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A NOTE ON FUNCTIONS WHICH OPERATE

ALAN G. KONHEIM AND BENJAMIN WEISS

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Let $\mathfrak{A}, \mathfrak{B}$ denote two families of functions $a, b: X \rightarrow Y$. A function $F: Z \subseteq Y \rightarrow Y$ is said to operate in $(\mathfrak{A}, \mathfrak{B})$ provided that for each $a \in \mathfrak{A}$ with range $(a) \subseteq Z$ we have $F(a) \in \mathfrak{B}$. Let G denote a locally compact Abelian group. In this paper we characterize the functions which operate in two cases:

(i) $\mathfrak{A} = \Phi_r(G)$ = positive definite functions on G with $\phi(e) = r$ and $\mathfrak{B} = \Phi_{i.d.,s}(G)$ = infinitely divisible positive definite functions on G with $\phi(e) = s$.

(ii) $\mathfrak{A} = \mathfrak{B} = \tilde{\Phi}_1(G) = \text{Log } \Phi_{i.d.,1}(G)$.

The determination of the class of functions that operate in $(\mathfrak{A}, \mathfrak{B})$ for other special families may be found in references [3]-[8]. Our goal here is to extend the results of [5, 6] and, at the same time, to obtain a new derivation of the results recently announced in [3].

G will denote a locally compact Abelian group and $B^+(G)$ the family of continuous, complex-valued, nonnegative-definite functions on G . Let

$$\begin{aligned} \Phi_r(G) &= \{\phi : \phi \in B^+(G) \text{ and } \phi(e) = r\}^1 \\ \Phi_{i.d.,r}(G) &= \{\phi : \phi \in \Phi_r(G) \text{ and } (\phi)^{1/n} \in B^+(G) \text{ for } n \geq 1\} \\ \tilde{\Phi}_r(G) &= \text{Log } \Phi_{i.d.,r}(G) = \{\log \phi : \phi \in \Phi_{i.d.,r}(G)\}. \end{aligned}$$

In the case where G is the real line $\Phi_1(G)$ is the class of characteristic functions, $\Phi_{i.d.,1}(G)$ the class of characteristic functions corresponding to the infinitely divisible distributions while $\tilde{\Phi}_1(G)$ is the class of logarithms of this latter class whose form is well known since Levy and Khintchine.

THEOREM 1. *If G has elements of arbitrarily high order then F operates on $(\Phi_r(G), \Phi_{i.d.,s}(G))$ if and only if*

$$F(z) = s \exp c(f(z/r) - 1) \quad (|z| \leq r)$$

where $c \geq 0$ and

$$f(z) = \sum_{n,m=0}^{\infty} a_{n,m} z^n \bar{z}^m \quad (|z| \leq 1)$$

with

¹ We denote the identity element of G by e .

$$a_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} a_{n,m} = 1 .$$

LEMMA 1. *Let*

$$h(s, t) = \sum_{n,m=0}^{\infty} b_{n,m} s^n t^m \quad (|s|, |t| \leq 1)$$

with

$$b_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} b_{n,m} = 1 .$$

Suppose that for each integer k , $k \geq 1$ we have

$$(h(s, t))^{1/k} = \sum_{n,m=0}^{\infty} b_{n,m}(k) s^n t^m \quad (|s|, |t| \leq 1)$$

with

$$b_{n,m}(k) \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} b_{n,m}(k) = 1 .$$

Then

$$h(s, t) = \exp c(g(s, t) - 1) \quad (|s|, |t| \leq 1)$$

where

$$g(s, t) = \sum_{n,m=0}^{\infty} g_{n,m} s^n t^m \quad (|s|, |t| \leq 1)$$

with

$$c \geq 0 \quad g_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} g_{n,m} = 1 .$$

Proof of Lemma 1. Since $(h(s, t))^{1/k}$ is to be a generating function with nonnegative coefficients we must have $h(0, 0) = b_{0,0} > 0$. For suitable $\varepsilon > 0$ we then have

$$0 < 1 - h(s, t) < 1 \quad (0 \leq s, t \leq \varepsilon) .$$

Thus $k(s, t) = \log \{1 - (1 - h(s, t))\}$ admits an expansion

$$k(s, t) = \sum_{n,m=0}^{\infty} k_{n,m} s^n t^m \quad (0 \leq s, t \leq \varepsilon) .$$

Clearly $k_{0,0} < 0$; we want to prove that all of the remaining coefficients $k_{n,m}$ are nonnegative. Assume on the contrary that

$$\{(n, m) : (n, m) \neq (0, 0) \quad \text{and} \quad k_{n,m} < 0\} \neq \emptyset .$$

Let (n_0, m_0) be a minimal element in this set (under the usual partial

ordering in the plane). We then write

$$k(s, t) = k_{0,0} + \sum_{\substack{0 \leq n \leq n_0 \\ 0 \leq m \leq m_0 \\ (n,m) \neq (0,0), (n_0, m_0)}} k_{n,m} s^n t^m + k_{n_0, m_0} s^{n_0} t^{m_0} + r_{n_0, m_0}(s, t).$$

It is easily seen that the

coefficient of $s^{n_0} t^{m_0}$ in $\exp \frac{1}{N} k(s, t) =$

$$\text{coefficient of } s^{n_0} t^{m_0} \text{ in } \exp \frac{1}{N} \left\{ k_{0,0} + \sum_{\substack{0 \leq n \leq n_0 \\ 0 \leq m \leq m_0 \\ (n,m) \neq (0,0), (n_0, m_0)}} k_{n,m} s^n t^m + k_{n_0, m_0} s^{n_0} t^{m_0} \right\}.$$

But this coefficient is of the form

$$\left\{ \frac{1}{N} k_{n_0, m_0} + \frac{1}{N^2} \sigma \left(\frac{1}{N} \right) \right\} \exp \frac{1}{N} k_{0,0}$$

where σ is a polynomial. For N sufficiently large this coefficient has the sign of k_{n_0, m_0} which provides a contradiction. Thus $k_{0,0} < 0$ and $k_{n,m} \geq 0 ((n, m) \neq (0, 0))$.

Proof of Theorem 1. By setting $\tilde{F}(z) = (1/s)F(rz)$ we may assume that $r = s = 1$. If F operates in $(\tilde{\Phi}_1(G), \tilde{\Phi}_{i.d.,i}(G))$ then $(F')^{1/k}$ operates in $\tilde{\Phi}_1(G)$ for each integer $k, k \geq 1$. Thus from [5]

$$(F(z))^{1/k} = \sum_{n,m=0}^{\infty} a_{n,m}(k) z^n \bar{z}^m (|z| \leq 1)$$

with

$$a_{n,m}(k) \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} a_{n,m}(k) = 1.$$

By virtue of Lemma 1 the proof is complete.

LEMMA 2. *If G has elements of arbitrarily high order then F operates in $\tilde{\Phi}_1(G)$ implies that for any $r, 0 < r < \infty$*

$$F(z) = c(r) \left\{ \sum_{n,m=0}^{\infty} a_{n,m}(r) (r+z)^n (r+\bar{z})^m - 1 \right\}$$

whenever $|z+r| \leq r$ where $c(r) \geq 0, a_{n,m}(r) \geq 0$ and

$$\sum_{n,m=0}^{\infty} a_{n,m}(r) r^{n+m} = 1.$$

Proof. We begin by observing that

$$\tilde{\Phi}_r(G) - r = \{ \phi - r : \phi \in \tilde{\Phi}_r(G) \} \subseteq \tilde{\Phi}_1(G).$$

Thus if $F_r(z) = F(z - r)$ then $\exp F_r$ operates in $(\mathcal{O}_r(G), \mathcal{O}_{i.d.,1}(G))$ which proves the lemma by Theorem 1.

THEOREM 2 [3]. *If G has elements of arbitrarily high order then F operates in $\tilde{\mathcal{O}}_1(G)$ if and only if*

$$F(z) = -\alpha + \beta z + \gamma \bar{z} + \int_0^\infty \int_0^\infty \{\exp(sz + t\bar{z}) - 1\} \mu(ds, dt) \quad (*)$$

$$\operatorname{Re} z \leq 0$$

where

- (i) α, β and γ are real and nonnegative,
- (ii) μ is a positive measure on $\{(s, t): 0 \leq s < \infty, 0 \leq t < \infty\}$ which is bounded (except perhaps at the origin) and for which

$$\int_0^\infty \int_0^\infty \frac{t + s}{1 + t + s} \mu(ds, dt) < \infty .$$

Proof. Since it is clear that functions of the form (*) operate on $\tilde{\mathcal{O}}_1(G)$ it suffices to prove the reverse implication. We begin by noting that if $0 < r < \rho$ then

$$c(r) \left\{ \sum_{n,m=0}^\infty a_{n,m}(r)(r + z)^n (r + w)^m - 1 \right\}$$

$$= c(\rho) \left\{ \sum_{n,m=0}^\infty a_{n,m}(\rho)(\rho + z)^n (\rho + w)^m - 1 \right\}$$

whenever $|z + r| \leq r$ and $|w + r| \leq r$, where F admits the expansion

$$F(z) = c(\rho) \left\{ \sum_{n,m=0}^\infty a_{n,m}(\rho)(\rho + z)^n (\rho + \bar{z})^m - 1 \right\}$$

$$|\rho + z| \leq \rho .$$

We now may uniquely define a function $\Psi(z, w)$ in $0 \leq z < \infty, 0 \leq w < \infty$ by

$$\Psi(z, w) = c(r) \left\{ 1 - \sum_{n,m=0}^\infty a_{n,m}(r)(r - z)^n (r - w)^m \right\}$$

provided $0 \leq w \leq r$ and $0 \leq z \leq r$. We note that

$$\frac{(-1)^{j+k-1} \partial^{j+k}}{\partial^j z \partial^k w} \Psi(z, w) \geq 0$$

$$0 \leq w < \infty \quad 0 \leq z < \infty$$

$$j, k \geq 0 \quad j + k > 0 .$$

It follows from a theorem of Bochner [2, p. 89] that

$$\Psi(z, w) = \alpha + \beta z + \gamma w + \int_0^\infty \int_0^\infty [1 - \exp - (sz + tw)] \mu(ds, dt)$$

where α, β, γ and μ have the desired properties.

We proceed now to give the connection between Theorem 2 and the results announced in [3].

DEFINITION. A continuous complex-valued function defined on a locally compact Abelian group G is said to *negative definite* if

$$\sum_{j=1}^n \sum_{i=1}^n \{f(x_i) + \overline{f(x_j)} - f(x_i x_j^{-1})\} a_i \bar{a}_j \geq 0$$

for any complex numbers $\{a_i\}$, any $\{x_i\} \subseteq G$ and for $n = 1, 2, \dots$. The class of such functions is denoted by $N(G)$. It was already noticed by Beurling and Deny [1] that $N(G) = -\tilde{\Phi}_1(G)$.² We include a brief proof for the reader's convenience.

LEMMA 3. *A continuous, complex-valued, function f on G is negative definitely if and only if $\exp(-f)$ is the Fourier transform of an infinitely divisible distribution on \hat{G} .*

Proof. (Necessity) By Bochner's theorem it suffices to show that $\exp(-(1/n)f)$ is a positive definite function on G for $n = 1, 2, \dots$. Since $(1/n)f$ is a negative definite function it suffices to check that $\exp(-f)$ is positive definite. Now

$$\begin{aligned} & \sum_{j=1}^n \sum_{i=1}^n \exp(-f(x_i x_j^{-1})) a_i \bar{a}_j \\ &= \sum_{j=1}^n \sum_{i=1}^n \exp\{f(x_i) + \overline{f(x_j)} - f(x_i x_j^{-1})\} \\ & \quad \cdot (a_i \exp(-f(x_i))) (\overline{a_j \exp(-f(x_j))}) . \end{aligned}$$

But the matrix

$$\exp(f(x_i) + f(x_j) - f(x_i x_j^{-1}))$$

is the limit of positive linear combinations of "element-wise" products of positive definite matrices. Since such products are again positive definite by Schur's theorem [9] we see that $\exp(-f)$ is indeed positive definite.

(*Sufficiency*) By DeFinetti's theorem and the fact that $N(G)$ is closed under pointwise limits it suffices to show that $1 - \phi \in N(G)$ for $\phi \in \Phi_1(G)$. We must therefore show

² Professor C. S. Herz has kindly pointed out that this result was actually first given by I. J. Schoenberg [9], albeit in a different context.

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^n \{1 - \phi(x_i) + 1 - \phi(x_j) - 1 + \phi(x_i x_j^{-1})\} a_i \bar{a}_j \\ & = \sum_{i=1}^n \sum_{j=1}^n \phi(x_i x_j^{-1}) a_i \bar{a}_j + \left| \sum_{i=1}^n a_i \right|^2 - 2 \operatorname{Re} \sum_{i=1}^n a_i \sum_{j=1}^n \overline{a_j \phi(x_j)} \geq 0. \end{aligned} \quad (**)$$

To prove (**) we first set $\phi(x) = \chi(x)$ where χ is a character of G noting that (**) becomes

$$\left| \sum_{i=1}^n a_i \chi(x_i) \right|^2 + \left| \sum_{i=1}^n a_i \right|^2 - 2 \operatorname{Re} \sum_{i=1}^n a_i \sum_{i=1}^n \overline{a_i \chi(x_i)} \geq 0.$$

For general ϕ we need only observe that by Bochner's theorem ϕ is in the closure of the convex hull spanned by the characters of G .

It is now clear that F operates on $N(G)$ if and only if \tilde{F} , defined by $\tilde{F}(z) = -F(-z)$, operates on $\tilde{\Phi}_1(G)$. Making this transformation Theorem 2 becomes identical with the main theorem of [3].

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