THE APPROXIMATION SOLUTION OF \( y' = F(x, y) \)

R. G. Huffstutler and Frederick Max Stein
THE APPROXIMATE SOLUTION OF $y' = F(x, y)$

R. G. HÜFFSTUTLER AND F. MAX STEIN

In general the exact solution of the differential system

$$y' = F(x, y), \quad y(0) = 0,$$

is either unattainable or is impractical to handle, even though a solution is known to exist and may even be obtained in certain cases. Thus some method of approximation is often employed. After the choice of approximating functions has been made, there still remains the questions of goodness of approximation and, if infinite processes are employed, the question of convergence.

The system described above is restricted in this paper to those cases in which $F(x, y)$ is an analytic function of $x$ and $y$ for $-1 \leq x \leq 1$ and all $y$. Then $F(x, y)$ can be written as a convergent power series

$$F(x, y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j.$$

By considering a sequence of $n$-th degree polynomials $\{P_n(x)\}$ which are $\varepsilon$-approximate solutions of the truncated system

$$L_k(y) = y' - \sum_{i=0, j=1}^{k} a_{ij} x^i y^j = F(x, 0), \quad y(0) = 0,$$

the solution of the original system can be uniformly approximated by polynomials which satisfy $P_n(0) = 0$ and which minimize

$$\| F(x, 0) - L_k[P_n(x)] \| = \sup_{0 \leq x \leq 1} | F(x, 0) - L_k[P_n(x)] |.$$

1. Introduction. The differential equation to be considered is the equation

(1) $$y' = F(x, y),$$

where $F(x, y)$ is an analytic function of $x$ and $y$ for $-1 \leq x \leq 1$ and all $y$; that is, $F(x, y)$ has the power series expansion

$$F(x, y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$$

which is valid for $-1 \leq x \leq 1$ and all $y$. Along with (1) we apply the initial condition

(2) $$y(0) = 0.$$

It is known that (1) possesses a unique solution satisfying the initial condition [1].

We wish to consider the equation (1) in the form
where \( F(x, 0) = \sum_{i=0}^{\infty} a_{i0} x^i \). Let \( R_k(x, y) = F(x, y) - F_k(x, y) \), where \( F_k(x, y) \) is defined by the equation
\[
F_k(x, y) = \sum_{i,j=0}^{k} a_{ij} x^i y^j. 
\]

Associated with (3) we consider the truncated system in either the form

\[
(3.k_1) \quad L_k(y) \equiv y' - \sum_{i=0, j=1}^{k} a_{ij} x^i y^j = F(x, 0) , \quad y(0) = 0 ,
\]
or

\[
(3.k_2) \quad y' - F_k(x, y) = R_k(x, 0) , \quad y(0) = 0 ,
\]
where \( R_k(x, 0) = \sum_{i=k+1}^{\infty} a_{i0} x^i \). Note that \( R_k(x, 0) \) is not the same as \( F(x, 0) \). However, since (3.k_1) and (3.k_2) are the same equations, a solution of one is a solution of the other. Observe also that the conditions assumed for (1) are sufficient to insure solutions for (3.k_1) or (3.k_2).

We wish to consider the best approximate solution of (1) by the use of polynomials \( P_n^k(x) \) as approximating functions in the sense that

\[
(4) \quad \| F(x, 0) - L_k[P_n^k(x)] \| = \sup_{0 \leq x \leq 1} | F(x, 0) - L_k[P_n^k(x)] |
\]
is a minimum for each fixed \( n \) and \( k \). Here the superscript \( k \) in \( P_n^k(x) \) represents the \( k \) of (3.k_1), and \( n \) is the degree of \( P_n^k(x) \). That is, we consider the conditions under which a sequence of minimizing polynomials of (4) will converge uniformly to the unique solution \( y \) of (1).

2. Preliminary results. In a paper to appear in the Proceedings [3], the authors consider the problem of best approximation of the solution of the equation

\[
(5) \quad N(y) \equiv y' - \sum_{k=1}^{m} f_k(x) y^k = R(x) , \quad x \in [0, 1] ,
\]
satisfying \( y(0) = 0 \), by the use of polynomials \( P_n(x) \) as approximating functions. The approximation there is a best approximation in the sense that the approximating polynomial is required to minimize

\[
(6) \quad \| R(x) - N[P_n(x)] \| = \sup_{0 \leq x \leq 1} | R(x) - N[P_n(x)] | .
\]

From [3] we have the following theorem:
THEOREM 1. If \( y(x) \) is the unique solution of (5) which satisfies the initial condition \( y(0) = 0 \) and which has a continuous first derivative, and if \( P_n(x) \) is a polynomial which minimizes (6) and is such that \( P_n(0) = 0 \), then \( P_n(x) \) and \( P_n'(x) \) converge uniformly throughout \([0, 1]\) to \( y(x) \) and \( y'(x) \) respectively as \( n \) increases without bound.

If we return now to equation (3.,k.), we see that the conditions of Theorem 1 are satisfied, and we know that for each fixed \( k \) there exists a sequence of polynomials \( P^k_n(x) \) which satisfy \( P^k_0(0) = 0 \), which minimize (4) for that fixed \( k \), and which uniformly approximate the solution of (3.,k.).

The following definition will be useful in our work to follow [1]. Let \( f(x, y) \) be a real-valued continuous function on a domain \( D \) in the \((x, y)\)-plane, and consider the equation

\[
y' = f(x, y) .
\]

An \( \varepsilon \)-approximate solution of (7) on an \( x \) interval \( I \) is a function \( \Phi \in C \) on \( I \) such that

(i) \( (x, \Phi(x)) \in D, \ x \in I, \)

(ii) \( \Phi \in C^1 \) on \( I \), except possibly for a finite set of points \( S \) on \( I \), where \( \Phi' \) may have simple discontinuities, and

(iii) \( |\Phi'(x) - f(x, \Phi(x))| \leq \varepsilon, \ X \in I - S. \)

Denote the set of all \( \varepsilon \)-approximate solutions of (3.,k.) by \( S^\varepsilon_k \). We know that given any \( \varepsilon > 0 \), there exists a \( K(\varepsilon) \) such that

\[
| \sum_{i,j=k+1}^{\infty} a_{ij} x^i y^j | < \varepsilon
\]

if \( k \geq K(\varepsilon) \), since the series is convergent, an observation we shall need in the following sequence of lemmas which lead to the proof of the principal result of this paper.

**Lemma 1.** If \( y(x) \) is any solution of (1), then \( y(x) \) is in \( S^\varepsilon \) for all \( k \) sufficiently large, say \( k \geq K(\varepsilon). \)

**Proof.** If \( y(x) \) is a solution of (1), then

\[
y' = F(x, y) = F^k(x, y) + R_k(x, y) .
\]

Given any \( \varepsilon > 0 \), then for sufficiently large \( k \), say \( k \geq K(\varepsilon) \), we have that \( |R_k(x, y)| \leq \varepsilon/2 \) (and hence \( |R_k(x, 0)| \leq \varepsilon/2 \)) by (8). Therefore

\[
| y' - F^k(x, y) - R_k(x, 0) | = | F(x, y) - F^k(x, y) - R_k(x, 0) |
\]

\[
\leq | R_k(x, y) | + | R_k(x, 0) | \leq \varepsilon/2 + \varepsilon/2 = \varepsilon .
\]

But this is precisely the statement of the fact that \( y(x) \) is in \( S^\varepsilon \) for
LEMMA 2. If $P^k_n$ is a minimizing polynomial of (4), and if the conditions of Theorem 1 are satisfied, then for $n$ sufficiently large, $P^k_n(x)$ is an $\varepsilon$-approximate solution of (3.k).

Proof. Since $F_k(x, y) = \sum_{i,j=0}^{k} a_{ij} x^i y^j$, then $F_k(x, y)$ is a continuous function of $y$. Thus, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|F_k(x, y_k) - F_k(x, P^k_n)| \leq \varepsilon/2,$$

provided that $|y_k - P^k_n| < \delta$, where $y_k$ is the solution of the $k$-th truncated equation (3.k). Since the conditions of Theorem 1 are satisfied,

$$|y_k - P^k_n| < \delta,$$

for $n$ sufficiently large, say $n > N(\varepsilon, k)$. Likewise, by Theorem 1, for $n > N(\varepsilon, k)$ we have that

$$|y_k' - (P^k_n)'| \leq \varepsilon/2,$$

for all $x \in [0, 1]$ and for $n \geq N$. Thus

$$|(P^k_n)' - F_k(x, P^k_n)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus $P^k_n(x)$ is an $\varepsilon$-approximate solution of (3.k) for $n$ sufficiently large.

LEMMA 3. If $P^k_n(x)$ is an $\varepsilon/3$ approximate solution of (3.k), then $P^k_n(x)$ is an $\varepsilon$-approximate solution of (1), provided that $k \geq K(\varepsilon)$.

Proof. Since $P^k_n(x)$ is an $\varepsilon/3$ approximate solution of (3.k), we have that

$$|(P^k_n)' - F(x, P^k_n)|$$

$$\leq |(P^k_n)' - F_k(x, P^k_n) - R_k(x, 0)| + |F_k(x, P^k_n) - F(x, P^k_n)|$$

$$+ |R_k(x, 0)|$$

$$= |(P^k_n)' - F_k(x, P^k_n) - R_k(x, 0)| + |R_k(x, P^k_n)| + |R_k(x, 0)|$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$
THE APPROXIMATE SOLUTION OF $y' = F(x, y)$

since for $k$ sufficiently large, say $k \geq K(\varepsilon)$,

$$\left| \sum_{i,j=k+1}^{\infty} a_{ij}x^iy^j \right| \leq \varepsilon/3 \quad \text{and} \quad \left| \sum_{i=k+1}^{\infty} a_{ii}x^i \right| \leq \varepsilon/3 .$$

**Lemma 4.** If $P^k(x)$ is an $\varepsilon/5$ approximate solution of $(3,k^2)$ where $k \geq K(\varepsilon)$, then for any $j$ such that $k \geq j \geq K(\varepsilon)$, $P^k_n(x)$ is in the class $S^j$.

**Proof.** Since $P^k_n(x)$ is an $\varepsilon/5$ approximate solution of $(3,k^2)$ we have that

$$\left| (P^k_n)' - F_j(x, P^k_n) - R_j(x, 0) \right| \leq \varepsilon/5 .$$

Also $|R_k(x, P^k_n)| \leq \varepsilon/5$ and $|R_j(x, P^k_n)| \leq \varepsilon/5$ by hypothesis. Thus by adding and subtracting $F_j(x, P^k_n)$ and $R_j(x, 0)$ we can write

$$\left| (P^k_n)' - F_j(x, P^k_n) - R_j(x, 0) \right| \leq \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon ,$$

where use has been made of the fact that

$$\left| F_j(x, P^k_n) - F_j(x, P^k_n) \right| = \left| R_k(x, P^k_n) - R_j(x, P^k_n) \right| .$$

Hence, $P^k_n$ is an $\varepsilon$-approximate solution of $(3,k^2)$ when $k = j$ and thus belongs to the class $S^j$.

It is useful to note that any solution of a given differential equation is an $\varepsilon$-approximate solution of that equation for all positive $\varepsilon$, and also if $y(x)$ is an $\varepsilon$-approximate solution of a given differential equation for all $\varepsilon > 0$, then $y(x)$ is a solution of that given differential equation.

3. The approximate solution of the differential equation.

We now consider the approximate solution of (1) by a sequence of polynomials $P^k_n(x)$ which minimize (4). Let $\{\varepsilon_k\}$ be a monotone null sequence of positive numbers with $\varepsilon_k < 1$. For each $\varepsilon_k$ there exists a $K_k(\varepsilon_k)$ such that whenever $h \geq K_k$,

$$\left| \sum_{i,j=k+1}^{\infty} a_{ij}x^iy^j \right| \leq \varepsilon_k/3 .$$

From Lemma 3 an $\varepsilon/3$ approximate solution $P^k_n(x)$ of $(3,k^2)$ is an $\varepsilon$-approximate solution of (1) provided that $k \geq K_k$. Then for $\varepsilon_k$ there exists a $K_k$ such that
are $\varepsilon_1$-approximate solutions of (1). Here we are denoting by $P^k_{n_1}, P^k_{n_2}, \ldots, P^k_{n_j}$ any $\varepsilon_1/3$ approximate solution of (3.2) when $k_2 = K_1 + j$. We also note that $P^k_{n_j+1}(0) = 0$; i.e., the polynomials satisfy the boundary condition for all $j = 0, 1, \ldots$.

For $\varepsilon_2$ there exists a $K'_2$ such that

$$P^k_{n_1}, P^k_{n_2}, \ldots, P^k_{n_j}$$

are $\varepsilon_2$-approximate solutions of (1). Choosing $K_2 = \max (K_1, K'_2)$ leads to the sequence $P^k_{n_1}, P^k_{n_2}, \ldots, P^k_{n_j}$, where $P^k_{n_j}$ is an $\varepsilon_2$-approximate solution of (3.2) when $k_2 = K_2 + j$, and for $j = 0, 1, \ldots$, and $P^k_{n_j+1}(0) = 0$. It should be noted that $P^k_{n_j}$ is an $\varepsilon_1$- and $\varepsilon_2$-approximate solution of (1).

Define inductively the sequence of solutions

$$P^k_{n_1}, P^k_{n_2}, \ldots, P^k_{n_j}$$

where $P^k_{n_j}$ is an $\varepsilon_1/3$ approximate solution of (3.2) for $k_2 = K_2 + j$ having the property that $P^k_{n_j+1}(0) = 0, j = 0, 1, \ldots$. Again $P^k_{n_j}$ is an $\varepsilon_1$-approximate solution of (1).

Consider the sequence

$$P^k_{n_1}, P^k_{n_2}, P^k_{n_3}, \ldots, P^k_{n_j}$$

Denote this sequence by $\{Y_i\}$. We observe that each $Y_i$ is at least an $\varepsilon_1$-approximate solution of (1), $i = 1, 2, \ldots$.

At this point we need a theorem given by Coddington and Levinson [1].

**Theorem 2.** Suppose that $f(x, y)$ is continuous in some domain $D$ of the $(x, y)$-plane and further that $f(x, y)$ satisfies a Lipschitz condition with respect to $y$ in $D$ with Lipschitz constant $K$. Let $\Phi_1$ and $\Phi_2$ be $\varepsilon_1$- and $\varepsilon_2$-approximate solutions of (7) of class $C^1$, at least piecewise on $[a, b]$, satisfying

$$|\Phi_1(x_0) - \Phi_2(x_0)| \leq \delta$$

for some $x_0$ such that $a \leq x_0 \leq b$, where $\delta$ is a nonnegative constant. If $\varepsilon = \varepsilon_1 + \varepsilon_2$, then for all $x \in [a, b]$

$$|\Phi_1(x) - \Phi_2(x)| \leq \delta e^{K|x-x_0|} + \frac{\varepsilon}{K} (e^{K|x-x_0|} - 1).$$

By a theorem given by Coppel [2] and stated in [3], we know that the solutions $y_k$ of (3.2) are uniformly bounded on $[0, 1]$. Hence the polynomials of the set $\{Y_i\}$ are uniformly bounded, and there exists a uniform Lipschitz constant $K$ for (1) for $x$ in $[0, 1]$. 

\[288 \text{ R. G. HUFFSTUTLER AND F. MAX STEIN}\]
Hence if $i^*$ and $j^*$ are chosen so large that $\varepsilon_{i^*}$ and $\varepsilon_{j^*}$ are both less than $\varepsilon/2C$ for arbitrary small $\varepsilon > 0$, where $C = (|e^K - 1|)/K$, then

$$| Y_{i^*}(x) - Y_{j^*}(x) | \leq 0 \cdot e^{K|x-0|} + \frac{\varepsilon_{i^*} + \varepsilon_{j^*}}{K} (e^{K|x-0|} - 1)$$

$$\leq \frac{\varepsilon}{C} \left( \frac{|e^K - 1|}{K} \right) = \varepsilon , \quad x \in [0, 1].$$

Thus, the sequence $\{Y_i(x)\}$ is a Cauchy sequence and hence converges uniformly on $[0, 1]$. Therefore there exists a continuous limit function $Y(x)$ on this interval such that $Y_i(x) \to Y(x)$ as $i \to \infty$ uniformly on $[0, 1]$. Recalling the fact that each member of $\{Y_i\}$ is at least an $\varepsilon$-approximate solution of (1), it is clear that $Y(x)$ is a solution of (1) satisfying $Y(0) = 0$, since it is an $\varepsilon$-approximate solution of (1) for all $\varepsilon > 0$. Applying the uniqueness of solutions of (1), it is necessary that $Y(x) = y(x)$.

Thus we have shown the existence of a sequence of polynomials, $\{Y_i(x)\}$, such that $Y_i(0) = 0$ and each $Y_i(x)$ is a minimizing polynomial of (4). Furthermore, this sequence converges uniformly to the solution of (1) as $i$ increases without bound. We state these results in the following theorem.

**Theorem 3.** If $y(x)$ is the unique solution of (1) having the value $y(0) = 0$, then $y(x)$ may be uniformly approximated on $[0, 1]$ by a sequence of polynomials $\{P^n_i(x)\}$ which are $\varepsilon$-approximate solutions of (3.1) which satisfy $P^n_0(0) = 0$ and which minimize condition (4).

**References**


**Colorado State University**

Mr. Huffstutler is now at West Texas State University

Received June 6, 1967. This work was partially supported by AFOSR Grants 185-65 and 67.
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.
<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>John Suemper Alin and Spencer Ernest Dickson</td>
<td>Goldie’s torsion theory and its derived functor</td>
<td>195</td>
</tr>
<tr>
<td>Steve Armentrout, Lloyd Lesley Lininger and</td>
<td>Equivalent decomposition of $R^3$</td>
<td>205</td>
</tr>
<tr>
<td>Donald Vern Meyer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>James Harvey Carruth</td>
<td>A note on partially ordered compacta</td>
<td>229</td>
</tr>
<tr>
<td>Charles E. Clark and Carl Eberhart</td>
<td>A characterization of compact connected planar lattices</td>
<td>233</td>
</tr>
<tr>
<td>Allan Clark and Larry Smith</td>
<td>The rational homotopy of a wedge</td>
<td>241</td>
</tr>
<tr>
<td>Donald Brooks Coleman</td>
<td>Semigroup algebras that are group algebras</td>
<td>247</td>
</tr>
<tr>
<td>John Eric Gilbert</td>
<td>Convolution operators on $L^p(G)$ and properties of locally compact</td>
<td>257</td>
</tr>
<tr>
<td>groups</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fletcher Gross</td>
<td>Groups admitting a fixed-point-free automorphism of order $2^n$</td>
<td>269</td>
</tr>
<tr>
<td>Jack Hardy and Howard E. Lacey</td>
<td>Extensions of regular Borel measures</td>
<td>277</td>
</tr>
<tr>
<td>R. G. Huffstutler and Frederick Max Stein</td>
<td>The approximation solution of $y' = F(x, y)$</td>
<td>283</td>
</tr>
<tr>
<td>Michael Joseph Kascic, Jr.</td>
<td>Polynomials in linear relations</td>
<td>291</td>
</tr>
<tr>
<td>Alan G. Konheim and Benjamin Weiss</td>
<td>A note on functions which operate</td>
<td>297</td>
</tr>
<tr>
<td>Warren Simms Loud</td>
<td>Self-adjoint multi-point boundary value problems</td>
<td>303</td>
</tr>
<tr>
<td>Kenneth Derwood Magill, Jr.</td>
<td>Topological spaces determined by left ideals of semigroups</td>
<td>319</td>
</tr>
<tr>
<td>Morris Marden</td>
<td>On the derivative of canonical products</td>
<td>331</td>
</tr>
<tr>
<td>J. L. Nelson</td>
<td>A stability theorem for a third order nonlinear differential equation</td>
<td>341</td>
</tr>
<tr>
<td>Raymond Moos Redheffer</td>
<td>Functions with real poles and zeros</td>
<td>345</td>
</tr>
<tr>
<td>Donald Zane Spicer</td>
<td>Group algebras of vector-valued functions</td>
<td>379</td>
</tr>
<tr>
<td>Myles Tierney</td>
<td>Some applications of a property of the functor $Ef$</td>
<td>401</td>
</tr>
</tbody>
</table>