

# Pacific Journal of Mathematics

**A NOTE ON FUNCTIONS WHICH OPERATE**

ALAN G. KONHEIM AND BENJAMIN WEISS

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Let  $\mathfrak{A}, \mathfrak{B}$  denote two families of functions  $a, b: X \rightarrow Y$ . A function  $F: Z \subseteq Y \rightarrow Y$  is said to operate in  $(\mathfrak{A}, \mathfrak{B})$  provided that for each  $a \in \mathfrak{A}$  with range  $(a) \subseteq Z$  we have  $F(a) \in \mathfrak{B}$ . Let  $G$  denote a locally compact Abelian group. In this paper we characterize the functions which operate in two cases:

(i)  $\mathfrak{A} = \Phi_r(G) =$  positive definite functions on  $G$  with  $\phi(e) = r$  and  $\mathfrak{B} = \Phi_{i.d.,s}(G) =$  infinitely divisible positive definite functions on  $G$  with  $\phi(e) = s$ .

(ii)  $\mathfrak{A} = \mathfrak{B} = \tilde{\Phi}_1(G) = \text{Log } \Phi_{i.d.,1}(G)$ .

The determination of the class of functions that operate in  $(\mathfrak{A}, \mathfrak{B})$  for other special families may be found in references [3]-[8]. Our goal here is to extend the results of [5, 6] and, at the same time, to obtain a new derivation of the results recently announced in [3].

$G$  will denote a locally compact Abelian group and  $B^+(G)$  the family of continuous, complex-valued, nonnegative-definite functions on  $G$ . Let

$$\begin{aligned} \Phi_r(G) &= \{\phi : \phi \in B^+(G) \text{ and } \phi(e) = r\}^1 \\ \Phi_{i.d.,r}(G) &= \{\phi : \phi \in \Phi_r(G) \text{ and } (\phi)^{1/n} \in B^+(G) \text{ for } n \geq 1\} \\ \tilde{\Phi}_r(G) &= \text{Log } \Phi_{i.d.,r}(G) = \{\log \phi : \phi \in \Phi_{i.d.,r}(G)\} . \end{aligned}$$

In the case where  $G$  is the real line  $\Phi_1(G)$  is the class of characteristic functions,  $\Phi_{i.d.,1}(G)$  the class of characteristic functions corresponding to the infinitely divisible distributions while  $\tilde{\Phi}_1(G)$  is the class of logarithms of this latter class whose form is well known since Levy and Khintchine.

**THEOREM 1.** *If  $G$  has elements of arbitrarily high order then  $F$  operates on  $(\Phi_r(G), \Phi_{i.d.,s}(G))$  if and only if*

$$F(z) = s \exp c(f(z/r) - 1) \quad (|z| \leq r)$$

where  $c \geq 0$  and

$$f(z) = \sum_{n,m=0}^{\infty} a_{n,m} z^n z^m \quad (|z| \leq 1)$$

with

---

<sup>1</sup> We denote the identity element of  $G$  by  $e$ .

$$a_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} a_{n,m} = 1 .$$

LEMMA 1. *Let*

$$h(s, t) = \sum_{n,m=0}^{\infty} b_{n,m} s^n t^m \quad (|s|, |t| \leq 1)$$

with

$$b_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} b_{n,m} = 1 .$$

Suppose that for each integer  $k, k \geq 1$  we have

$$(h(s, t))^{1/k} = \sum_{n,m=0}^{\infty} b_{n,m}(k) s^n t^m \quad (|s|, |t| \leq 1)$$

with

$$b_{n,m}(k) \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} b_{n,m}(k) = 1 .$$

Then

$$h(s, t) = \exp c(g(s, t) - 1) \quad (|s|, |t| \leq 1)$$

where

$$g(s, t) = \sum_{n,m=0}^{\infty} g_{n,m} s^n t^m \quad (|s|, |t| \leq 1)$$

with

$$c \geq 0 \quad g_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} g_{n,m} = 1 .$$

*Proof of Lemma 1.* Since  $(h(s, t))^{1/k}$  is to be a generating function with nonnegative coefficients we must have  $h(0, 0) = b_{0,0} > 0$ . For suitable  $\varepsilon > 0$  we then have

$$0 < 1 - h(s, t) < 1 \quad (0 \leq s, t \leq \varepsilon) .$$

Thus  $k(s, t) = \log \{1 - (1 - h(s, t))\}$  admits an expansion

$$k(s, t) = \sum_{n,m=0}^{\infty} k_{n,m} s^n t^m \quad (0 \leq s, t \leq \varepsilon) .$$

Clearly  $k_{0,0} < 0$ ; we want to prove that all of the remaining coefficients  $k_{n,m}$  are nonnegative. Assume on the contrary that

$$\{(n, m) : (n, m) \neq (0, 0) \quad \text{and} \quad k_{n,m} < 0\} \neq \phi .$$

Let  $(n_0, m_0)$  be a minimal element in this set (under the usual partial

ordering in the plane). We then write

$$k(s, t) = k_{0,0} + \sum_{\substack{0 \leq n \leq n_0 \\ 0 \leq m \leq m_0 \\ (n, m) \neq (0, 0), (n_0, m_0)}} k_{n,m} s^n t^m + k_{n_0, m_0} s^{n_0} t^{m_0} + r_{n_0, m_0}(s, t).$$

It is easily seen that the

coefficient of  $s^{n_0} t^{m_0}$  in  $\exp \frac{1}{N} k(s, t) =$

$$\text{coefficient of } s^{n_0} t^{m_0} \text{ in } \exp \frac{1}{N} \left\{ k_{0,0} + \sum_{\substack{0 \leq n \leq n_0 \\ 0 \leq m \leq m_0 \\ (n, m) \neq (0, 0), (n_0, m_0)}} k_{n,m} s^n t^m + k_{n_0, m_0} s^{n_0} t^{m_0} \right\}.$$

But this coefficient is of the form

$$\left\{ \frac{1}{N} k_{n_0, m_0} + \frac{1}{N^2} \sigma \left( \frac{1}{N} \right) \right\} \exp \frac{1}{N} k_{0,0}$$

where  $\sigma$  is a polynomial. For  $N$  sufficiently large this coefficient has the sign of  $k_{n_0, m_0}$  which provides a contradiction. Thus  $k_{0,0} < 0$  and  $k_{n,m} \geq 0$  ( $(n, m) \neq (0, 0)$ ).

*Proof of Theorem 1.* By setting  $\tilde{F}(z) = (1/s)F(rz)$  we may assume that  $r = s = 1$ . If  $F$  operates in  $(\Phi_1(G), \Phi_{i.d.,1}(G))$  then  $(F)^{1/k}$  operates in  $\Phi_1(G)$  for each integer  $k, k \geq 1$ . Thus from [5]

$$(F(z))^{1/k} = \sum_{n, m=0}^{\infty} a_{n,m}(k) z^n \bar{z}^m (|z| \leq 1)$$

with

$$a_{n,m}(k) \geq 0 \quad \text{and} \quad \sum_{n, m=0}^{\infty} a_{n,m}(k) = 1.$$

By virtue of Lemma 1 the proof is complete.

**LEMMA 2.** *If  $G$  has elements of arbitrarily high order then  $F$  operates in  $\tilde{\Phi}_1(G)$  implies that for any  $r, 0 < r < \infty$*

$$F(z) = c(r) \left\{ \sum_{n, m=0}^{\infty} a_{n,m}(r) (r+z)^n (r+\bar{z})^m - 1 \right\}$$

whenever  $|z+r| \leq r$  where  $c(r) \geq 0, a_{n,m}(r) \geq 0$  and

$$\sum_{n, m=0}^{\infty} a_{n,m}(r) r^{n+m} = 1.$$

*Proof.* We begin by observing that

$$\Phi_r(G) - r = \{ \phi - r : \phi \in \Phi_r(G) \} \subseteq \tilde{\Phi}_1(G).$$

Thus if  $F_r(z) = F(z - r)$  then  $\exp F_r$  operates in  $(\Phi_r(G), \Phi_{i.d.,1}(G))$  which proves the lemma by Theorem 1.

**THEOREM 2 [3].** *If  $G$  has elements of arbitrarily high order then  $F$  operates in  $\tilde{\Phi}_1(G)$  if and only if*

$$F(z) = -\alpha + \beta z + \gamma \bar{z} + \int_0^\infty \int_0^\infty \{\exp(sz + t\bar{z}) - 1\} \mu(ds, dt) \quad (*)$$

$$\operatorname{Re} z \leq 0$$

where

- (i)  $\alpha, \beta$  and  $\gamma$  are real and nonnegative,
- (ii)  $\mu$  is a positive measure on  $\{(s, t): 0 \leq s < \infty, 0 \leq t < \infty\}$  which is bounded (except perhaps at the origin) and for which

$$\int_0^\infty \int_0^\infty \frac{t + s}{1 + t + s} \mu(ds, dt) < \infty .$$

*Proof.* Since it is clear that functions of the form (\*) operate on  $\tilde{\Phi}_1(G)$  it suffices to prove the reverse implication. We begin by noting that if  $0 < r < \rho$  then

$$c(r) \left\{ \sum_{n,m=0}^\infty a_{n,m}(r)(r+z)^n(r+w)^m - 1 \right\}$$

$$= c(\rho) \left\{ \sum_{n,m=0}^\infty a_{n,m}(\rho)(\rho+z)^n(\rho+w)^m - 1 \right\}$$

whenever  $|z+r| \leq r$  and  $|w+r| \leq r$ , where  $F$  admits the expansion

$$F(z) = c(\rho) \left\{ \sum_{n,m=0}^\infty a_{n,m}(\rho)(\rho+z)^n(\rho+\bar{z})^m - 1 \right\}$$

$$|\rho+z| \leq \rho .$$

We now may uniquely define a function  $\Psi(z, w)$  in  $0 \leq z < \infty, 0 \leq w < \infty$  by

$$\Psi(z, w) = c(r) \left\{ 1 - \sum_{n,m=0}^\infty a_{n,m}(r)(r-z)^n(r-w)^m \right\}$$

provided  $0 \leq w \leq r$  and  $0 \leq z \leq r$ . We note that

$$\frac{(-1)^{j+k-1} \partial^{j+k}}{\partial^j z \partial^k w} \Psi(z, w) \geq 0$$

$$0 \leq w < \infty \quad 0 \leq z < \infty$$

$$j, k \geq 0 \quad j+k > 0 .$$

It follows from a theorem of Bochner [2, p. 89] that

$$\Psi(z, w) = \alpha + \beta z + \gamma w + \int_0^\infty \int_0^\infty [1 - \exp - (sz + tw)] \mu(ds, dt)$$

where  $\alpha, \beta, \gamma$  and  $\mu$  have the desired properties.

We proceed now to give the connection between Theorem 2 and the results announced in [3].

DEFINITION. A continuous complex-valued function defined on a locally compact Abelian group  $G$  is said to *negative definite* if

$$\sum_{j=1}^n \sum_{i=1}^n \{f(x_i) + \overline{f(x_j)} - f(x_i x_j^{-1})\} a_i \bar{a}_j \geq 0$$

for any complex numbers  $\{a_i\}$ , any  $\{x_i\} \subseteq G$  and for  $n = 1, 2, \dots$ . The class of such functions is denoted by  $N(G)$ . It was already noticed by Beurling and Deny [1] that  $N(G) = -\tilde{\Phi}_1(G)$ .<sup>2</sup> We include a brief proof for the reader's convenience.

LEMMA 3. A continuous, complex-valued, function  $f$  on  $G$  is negative definitely if and only if  $\exp(-f)$  is the Fourier transform of an infinitely divisible distribution on  $\hat{G}$ .

*Proof.* (Necessity) By Bochner's theorem it suffices to show that  $\exp(-(1/n)f)$  is a positive definite function on  $G$  for  $n = 1, 2, \dots$ . Since  $(1/n)f$  is a negative definite function it suffices to check that  $\exp(-f)$  is positive definite. Now

$$\begin{aligned} & \sum_{j=1}^n \sum_{i=1}^n \exp(-f(x_i x_j^{-1})) a_i \bar{a}_j \\ &= \sum_{j=1}^n \sum_{i=1}^n \exp\{f(x_i) + \overline{f(x_j)} - f(x_i x_j^{-1})\} \\ & \quad \cdot (a_i \exp(-f(x_i))) (\bar{a}_j \exp(-\overline{f(x_j)})) . \end{aligned}$$

But the matrix

$$\exp(f(x_i) + f(x_j) - f(x_i x_j^{-1}))$$

is the limit of positive linear combinations of "element-wise" products of positive definite matrices. Since such products are again positive definite by Schur's theorem [9] we see that  $\exp(-f)$  is indeed positive definite.

(Sufficiency) By DeFinetti's theorem and the fact that  $N(G)$  is closed under pointwise limits it suffices to show that  $1 - \phi \in N(G)$  for  $\phi \in \Phi_1(G)$ . We must therefore show

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<sup>2</sup> Professor C. S. Herz has kindly pointed out that this result was actually first given by I. J. Schoenberg [9], albeit in a different context.

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^n \{1 - \phi(x_i) + 1 - \phi(x_j) - 1 + \phi(x_i x_j^{-1})\} a_i \bar{a}_j \\ &= \sum_{i=1}^n \sum_{j=1}^n \phi(x_i x_j^{-1}) a_i \bar{a}_j + \left| \sum_{i=1}^n a_i \right|^2 - 2 \operatorname{Re} \sum_{i=1}^n a_i \sum_{j=1}^n \overline{a_j \phi(x_j)} \geq 0. \end{aligned} \quad (**)$$

To prove (\*\*) we first set  $\phi(x) = \chi(x)$  where  $\chi$  is a character of  $G$  noting that (\*\*) becomes

$$\left| \sum_{i=1}^n a_i \chi(x_i) \right|^2 + \left| \sum_{i=1}^n a_i \right|^2 - 2 \operatorname{Re} \sum_{i=1}^n a_i \sum_{i=1}^n \overline{a_i \chi(x_i)} \geq 0.$$

For general  $\phi$  we need only observe that by Bochner's theorem  $\phi$  is in the closure of the convex hull spanned by the characters of  $G$ .

It is now clear that  $F$  operates on  $N(G)$  if and only if  $\tilde{F}$ , defined by  $\tilde{F}(z) = -F(-z)$ , operates on  $\tilde{\mathcal{D}}_1(G)$ . Making this transformation Theorem 2 becomes identical with the main theorem of [3].

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