $X$, and $\sigma$ is a regular (in the topological sense) sub-topology of $\tau$. The space $C(X, \tau)$ is the (partially ordered) Banach space of all continuous real-valued functions on $(X, \sigma)$, in the supremum norm. The space $C(X, \sigma)$ is similarly defined. By constructing a one-to-one correspondence between the collection of regular Borel measures on $(X, \sigma)$ and the collection of positive linear functionals on $C(X, \sigma)$ it is shown that every regular Borel measure on $(X, \sigma)$ can be extended to a regular Borel measure on $(X, \tau)$. This result is used to prove the existence of nonatomic regular Borel measures on compact Hausdorff spaces with perfect sub-sets.

The concept of a "partition space" plays a central role in this development.

DEFINITION 1. Let $X$ be a topological space. A topological space $Y$ is said to be a partition space of $X$ if there is an onto function $f: X \to Y$ such that the topology for $X$ is the smallest topology for which $f$ is continuous.

A partition space is a special kind of quotient space [6: 94]. Every topological space is a partition space of itself, and a partition space of a compact space is compact. It will be important to know when a topological space has a Hausdorff partition space. For a similar result about quotient spaces see [6: 98]. A proof is given here because the notation of the proof is used later on.

THEOREM 2. A topological space $X$ has a Hausdorff partition space if and only if, for any two points $x$ and $y$ in $X$, if there is an open set $U$ such that $x \in U$ and $y \notin U$, then there are disjoint neighborhoods of $x$ and $y$. 

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Proof. Let $Y$ be a Hausdorff partition space of $X$ and let $f: X \to Y$ be the function in Definition 1. Suppose $x$ and $y$ are two points of $X$, and $U$ is an open set such that $x \in U$ and $y \in U$. There is an open set $E$ in $Y$ such that $U = f^{-1}(E)$. Then $f(x) \in E$ and $f(y) \in E$. Thus $f(x) \neq f(y)$, and there are disjoint neighborhoods $E_1$ of $f(x)$ and $E_2$ of $f(y)$. Then $f^{-1}(E_1)$ and $f^{-1}(E_2)$ are disjoint neighborhoods of $x$ and $y$.

Conversely, for each $x \in X$, let $N_x$ be the set all elements $y \in X$ such that, for each open set $U$, $y \in U$ if and only if $x \in U$. Let $Y = \{N_x : x \in X\}$. Define a function $f: X \to Y$ by $x \to N_x$ for every $x \in X$, and give $Y$ the largest topology for which $f$ is continuous (that is, a subset $B \subset Y$ is open if and only if $f^{-1}(B)$ is open). Then $Y$ is a Hausdorff partition space of $X$, because $f$ is certainly continuous, and if $U$ is open in $X$, then $U = \bigcup \{N_x : x \in U\}$ implies $f^{-1}(N_x : x \in U) = U$.

**Corollary 3.** Every regular topological space has a Hausdorff partition space. In particular, every compact regular space has a compact Hausdorff partition space.

From now on, the "partition space" of a topological space $X$ will mean the partition space $Y = \{N_x : x \in X\}$ defined in the proof of Theorem 2.

The proof of the following theorem is straight-forward computation and hence will be omitted.

**Theorem 4.** Let $Y$ be the partition space of a topological space $X$, and $B_X$ and $B_Y$ be the classes of Borel sets in $X$ and $Y$, respectively. Then

$$B_X = \{f^{-1}(E) : E \in B_Y\}.$$  

If $\mu$ and $\nu$ are real-valued functions on $B_X$ and $B_Y$ such that $\mu(f^{-1}(E)) = \nu(E)$ for every $E \in B_Y$, then $\mu$ is a regular Borel measure on $X$ if and only if $\nu$ is a regular Borel measure on $Y$.

**Corollary 5.** In the notation of Theorem 4, to every regular Borel measure $\mu$ on $X$ assign a unique regular Borel measure $\nu_\mu$ on $Y$ by means of the formula

$$\nu_\mu(E) = \mu(f^{-1}(E)), (E \in B_Y).$$

Then the mapping $\mu \to \nu_\mu$ is a one-to-one correspondence between the collection of regular Borel measures on $X$ and the collection of regular Borel measures on $Y$.

**Theorem 6.** Let $Y$ be the partition space of a topological space
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X, and C(X) and C(Y) be the spaces of continuous real-valued functions on X and Y, respectively. Then

\[ C(X) = \{g \circ f : g \in C(Y)\}. \]

If \( I(g \circ f) = J(g) \) for every \( g \in C(Y) \), then \( I \) is a positive linear functional on \( C(X) \) if and only if \( J \) is a positive linear functional on \( C(Y) \).

**Proof.** Clearly \( \{g \circ f : g \in C(Y)\} \subset C(X) \). On the other hand, take \( h \in C(X) \). If \( N_x \) is a fixed point of \( Y \) and \( y_1, y_2 \in N_x \), then \( h(y_1) = h(y_2) \) (otherwise, there are disjoint neighborhoods \( E_1 \) and \( E_2 \) of \( h(y_1) \) and \( h(y_2) \), and \( h^{-1}(E_1) \) and \( h^{-1}(E_2) \) would be disjoint neighborhoods of \( y_1 \) and \( y_2 \)). Thus \( g(N_x) = h(x) \) defines a real-valued function \( g \) on \( Y \). Clearly \( g \in C(Y) \) and \( g \circ f = h \). This shows \( C(X) = \{g \circ f : g \in C(Y)\} \).

The second part is immediate.

**Corollary 7.** In the notation of Theorem 6, to every positive linear functional \( I \) on \( C(X) \) assign a unique positive linear functional \( J_I \) on \( C(Y) \) by means of the formula

\[ J_I(g) = I(g \circ f), \quad (g \in C(Y)). \]

Then the mapping \( I \rightarrow J_I \) is a one-to-one correspondence between the collection of positive linear functionals on \( C(X) \) and the collection of positive linear functionals on \( C(Y) \).

Let \( X \) be a compact regular space. To every regular Borel measure \( \mu \) on \( X \) assign a unique positive linear functional \( I_\mu \) on \( C(X) \) as follows. If \( Y \) is the partition space of \( X \), a regular Borel measure \( \mu \) on \( X \) gives rise (by Corollary 5) to a regular Borel measure \( \nu \) on \( Y \). Then the formula

\[ J_\mu(g) = \int_Y g d\nu \quad (g \in C(Y)) \]

defines a positive linear functional \( J_\mu \) on \( C(Y) \) which (by Corollary 7) defines a positive linear functional \( I_\mu \) on \( C(X) \).

**Theorem 8.** For a compact regular space \( X \), the mapping \( \mu \rightarrow I_\mu \) is a one-to-one correspondence between the collection of regular Borel measures on \( X \) and the collection of positive linear functionals on \( C(X) \).

**Proof.** The Riesz Representation Theorem for compact Hausdorff spaces [4:177–178] shows that the mapping \( \nu \rightarrow J_\nu \) is a one-to-one
correspondence between the collection of regular Borel measures on $Y$ and the collection of positive linear functionals on $C(Y)$. Then Corollaries 5 and 7 complete the proof.

The next theorem (which generalizes a proof in [9]) is the main result of this paper.

**Theorem 9.** Let $\tau$ be a compact Hausdorff topology for a set $X$, and let $\sigma$ be a regular topology for $X$ such that $\sigma \subset \tau$. Then every regular Borel measure on $(X, \sigma)$ can be extended to a regular Borel measure on $(X, \tau)$.

**Proof.** Let $\mu$ be a regular Borel measure on $(X, \sigma)$ and $I$ be the positive linear functional on $(X, \sigma)$ corresponding to $\mu$ by the mapping in Theorem 8. By [8, p. 18], $I$ can be extended to a positive linear functional $I^*$ on $C(X, \tau)$. Let $\mu^*$ be the regular Borel measure on $(X, \tau)$ such that

$$I^*(g) = \int_X g d\mu^*$$

for all $g \in C(X, \tau)$. It is shown that $\mu^*$ extends $\mu$.

Let $Y$ be the partition space of $(X, \sigma)$, and $\nu$ be the regular Borel measure on $Y$ defined by $\nu(E) = \mu(f^{-1}(E))$ for every Borel set $E$ in $Y$.

Let $J$ be the positive linear functional on $C(Y)$ corresponding to $\nu$. Let $U$ be a member of $\sigma$. Then there is an open set $V$ in $Y$ such that $f^{-1}(V) = U$. Now, $\mu(U) = \nu(V) =$

$$\sup \{J(h) \mid h \in C(Y), 0 \leq h \leq 1, h(y) = 0 \quad \text{if} \quad y \in V\}$$

$$= \sup \{J(k) \mid k \in C(X, \sigma), 0 \geq k \leq 1, k(x) = 0 \quad \text{if} \quad x \in U\}$$

$$\leq \sup \{I^*(k) \mid k \in C(X, \tau), 0 \leq k \leq 1, k(x) = 0 \quad \text{if} \quad x \in U\}$$

$$= \mu^*(U).$$

To show the reverse inequality, let $\varepsilon > 0$ and let $K$ be a closed set in $(X, \tau)$ such that $K \subset U$ and $\mu^*(U) < \varepsilon + \mu^*(K)$. Since $(X, \sigma)$ is regular, for each $x$ in $K$, there is a set $V(x)$ such that $V(x)$ is closed in $(X, \sigma)$ and $x \in V(x) \subset U$. Since a compact regular space in normal [6: 141], for each $x \in K$, there is a $g_x \in C(X, \sigma)$ such that $CH_{V(x)} \leq g_x \leq CH_U$ ($CH$ is the characteristic function). Let $U(x) = \{y \in X \mid g_x(y) + \varepsilon > 1\}$. Then $\{U(x) \mid x \in K\}$ is a family of open subsets $g(X, \tau)$ which covers $K$, and there are $X_1, \ldots, X_n$ in $K$ such that $\{U(X_i) \mid i = 1, \ldots, n\}$ covers $K$. Let $g = \max \{g_{x_i} \mid i = 1, \ldots, n\}$. Then $g \in C(X, \sigma)$ and $\mu^*(U) < \varepsilon + \mu^*(K) = \varepsilon + \int_X CH_g d\mu^* \leq \varepsilon + \int_X (g + \varepsilon) d\mu^* = \varepsilon + \varepsilon \mu^*(X) + I^*(g) \leq 2\varepsilon + \mu(U)$ (since $0 \leq g \leq CH_U$). Thus $\mu^*(U) = \varepsilon$.

\footnote{We wish to thank the referee for pointing out a simplification of the proof.}
&mu;(U) and by the regularity of &mu;, &mu^* extends &mu.

It is now shown how this can be applied to relationships between measures on X and Y and mappings from X and Y. In particular, it is shown that if X, Y are compact Hausdorff spaces and f: X \to Y is a continuous onto map, then each regular Borel measure on Y generates a regular Borel measure on X.

**Theorem 10.** Let X and Y be a compact Hausdorff spaces, f be a continuous function from X onto Y, and \( \nu \) be a regular Borel measure on Y. Then there is a regular Borel measure \( \mu \) on X such that

\[
\mu(f^{-1}(E)) = \nu(E)
\]

for every Borel set E in Y. Moreover, if \( \nu \) is nonatomic, then so is \( \mu \).

**Proof.** Let \( \tau \) denote the topology for X, and let

\[
\sigma = \{ f^{-1}(U) : U \text{ open in } Y \}.
\]

Then \( \sigma \) is a regular topology for X, \( \sigma \subset \tau \), \( C(X, \sigma) \) is a linear subspace of \( C(X, \tau) \), and \( C(X, \sigma) \) contains the constant functions. Thus Theorem 9 implies that every regular Borel measure on \( (X, \sigma) \) can be extended to a regular Borel measure on \( (X, \tau) \).

For every Borel set E in Y, define \( \mu_0(f^{-1}(E)) = \nu(E) \). The proof of Theorem 4 shows that \( \mu_0 \) is a regular Borel measure on \( (X, \sigma) \). Let \( \mu \) be a regular Borel measure on \( (X, \tau) \) which extends \( \mu_0 \). Then

\[
\mu(f^{-1}(E)) = \nu(E)
\]

for every Borel set E in Y. If \( \nu \) is nonatomic, then, for every \( x \in X \),

\[
0 \leq \mu(x) \leq \mu(f^{-1}f(x)) = \nu(f(x)) = 0
\]

implies \( \mu(x) = 0 \), and thus \( \mu \) is nonatomic.

**Corollary 11.** [9]. If X is a compact Hausdorff space with a nonempty perfect set, then there is a nonzero, nonatomic regular Borel measure on X.

**Proof.** There is a continuous map of X onto [0, 1]. Thus, in Theorem 10 one can use Lebesgue measure for \( \nu \).

**Corollary 12.** Let X be a compact Hausdorff space, \( \{P_n\} \) be a disjoint sequence of perfect subsets of X, and \( \{a_n\} \) be a sequence of nonnegative real numbers such that \( \sum a_n < \infty \). Then there is a nonatomic regular Borel measure \( \mu \) on X such that \( \mu(P_n) = a_n \) for every n.
Proof. For each $n$, let $\nu_n$ be a nonatomic regular Borel measure on $P_n$ such that $\nu_n(P_n) = a_n$, and define $\mu_n(E) = \nu_n(E \cap P_n)$ for every Borel set $E$ in $X$. Then $\mu = \sum \mu_n$ is a nonatomic regular Borel measure on $X$, and $\mu(P_n) = a_n$ for every $n$.

For the last theorem some additional terminology is needed. Let $X$ and $Y$ be compact Hausdorff spaces. By $M(X)$ is meant the Banach lattice of all regular Borel measures on $X$ under the total variation norm. Of course, $M(X)$ is precisely the Banach space dual of $C(X)$. If $\nu$ is a regular Borel measure on $Y$, by $L'(\nu)$ is meant the Banach lattice of all $\nu$-integrable functions on $Y$, under the integral norm.

**Theorem 13.** If there is a continuous map $f$ of $X$ onto $Y$, then $L'(\nu)$ is linearly isometric to a subspace of $M(X)$.

Proof. Let $\mu$ be the regular Borel measure associated with $\nu$ of Theorem 11. Let $N$ be the normed linear space whose elements are the continuous functions on $Y$, but whose norm is the integral norm with respect to $\nu$. Then $N$ is dense in $L'(\nu)$. Define the linear operator $A: N \to M(X)$ by $(Ag)(h) = \int_X h(g \circ f) \, d\mu$ for all $g \in N$, $h \in C(X)$. Now, $\| Ag \| = \int_X |g \circ f| \, d\mu = \| g \|$ and $A$ is an isometry of $N$ into $M(X)$. Since $N$ is dense in $L'(\nu)$, $A$ can be uniquely extended to an isometry of $L'(\nu)$ into $M(X)$.

**Bibliography**


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