Pacific Journal of Mathematics

A NOTE ON FUNCTIONS WHICH OPERATE

Alan G. Konheim and Benjamin Weiss

Vol. 24, No. 2 June 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 2, 1968

A NOTE ON FUNCTIONS WHICH OPERATE

ALAN G. KONHEIM AND BENJAMIN WEISS

Let $\mathfrak{A}$, $\mathscr{B}$ denote two families of functions $a, b: X \to Y$. A function $F: Z \subseteq Y \to Y$ is said to operate in $(\mathfrak{A}, \mathscr{B})$ provided that for each $a \in \mathfrak{A}$ with range $(a) \subseteq Z$ we have $F(a) \in \mathscr{B}$. Let G denote a locally compact Abelian group. In this paper we characterize the functions which operate in two cases:

(i) $\mathfrak{A} = \Phi_r(G) =$ positive definite functions on G with $\phi(e) = r$ and $\mathscr{B} = \Phi_{i.d.,s}(G) =$ infinitely divisible positive definite functions on G with $\phi(e) = s$.

(ii) $\mathfrak{A} = \mathscr{B} = \widetilde{\Phi}_1(G) = \operatorname{Log} \Phi_{i.d.,1}(G)$.

The determination of the class of functions that operate in $(\mathfrak{A}, \mathscr{B})$ for other special families may be found in refernces [3]-[8]. Our goal here is to extend the results of [5, 6] and, at the same time, to obtain a new derivation of the results recently announced in [3].

G will denote a locally compact Abelian group and $B^+(G)$ the family of continuous, complex-valued, nonnegative-definite functions on G. Let

$$\begin{aligned} \Phi_r(G) &= \{\phi : \phi \in B^+(G) \text{ and } \phi(e) = r\}^1 \\ \Phi_{i.d.,r}(G) &= \{\phi : \phi \in \Phi_r(G) \text{ and } (\phi)^{1/n} \in B^+(G) \text{ for } n \geqq 1\} \\ \widetilde{\Phi}_r(G) &= \operatorname{Log} \Phi_{i.d.,r}(G) = \{\log \phi : \phi \in \Phi_{i.d.,r}(G)\} . \end{aligned}$$

In the case where G is the real line $\Phi_1(G)$ is the class of characteristic functions, $\Phi_{i.d.,1}(G)$ the class of characteristic functions corresponding to the infinitely devisible distributions while $\widetilde{\Phi}_1(G)$ is the class of logarithms of this latter class whose form is well known since Levy and Khintchine.

THEOREM 1. *If G has elements of arbitrarily high order then F operates on $(\Phi_r(G), \Phi_{i.d.,s}(G))$ if and only if*

$$F(z) = s \exp c(f(z/r) - 1) \qquad (|z| \leqq r)$$

where $c \geqq 0$ and

$$f(z) = \sum_{n,m=0}^{\infty} a_{n,m} z^n z^m \qquad (|z| \leqq 1)$$

with

[1] We denote the identity element of G by e.

$$a_{n,m} \geqq 0 \quad and \quad \sum_{n,m=0}^{\infty} a_{n,m} = 1 .$$

LEMMA 1. *Let*

$$h(s, t) = \sum_{n,m=0}^{\infty} b_{n,m} s^n t^m \qquad (|s|, |t| \leqq 1)$$

with

$$b_{n,m} \geqq 0 \quad and \quad \sum_{n,m=0}^{\infty} b_{n,m} = 1 .$$

Suppose that for each integer k, $k \geqq 1$ we have

$$(h(s, t))^{1/k} = \sum_{n,m=0}^{\infty} b_{n,m}(k) s^n t^m \qquad (|s|, |t| \leqq 1)$$

with

$$b_{n,m}(k) \geqq 0 \quad and \quad \sum_{n,m=0}^{\infty} b_{n,m}(k) = 1 .$$

Then

$$h(s, t) = \exp c(g(s, t) - 1)) \qquad (|s|, |t| \leqq 1)$$

where

$$g(s, t) = \sum_{n,m=0}^{\infty} g_{n,m} s^n t^m \qquad (|s|, |t| \leqq 1)$$

with

$$c \geqq 0 \;\; g_{n,m} \geqq 0 \quad and \quad \sum_{n,m=0}^{\infty} g_{n,m} = 1 .$$

Proof of Lemma 1. Since $(h(s, t))^{1/k}$ is to be a generating function with nonnegative coefficients we must have $h(0, 0) = b_{0,0} > 0$. For suitable $\varepsilon > 0$ we then have

$$0 < 1 - h(s, t) < 1 \qquad (0 \leqq s, t \leqq \varepsilon) .$$

Thus $k(s, t) = \log\{1 - (1 - h(s, t))\}$ admits an expansion

$$k(s, t) = \sum_{n,m=0}^{\infty} k_{n,m} s^n t^m \qquad (0 \leqq s, t \leqq \varepsilon) .$$

Clearly $k_{0,0} < 0$; we want to prove that all of the remaining coefficients $k_{n,m}$ are nonnegative. Assume on the contrary that

$$\{(n, m) : (n, m) \neq (0, 0) \quad \text{and} \quad k_{n,m} < 0\} \neq \phi .$$

Let (n_0, m_0) be a minimal element in this set (under the usual partial ordering in the plane). We then write

$$k(s, t) = k_{0,0} + \sum_{\substack{0 \leqq n \leqq n_0 \\ 0 \leqq m \leqq m_0 \\ (n,m) \neq (0,0),(n_0,m_0)}} k_{n,m} s^n t^m + k_{n_0,m_0} s^{n_0} t^{m_0} + r_{n_0,m_0}(s, t) .$$

It is easily seen that the

coefficient of $s^{n_0}t^{m_0}$ in $\exp \dfrac{1}{N} k(s, t) =$

$$\text{coefficient of } s^{n_0}t^{m_0} \text{ in } \exp \frac{1}{N}\left\{ k_{0,0} + \sum_{\substack{0 \leqq n \leqq n_0 \\ 0 \leqq m \leqq m_0 \\ (n,m) \neq (0,0),(n_0,m_0)}} k_{n,m} s^n t^m + k_{n_0,m_0} s^{n_0} t^{m_0} \right\} .$$

But this coefficient is of the form

$$\left\{ \frac{1}{N} k_{n_0,m_0} + \frac{1}{N^2} \sigma\left(\frac{1}{N}\right) \right\} \exp \frac{1}{N} k_{0,0}$$

where σ is a polynomial. For N sufficiently large this coefficient has the sign of k_{n_0,m_0} which provides a contradiction. Thus $k_{0,0} < 0$ and $k_{n,m} \geqq 0$ $((n, m) \neq (0, 0))$.

Proof of Theorem 1. By setting $\tilde{F}(z) = (1/s)F(rz)$ we may assume that $r = s = 1$. If F operates in $(\Phi_1(G), \Phi_{i.d.,1}(G))$ then $(F)^{1/k}$ operates in $\Phi_1(G)$ for each integer k, $k \geqq 1$. Thus from [5]

$$(F(z))^{1/k} = \sum_{n,m=0}^{\infty} a_{n,m}(k) z^n \overline{z^m} (|z| \leqq 1)$$

with

$$a_{n,m}(k) \geqq 0 \quad \text{and} \quad \sum_{n,m=0}^{\infty} a_{n,m}(k) = 1 .$$

By virtue of Lemma 1 the proof is complete.

LEMMA 2. *If G has elements of arbitrarily high order then F operates in $\tilde{\Phi}_1(G)$ implies that for any r, $0 < r < \infty$*

$$F(z) = c(r)\left\{ \sum_{n,m=0}^{\infty} a_{n,m}(r)(r + z)^n (r + \bar{z})^m - 1 \right\}$$

whenever $|z + r| \leqq r$ where $c(r) \geqq 0$, $a_{n,m}(r) \geqq 0$ and

$$\sum_{n,m=0}^{\infty} a_{n,m}(r) r^{n+m} = 1 .$$

Proof. We begin by observing that

$$\Phi_r(G) - r = \{\phi - r : \phi \in \Phi_r(G)\} \subseteqq \tilde{\Phi}_1(G) .$$

Thus if $F_r(z) = F(z - r)$ then $\exp F_r$ operates in $(\Phi_r(G), \Phi_{i.d.,1}(G))$ which proves the lemma by Theorem 1.

THEOREM 2 [3]. *If G has elements of arbitrarily high order then F operates in $\tilde{\Phi}_1(G)$ if and only if*

$$F(z) = -\alpha + \beta z + \gamma \bar{z} + \int_0^\infty \int_0^\infty \{\exp(sz + t\bar{z}) - 1\} \mu(ds, dt) \qquad (*)$$
$$\operatorname{Re} z \leqq 0$$

where

(i) *α, β and γ are real and nonnegative,*

(ii) *μ is a positive measure on $\{(s, t): 0 \leqq s < \infty, 0 \leqq t < \infty\}$ which is bounded (except perhaps at the origin) and for which*

$$\int_0^\infty \int_0^\infty \frac{t + s}{1 + t + s} \mu(ds, dt) < \infty .$$

Proof. Since it is clear that functions of the form (*) operate on $\tilde{\Phi}_1(G)$ it suffices to prove the reverse implication. We begin by noting that if $0 < r < \rho$ then

$$c(r)\left\{\sum_{n,m=0}^{\infty} a_{n,m}(r)(r + z)^n (r + w)^m - 1\right\}$$
$$= c(\rho)\left\{\sum_{n,m=0}^{\infty} a_{n,m}(\rho)(\rho + z)^n (\rho + w)^m - 1\right\}$$

whenever $|z + r| \leqq r$ and $|w + r| \leqq r$, where F admits the expansion

$$F(z) = c(\rho)\left\{\sum_{n,m=0}^{\infty} a_{n,m}(\rho)(\rho + z)^n (\rho + \bar{z})^m - 1\right\}$$
$$|\rho + z| \leqq \rho .$$

We now may uniquely define a function $\Psi(z, w)$ in $0 \leqq z < \infty$, $0 \leqq w < \infty$ by

$$\Psi(z, w) = c(r)\left\{1 - \sum_{n,m=0}^{\infty} a_{n,m}(r)(r - z)^n (r - w)^m\right\}$$

provided $0 \leqq w \leqq r$ and $0 \leqq z \leqq r$. We note that

$$\frac{(-1)^{j+k-1}\partial^{j+k}}{\partial^j z \partial^k w} \Psi(z, w) \geqq 0$$
$$0 \leqq w < \infty \qquad 0 \leqq z < \infty$$
$$j, k \geqq 0 \qquad j + k > 0 .$$

It follows from a theorem of Bochner [2, p. 89] that

$$\Psi(z, w) = \alpha + \beta z + \gamma w + \int_0^\infty \int_0^\infty [1 - \exp - (sz + tw)] \mu(ds, dt)$$

where α, β, γ and μ have the desired properties.

We proceed now to give the connection between Theorem 2 and the results announced in [3].

DEFINITION. A continuous complex-valued function defined on a locally compact Abelian group G is said to *negative definite* if

$$\sum_{j=1}^{n} \sum_{i=1}^{n} \{f(x_i) + \overline{f(x_j)} - f(x_i x_j^{-1})\} a_i \bar{a}_j \geqq 0$$

for any complex numbers $\{a_i\}$, any $\{x_i\} \subseteqq G$ and for $n = 1, 2, \cdots$. The class of such functions is denoted by $N(G)$. It was already noticed by Beurling and Deny [1] that $N(G) = -\tilde{\Phi}_1(G)$.[2] We include a brief proof for the reader's convenience.

LEMMA 3. *A continuous, complex-valued, function f on G is negative definitely if and only if* $\exp(-f)$ *is the Fourier transform of an infinitely divisible distribution on* $\hat{G}$.

Proof. (*Necessity*) By Bochner's theorem it suffices to show that $\exp(-(1/n)f)$ is a positive definite function on G for $n = 1, 2, \cdots$. Since $(1/n)f$ is a negative definite function it suffices to check that $\exp(-f)$ is positive definite. Now

$$\begin{aligned}\sum_{j=1}^{n} \sum_{i=1}^{n} &\exp(-f(x_i x_j^{-1})) a_i \bar{a}_j \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} \exp\{f(x_i) + \overline{f(x_j)} - f(x_i x_j^{-1})\} \\ &\qquad \cdot (a_i \exp(-f(x_i))) \overline{(a_j \exp(-f(x_j)))} .\end{aligned}$$

But the matrix

$$\exp(f(x_i) + f(x_j) - f(x_i x_j^{-1}))$$

is the limit of positive linear combinations of "element-wise" products of positive definite matrices. Since such products are again positive definite by Schur's theorem [9] we see that $\exp(-f)$ is indeed positive definite.

(*Sufficiency*) By DeFinetti's theorem and the fact that $N(G)$ is closed under pointwise limits it suffices to show that $1 - \phi \in N(G)$ for $\phi \in \Phi_1(G)$. We must therefore show

[2] Professor C. S. Herz has kindly pointed out that this result was actually first given by I. J. Schoenberg [9], albeit in a different context.

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\{1-\phi(x_i)+1-\phi(x_j)-1+\phi(x_ix_j^{-1})\}a_i\bar{a}_j\\ &\quad=\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(x_ix_j^{-1})a_i\overline{a_j}+\left|\sum_{i=1}^{n}a_i\right|^2-2\operatorname{Re}\sum_{i=1}^{n}a_i\sum_{j=1}^{n}\overline{a_j\phi(x_j)}\geqq 0\ .\end{aligned}\tag{**}$$

To prove (**) we first set $\phi(x)=\chi(x)$ where χ is a character of G noting that (**) becomes

$$\left|\sum_{i=1}^{n}a_i\chi(x_i)\right|^2+\left|\sum_{i=1}^{n}a_i\right|^2-2\operatorname{Re}\sum_{i=1}^{n}a_i\sum_{i=1}^{n}\overline{a_i\chi(x_i)}\geqq 0\ .$$

For general ϕ we need only observe that by Bochner's theorem ϕ is in the closure of the convex hull spanned by the characters of G.

It is now clear that F operates on $N(G)$ if and only if $\widetilde{F}$, defined by $\widetilde{F}(z)=-F(-z)$, operates on $\widetilde{\Phi}_1(G)$. Making this transformation Theorem 2 becomes identical with the main theorem of [3].

REFERENCES

1. A. Beurling and J. Deny, *Dirichlet spaces*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 208-215.

2. S. Bochner, *Harmonic analysis and the theory of probability*, University of California Press, 1960.

3. K. Harzallah, *Fonctions cpérant sur les founctions définies négatives à valeurs complexes*, C. R. Acad. Sc. **262** (1966), 824-826.

4. H. Helson, J. P. Kahane, Y. Katznelson and W. Rudin, *The functions which operate on Fourier transforms*, Acta Math. **102** (1959), 135-157.

5. C. S. Herz, *Fonctions opérant sur les fonctions définies-positives*, Ann. Inst. Fourier **3** (1963), 161-180.

6. A. G. Konheim and B. Weiss, *Functions which operate on characteristic functions*, Pacific. J. Math. **15** (1965), 1279-1293.

7. W. Rudin, *Positive definite sequences and absolutely monotonic functions*, Duke Math. J. **26** (1959), 617-622.

8. I. J. Schoenberg, *Positive definite functions on spheres*, Duke Math. J. **9** (1942), 96-108.

9. ———, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. **44** (1938), 522-536.

10. I. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unedlichvielen Veranderlichen*, J. für Math. **140** (1911), 1-28.

Received July 12, 1966. The research of A. C. Konheim was supported by the United States Air Force under Contract No. AF 49(638)-1682.

IMB WATSON RESEARCH CENTER
YORKTOWN HEIGHTS, NEW YORK

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 24, No. 2 June, 1968