A NOTE ON FUNCTIONS WHICH OPERATE

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Let $\mathcal{A}$, $\mathcal{B}$ denote two families of functions $a, b: X \to Y$. A function $F: Z \subseteq Y \to Y$ is said to operate in $(\mathcal{A}, \mathcal{B})$ provided that for each $a \in \mathcal{A}$ with range $(a) \subseteq Z$ we have $F(a) \in \mathcal{B}$. Let $G$ denote a locally compact Abelian group. In this paper we characterize the functions which operate in two cases:

(i) $\mathcal{A} = \Phi_r(G) =$ positive definite functions on $G$ with $\phi(e) = r$ and $\mathcal{B} = \Phi_{i.d.}(G) =$ infinitely divisible positive definite functions on $G$ with $\phi(e) = s$.

(ii) $\mathcal{A} = \mathcal{B} = \tilde{\Phi}_r(G) =$ Log $\Phi_{i.d.}(G)$.

The determination of the class of functions that operate in $(\mathcal{A}, \mathcal{B})$ for other special families may be found in references [3]–[8]. Our goal here is to extend the results of [5, 6] and, at the same time, to obtain a new derivation of the results recently announced in [3].

$G$ will denote a locally compact Abelian group and $B^+(G)$ the family of continuous, complex-valued, nonnegative-definite functions on $G$. Let

$$
\Phi_r(G) = \{\phi : \phi \in B^+(G) \text{ and } \phi(e) = r\}^1
$$

$$
\Phi_{i.d.}(G) = \{\phi : \phi \in \Phi_r(G) \text{ and } (\phi)_{1/n} \in B^+(G) \text{ for } n \geq 1\}
$$

$$
\tilde{\Phi}_r(G) = \text{Log } \Phi_{i.d.}(G) = \{\text{Log } \phi : \phi \in \Phi_{i.d.}(G)\}.
$$

In the case where $G$ is the real line $\Phi_r(G)$ is the class of characteristic functions, $\Phi_{i.d.}(G)$ the class of characteristic functions corresponding to the infinitely divisible distributions while $\tilde{\Phi}_r(G)$ is the class of logarithms of this latter class whose form is well known since Levy and Khintchine.

**Theorem 1.** If $G$ has elements of arbitrarily high order then $F$ operates on $(\Phi_r(G), \Phi_{i.d.}(G))$ if and only if

$$
F(z) = s \exp c(f(z/r) - 1) \quad (|z| \leq r)
$$

where $c \geq 0$ and

$$
f(z) = \sum_{n, m=0}^{\infty} a_{n, m} z^n z^m \quad (|z| \leq 1)
$$

with

1 We denote the identity element of $G$ by $e$. 297
\[ a_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} a_{n,m} = 1. \]

**Lemma 1.** Let
\[ h(s, t) = \sum_{n,m=0}^{\infty} b_{n,m}s^n t^m \quad (|s|, |t| \leq 1) \]
with
\[ b_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} b_{n,m} = 1. \]

Suppose that for each integer \( k, k \geq 1 \) we have
\[ (h(s, t))^{1/k} = \sum_{n,m=0}^{\infty} b_{n,m}(k)s^n t^m \quad (|s|, |t| \leq 1) \]
with
\[ b_{n,m}(k) \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} b_{n,m}(k) = 1. \]

Then
\[ h(s, t) = \exp c(g(s, t) - 1)) \quad (|s|, |t| \leq 1) \]
where
\[ g(s, t) = \sum_{n,m=0}^{\infty} g_{n,m}s^n t^m \quad (|s|, |t| \leq 1) \]
with
\[ c \geq 0 \quad g_{n,m} \geq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} g_{n,m} = 1. \]

**Proof of Lemma 1.** Since \((h(s, t))^{1/k}\) is to be a generating function with nonnegative coefficients we must have \(h(0, 0) = b_{0,0} > 0\). For suitable \(\varepsilon > 0\) we then have
\[ 0 < 1 - h(s, t) < 1 \quad (0 \leq s, t \leq \varepsilon). \]
Thus \(k(s, t) = \log \{1 - (1 - h(s, t))\}\) admits an expansion
\[ k(s, t) = \sum_{n,m=0}^{\infty} k_{n,m}s^n t^m \quad (0 \leq s, t \leq \varepsilon). \]

Clearly \(k_{0,0} < 0\); we want to prove that all of the remaining coefficients \(k_{n,m}\) are nonnegative. Assume on the contrary that
\[ \{(n, m) : (n, m) \neq (0, 0) \quad \text{and} \quad k_{n,m} < 0\} \neq \emptyset. \]
Let \((n_0, m_0)\) be a minimal element in this set (under the usual partial
ordering in the plane). We then write

\[ k(s, t) = k_{0,0} + \sum_{0 \leq n \leq n_0, 0 \leq m \leq m_0} k_{n,m} s^n t^m + k_{n_0,m_0} s^n t^{m_0} + r_{n_0,m_0}(s, t). \]

It is easily seen that the coefficient of \( s^n t^{m_0} \) in \( \exp \frac{1}{N} k(s, t) \) equals

\[ \frac{1}{N} k_{n_0,m_0} + \sum_{0 \leq n \leq n_0, 0 \leq m \leq m_0} k_{n,m} s^n t^m + k_{n_0,m_0} s^n t^{m_0}. \]

But this coefficient is of the form

\[ \left\{ \frac{1}{N} k_{n_0,m_0} + \frac{1}{N^2} \sigma \left( \frac{1}{N} \right) \right\} \exp \frac{1}{N} k_{0,0}. \]

where \( \sigma \) is a polynomial. For \( N \) sufficiently large this coefficient has the sign of \( k_{n_0,m_0} \) which provides a contradiction. Thus \( k_{0,0} < 0 \) and \( k_{n,m} \geq 0 \) \( ((n, m) \neq (0, 0)) \).

**Proof of Theorem 1.** By setting \( \tilde{F}(z) = (1/s)F(rz) \) we may assume that \( r = s = 1 \). If \( F \) operates in \( (\Phi_1(G), \Phi_{t,d,1}(G)) \) then \( (F)^{1/k} \) operates in \( \Phi_1(G) \) for each integer \( k, k \geq 1 \). Thus from [5]

\[ (F(z))^{1/k} = \sum_{n, m = 0}^{\infty} a_{n,m}(k)z^n \bar{z}^m (|z| \leq 1) \]

with

\[ a_{n,m}(k) \geq 0 \quad \text{and} \quad \sum_{n, m = 0}^{\infty} a_{n,m}(k) = 1. \]

By virtue of Lemma 1 the proof is complete.

**Lemma 2.** If \( G \) has elements of arbitrarily high order then \( F \) operates in \( \bar{\Phi}_1(G) \) implies that for any \( r, 0 < r < \infty \)

\[ F(z) = c(r) \left\{ \sum_{n, m = 0}^{\infty} a_{n,m}(r)(r + z)^n(r + \bar{z})^m - 1 \right\} \]

whenever \( |z + r| \leq r \) where \( c(r) \geq 0, a_{n,m}(r) \geq 0 \) and

\[ \sum_{n, m = 0}^{\infty} a_{n,m}(r)r^n m^m = 1. \]

**Proof.** We begin by observing that

\[ \Phi_*,(G) - r = \{ \phi - r : \phi \in \Phi_*(G) \} \subseteq \bar{\Phi}_1(G). \]
Thus if $F_r(z) = F(z - r)$ then $\exp F_r$ operates in $\left(\Phi_r(G), \Phi_{i,d}(G)\right)$ which proves the lemma by Theorem 1.

**Theorem 2** [3]. If $G$ has elements of arbitrarily high order then $F$ operates in $\tilde{\Phi}_1(G)$ if and only if

$$F(z) = -\alpha + \beta z + \gamma \bar{z} + \int_0^\infty \int_0^\infty \exp(sz + t\bar{z}) - 1 \mu(ds, dt) \quad (*)$$

Re $z \leq 0$

where

(i) $\alpha, \beta$ and $\gamma$ are real and nonnegative,

(ii) $\mu$ is a positive measure on $\{(s, t): 0 \leq s < \infty, 0 \leq t < \infty\}$ which is bounded (except perhaps at the origin) and for which

$$\int_0^\infty \int_0^\infty \frac{t + s}{1 + t + s} \mu(ds, dt) < \infty .$$

**Proof.** Since it is clear that functions of the form (*) operate on $\tilde{\Phi}_1(G)$ it suffices to prove the reverse implication. We begin by noting that if $0 < r < \rho$ then

$$c(r) \left\{ \sum_{n,m=0}^{\infty} a_{n,m}(r)(r + z)^n(r + w)^m - 1 \right\}$$

$$= c(\rho) \left\{ \sum_{n,m=0}^{\infty} a_{n,m}(\rho)(\rho + z)^n(\rho + w)^m - 1 \right\}$$

whenever $|z + r| \leq r$ and $|w + r| \leq r$, where $F$ admits the expansion

$$F(z) = c(\rho) \left\{ \sum_{n,m=0}^{\infty} a_{n,m}(\rho)(\rho + z)^n(\rho + \bar{z})^m - 1 \right\}$$

$|\rho + z| \leq \rho .$

We now may uniquely define a function $\Psi(z, w)$ in $0 \leq z < \infty$, $0 \leq w < \infty$ by

$$\Psi(z, w) = c(r) \left\{ 1 - \sum_{n,m=0}^{\infty} a_{n,m}(r)(r - z)^n(r - w)^m \right\}$$

provided $0 \leq w \leq r$ and $0 \leq z \leq r$. We note that

$$\frac{(-1)^{j+k-1}z^j\bar{z}^k}{\partial^j z \partial^k \bar{w}} \Psi(z, w) \geq 0$$

$0 \leq w < \infty \quad 0 \leq z < \infty$

$$j, k \geq 0 \quad j + k > 0 .$$

It follows from a theorem of Bochner [2, p. 89] that
\( \Psi(z, w) = \alpha + \beta z + \gamma w + \int_{0}^{\infty} \int_{0}^{\infty} [1 - \exp - (sz + tw)] \mu(ds, dt) \)

where \( \alpha, \beta, \gamma \) and \( \mu \) have the desired properties.

We proceed now to give the connection between Theorem 2 and the results announced in [3].

**Definition.** A continuous complex-valued function defined on a locally compact Abelian group \( G \) is said to **negative definite** if

\[
\sum_{j=1}^{n} \sum_{i=1}^{n} \{f(x_i) + \bar{f}(x_j) - f(x_i x_j^{-1})\} a_i \bar{a}_j \geq 0
\]

for any complex numbers \( \{a_i\} \), any \( \{x_i\} \subseteq G \) and for \( n = 1, 2, \ldots \). The class of such functions is denoted by \( N(G) \). It was already noticed by Beurling and Deny [1] that \( N(G) = \Phi(G) \). We include a brief proof for the reader's convenience.

**Lemma 3.** A continuous, complex-valued, function \( f \) on \( G \) is negative definitely if and only if \( \exp(-f) \) is the Fourier transform of an infinitely divisible distribution on \( G \).

**Proof.** (*Necessity*) By Bochner's theorem it suffices to show that \( \exp(- (1/n)f) \) is a positive definite function on \( G \) for \( n = 1, 2, \ldots \).

Since \( (1/n)f \) is a negative definite function it suffices to check that \( \exp(-f) \) is positive definite. Now

\[
\sum_{j=1}^{n} \sum_{i=1}^{n} \exp(-f(x_i x_j^{-1})) a_i \bar{a}_j
\]

\[
= \sum_{j=1}^{n} \sum_{i=1}^{n} \exp\{f(x_i) + \bar{f}(x_j) - f(x_i x_j^{-1})\}
\cdot (a_i \exp(-f(x_i))(a_j \exp(-f(x_j)))
\]

But the matrix

\[
\exp(f(x_i) + f(x_j) - f(x_i x_j^{-1}))
\]

is the limit of positive linear combinations of "element-wise" products of positive definite matrices. Since such products are again positive definite by Schur's theorem [9] we see that \( \exp(-f) \) is indeed positive definite.

(*Sufficiency*) By DeFinetti's theorem and the fact that \( N(G) \) is closed under pointwise limits it suffices to show that \( 1 - \phi \in N(G) \) for \( \phi \in \Phi(G) \). We must therefore show

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Professor C. S. Herz has kindly pointed out that this result was actually first given by I. J. Schoenberg [9], albeit in a different context.
To prove (**) we first set $\phi(x) = \chi(x)$ where $\chi$ is a character of $G$ noting that (**) becomes

$$\left| \sum_{i=1}^{n} a_i \chi(x_i) \right|^2 + \left| \sum_{i=1}^{n} a_i \right|^2 - 2 \Re \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \phi(x_j) \geq 0 .$$

For general $\phi$ we need only observe that by Bochner’s theorem $\phi$ is in the closure of the convex hull spanned by the characters of $G$.

It is now clear that $F$ operates on $N(G)$ if and only if $\hat{F}^3$, defined by $\hat{F}^3(z) = -F(-z)$, operates on $\Phi_1(G)$. Making this transformation Theorem 2 becomes identical with the main theorem of [3].

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