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**A STABILITY THEOREM FOR A THIRD ORDER NONLINEAR
DIFFERENTIAL EQUATION**

J. L. NELSON

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A stability theorem and a corollary are proved for a nonlinear nonautonomous third order differential equation. A remark shows that the results do not hold for the linear case.

THEOREM. Let $p'(t)$ and $q(t)$ be continuous and $q(t) \geq 0$, $p(t) < 0$ with $p'(t) \geq 0$. For any A and B suppose

$$A + Bt - \int_{t_1}^t q(s)ds < 0$$

for large t where $Q(t) = \int_{t_0}^t q(s)ds$, then any nonoscillatory solution $x(t)$ of the equation

$$\ddot{x} = p(t)\dot{x} + q(t)x^{2n+1} = 0, n = 1, 2, 3, \dots,$$

has the following properties;

$$\begin{aligned} \operatorname{sgn} x &= \operatorname{sgn} \ddot{x}, \neq \operatorname{sgn} \dot{x}, \lim_{t \rightarrow \infty} \ddot{x}(t) \\ &= \lim_{t \rightarrow \infty} \dot{x}(t) = 0, \lim_{t \rightarrow \infty} |x(t)| = L \geq 0, \end{aligned}$$

and $x(t)\dot{x}(t), \ddot{x}(t)$ are monotone functions.

COROLLARY. If $q(t) > \epsilon > 0$ for large t , then $\lim_{t \rightarrow \infty} x(t) = 0$.

In this paper, a nonoscillatory solution $x(t)$ of a differential equation is one that is continuable for large t and for which there exists a t_0 such that if $t > t_0$ then $x(t) \neq 0$. Under above conditions on $p(t)$ and $q(t)$ there always exist continuable nonoscillatory solutions of the equation

$$(1) \quad \ddot{x} + p(t)\dot{x} + q(t)x^{2n+1} = 0.$$

This follows from an exercise in [1] by letting

$$x(t) = y_1(t), \dot{x}(t) = -y_2(t), \ddot{x}(t) = y_3(t),$$

so that

$$\begin{aligned} \dot{y}_1 &= -y_2 \\ \dot{y}_2 &= -y_3 \\ \dot{y}_3 &= -[q(t)y_1^{2n+1} - p(t)y_2]. \end{aligned}$$

Equation (1) can then be written as the system $\bar{y}' = -f(t, \bar{y})$ where $f(t, \bar{y}) = \bar{y}$, $f(t, \bar{y})$ continuous for $t \geq 0$, $y_1, y_2, y_3, \geq 0$ and $f_k(t, \bar{y}) \geq 0$, $k = 1, 2, 3$, for $y_k > 0$. In fact $\|\bar{y}(0)\|$ may be prescribed.

THEOREM 1.¹ *If p and q satisfy the following conditions for large t ,*

- (i) $q(t) \geq 0$ and q continuous,
 - (ii) $p(t) < 0$ with $p'(t) \geq 0$ and continuous,
 - (iii) for any A and B , $A + Bt - \int_{t_0}^t Q(s)ds < 0$ for large t where $Q(t) = \int_{t_0}^t q(s)ds$,
- then for any nonoscillatory solution $x(t)$ of (1) the following properties hold for large t :

- (a) $\operatorname{sgn} x = \operatorname{sgn} \ddot{x} \neq \operatorname{sgn} \dot{x}$, where $\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$.
- (b) $\lim_{t \rightarrow \infty} \ddot{x}(t) = \lim_{t \rightarrow \infty} \dot{x}(t) = 0$, $\lim_{t \rightarrow \infty} |x(t)| = L \geq 0$.
- (c) $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ are monotone functions.

Proof. Suppose $x(t)$ is a solution that does not oscillate. Let a be a large positive number such that $x(t) \neq 0$ for $t \geq a$.

Since $-x(t)$ is also a solution of (1), without loss of generality, assume that $x(t) > 0$ for $t \geq a$. (1) may be written in the form

$$(2) \quad \frac{\ddot{x}(t)}{x^{2n+1}(t)} + \frac{p(t)\dot{x}(t)}{x^{2n+1}(t)} = -q(t) \text{ for } t \geq a.$$

An integration from a to t , an integration by parts, and another integration from a to t yield

$$(3) \quad \begin{aligned} & \frac{\dot{x}(t)}{x^{2n+1}(t)} + \frac{2n+1}{2} \int_a^t \frac{(\dot{x}(s))^2}{x^{2n+2}(s)} ds \\ & + (2n+1)(n+1) \int_a^t \frac{(t-s)(\dot{x}(s))^3}{x^{2n+3}(s)} ds \\ & - \frac{1}{2n} \int_a^t \frac{p(s)}{x^{2n}(s)} ds + \frac{1}{2n} \int_a^t \frac{(t-s)p'(s)}{x^{2n}(s)} ds \\ & = M + Kt - \int_a^t Q(s)ds. \end{aligned}$$

Assertion 1. For any $t_a > a$, $\dot{x}(t)$ cannot be nonnegative for all $t > t_a$. Suppose that $\dot{x}(t) \geq 0$ for all $t > t_a$. Let t_p be so large that the conditions of the theorem hold for all $t \geq t_p$ and $t_p \geq t_a$. For $t \geq t_p$ the following holds

$$(4) \quad \begin{aligned} & \frac{\dot{x}(t)}{x^{2n+1}(t)} + (2n+1)(n+1) \int_{t_p}^t \frac{(t-s)(\dot{x}(s))^3}{x^{2n+3}(s)} ds - \frac{1}{2n} \int_{t_p}^t \frac{p(s)}{x^{2n}(s)} ds \\ & + \frac{1}{2n} \int_{t_p}^t \frac{(t-s)p'(s)}{x^{2n}(s)} ds \leq \bar{M} + Kt - \int_a^t Q(s)ds, \end{aligned}$$

¹ This theorem appears in the author's Ph. D. dissertation written at the University of Missouri under the direction of W. R. Utz.

where all constants are combined and named \bar{M} . For sufficiently large t the right side, $\bar{M} + \bar{K}t - \int_0^t Q(s)ds$, is negative and the left side positive, this is clearly impossible.

There are two possibilities for $\dot{x}(t)$.

Case 1. $\dot{x}(t) < 0$ for $t > \bar{t}$, for some \bar{t} .

Case 2. For each $t \in (a, \infty)$ there is a $\bar{t} > t$ such that $\dot{x}(\bar{t}) \geq 0$.

Assertion 2. Case 2 is impossible.

Let t_1 be a large t such that $\dot{x}(t_1) \geq 0$. There exists a number $t_2 > t_1$ such that $\dot{x}(t_2) < 0$. Let r be the greatest zero of $\dot{x}(t)$ less than t_2 . There exists a number $t_3 > t_2$ such that $\dot{x}(t_3) \geq 0$. Let s be the smallest zero of $\dot{x}(t)$ greater than t_2 . Multiply the original differential Equation (1) by $\dot{x}(t)$ to obtain

$$\ddot{x}(t)\dot{x}(t) + p(t)[\dot{x}(t)]^2 + q(t)x^{2n+1}(t) = 0 ,$$

integrating from r to s and using integration by parts on the first integral gives

$$- \int_r^s [\ddot{x}(t)]^2 dt + \int_r^s p(t)[\dot{x}(t)]^2 dt + \int_r^s q(t)x^{2n+1}(t)\dot{x}(t) dt = 0 .$$

The left side is negative, this is clearly impossible and Assertion 2 is proved. Therefore, there exists a \bar{t} such that $\dot{x}(t) < 0$ for $t > \bar{t}$.

Consider Equation (1) written in the form

$$\ddot{x}(t) = -p(t)\dot{x}(t) - q(t)x^{2n+1}(t) ,$$

the right side is negative for large t . Therefore, $\ddot{x}(t) < 0$ for $t > \bar{t}$. This implies that $\ddot{x}(t)$ is a decreasing function and $\dot{x}(t)$ is concave downward for $t > \bar{t}$. Since $\ddot{x}(t)$ is eventually of one sign, there are three possibilities for $\dot{x}(t)$.

Case 1. $\lim_{t \rightarrow \infty} \dot{x}(t) = -\infty$

Case 2. $\lim_{t \rightarrow \infty} \dot{x}(t) = c < 0$

Case 3. $\lim_{t \rightarrow \infty} \dot{x}(t) = 0$.

Case 1 is impossible since it implies that $x(t)$ is negative for large t . Case 2 also implies that $x(t)$ is negative for large t . Therefore, the only remaining possibility is

$$\lim_{t \rightarrow \infty} \dot{x}(t) = 0 .$$

Since $\ddot{x}(t)$ is decreasing and must remain positive for large t , $\dot{x}(t)$ is eventually monotone increasing. Since $\ddot{x}(t)$ is monotone decreasing and positive, $\lim_{t \rightarrow \infty} (\ddot{x})t$ exists. Suppose that $\lim_{t \rightarrow \infty} \ddot{x}(t) = c > 0$. Then $x(t)$ eventually has slope larger than $c/2$, this is impossible since $\dot{x}(t) < 0$ for large t . Therefore, $\lim_{t \rightarrow \infty} \ddot{x}(t) = 0$. Thus $x(t)$ is positive,

decreasing and concave upward for large t .

COROLLARY. *If $q(t) > \varepsilon > 0$ for large t , then $\lim_{t \rightarrow \infty} x(t) = 0$.*

Proof. Suppose $\lim_{t \rightarrow \infty} x(t) = L, L \neq 0$. Since $-x(t)$ is a solution whenever $x(t)$ is a solution, it can be assumed without loss of generality that $L > 0$. Consider Equation (1) in the form

$$\ddot{x}(t) = -p(t)\dot{x}(t) - q(t)x^{2n+1}(t).$$

Since $\lim_{t \rightarrow \infty} \dot{x}(t) = 0$ and $\lim_{t \rightarrow \infty} p(t) = p$, where $p \leq 0$, given any α such that

$$0 < \frac{\alpha}{2} < L^{2n+1}, \text{ for large } t$$

$L^{2n+1} - \alpha/2 < x^{2n+1}(t) < L^{2n+1} + \alpha/2$ and $p(t)\dot{x}(t) > 0$. Therefore, $\ddot{x}(t) = -p(t)\dot{x}(t) - q(t)x^{2n+1}(t) < -\varepsilon(L^{2n+1} - \alpha/2) < 0$ and $\ddot{x}(t)$ must then tend to $-\infty$ as t tends to $+\infty$, this is impossible. This $L = 0$.

REMARK. The following example illustrates the theorem.

$$\ddot{x} - \frac{1}{2}\dot{x} + \frac{e^{2t}}{2}x^3 = 0.$$

$x = e^{-t}$ is a solution with the required properties.

REMARK. The theorem does not hold for $n = 0$, i. e., in the linear case.

Proof. Consider $\ddot{x} - 2\dot{x} + x = 0, x = e^t$ is a solution.

REFERENCE

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