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Lawrence Peter Belluce and Surender Kumar Jain

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# PRIME RINGS WITH A ONE-SIDED IDEAL SATISFYING A POLYNOMIAL IDENTITY

L. P. BELLUCE AND S. K. JAIN

**It is known that the existence of a nonzero commutative one-sided ideal in a prime ring implies that the whole ring is commutative. Since rings satisfying a polynomial identity are natural generalizations of commutative rings the question arises as to what extent the above mentioned result can be extended to include these generalizations. That is, if $R$ is a prime ring and $I$ a nonzero one-sided ideal which satisfies a polynomial identity does $R$ satisfy a polynomial identity?**

**This paper initiates an investigation of this problem. A counter example, given later, will show that the answer to the above question may be negative, even when $R$ is a simple primitive ring with nonzero socle. The main theorem of this paper is Theorem 3 which states:**

**Let $R$ be a prime ring having a nonzero right ideal which satisfies a polynomial identity. Then, a necessary and sufficient condition that $R$ satisfy a polynomial identity is that $R$ have zero right singular ideal and $\hat{R}$, the right quotient ring of $R$, have at most finitely many orthogonal idempotents.**

2. In the following given a ring $R$, $R^{\Delta}({}^{\Delta}R)$ denotes the right (left) *singular ideal* of $R$. Thus $R^{\Delta} = \{x \mid x \in R, x^r \in L^{\Delta}(R)\}$ where $L^{\Delta}(R)$ denotes the set of right ideals of $R$ that meet, in a nonzero fashion, all right ideals of $R$. Similarly for ${}^{\Delta}R$ and ${}^{\Delta}L(R)$.

If $Q$ is a ring such that $R$ is a subring of $Q$ and $qR \cap R \neq 0$ for each $q \in Q$ then $Q$ is called a right quotient ring for $R$. Moreover if $Q = \{ab^{-1} \mid a, b \in R, b \text{ regular}\}$ then $Q$ is called a classical right quotient ring. Following [2] we say that a ring $R$ is right quotient simple if and only if it has a classical right quotient ring $Q$ with $Q \cong D_n$, $D_n$ a ring of $n \times n$ matrices over a division ring $D$.

From [4] we know that if $R$ is a prime ring with $R^{\Delta} = 0$ then $R$ has a unique maximal right quotient ring $\hat{R}$ where $\hat{R}$ is a prime regular ring. Moreover, letting $L(R)$ denote the lattice of right ideals of $R$, there is a mapping $s$: $A \to A^s$ of $L(R)$ which is a closure operation satisfying $0^s = 0$, $(A \cap B)^s = A^s \cap B^s$ and $(x^{-1} A)^s = x^{-1} A^s$. The set $L^s(R)$ of closed ideals of $R$ can be made into a lattice in a natural way and it is shown in [4] that $L^s(R) \cong L^s(\hat{R})$ under the mapping $A \to A \cap R$, $A \in L^s(\hat{R})$. We shall have occasion to use the following realization of $\hat{R}$. Let $E = \bigcup_{A \in L^{\Delta}(R)} \mathrm{Hom}_R(A, R)$. On $E$

define the relation, $\alpha \equiv \beta$ if for some $A \in L^{\Delta}(R)$, $A \subseteqq \mathrm{Dom}\,\alpha \cap \mathrm{Dom}\,\beta$ and $\alpha(x) = \beta(x)$ for each $x \in A$. It is shown in [5] that $\equiv$ is an equivalence relation and that $E/\equiv$ is a ring and in fact is $\hat{R}$.

The above remarks apply similarly to a prime ring $R$ for which ${}^{\Delta}R = 0$.

3. In this section occur the basic results of this paper. We will have occasion to use the result of Posner [8] stating that if $R$ is a prime ring with polynomial identity then $\hat{R}$ is a classical two-sided quotient ring having the same multilinear identities as $R$. That part of Posners argument that shows if $R$ has a polynomal identity then so does $\hat{R}$ is a very complicated argument and we take this opportunity to present a simple alternative argument.

LEMMA 1. *Let $R$ be a prime ring with polynomial identity. Then $\hat{R}$ has a polynomial identity.*

*Proof.* From Posner [8] we know that $R$ has left and right quotient conditions and hence $R$ is right quotient simple, with $\hat{R} \cong D_n$. By a theorem of Faith and Utumi [2] $R$ contains an integral domain $K$ with right quotient ring $\hat{K} \cong \hat{D}$. Since $K$ satisfies a polynomial identity we have by Amitsur [1] that $\hat{K}$ also has a polynomial identity. Thus $D$, and hence $D_n$, is finite dimensional over its center; thus $D_n$, so $\hat{R}$, has a standard identity.

LEMMA 2. *Let $R$ be a prime ring with $R^{\Delta} = 0$, let $A \in L^{\Delta}(R)$ and let $\alpha \in \mathrm{Hom}_R(R, R)$, $R$ considered as a right $R$-module. If $\alpha(A) = 0$ then $\alpha = 0$.*

*Proof.* Let $x \in R$; then we have that $x^{-1}A \in L^{\Delta}(R)$. If $r \in x^{-1}A$ then $xr \in A$ and thus $\alpha(xr) = 0$. Since $\alpha$ is a right $R$-endomorphism, $\alpha(xr) = \alpha(x)\cdot r$: It follows that $\alpha(x)\cdot x^{-1}A = 0$, hence $x^{-1}A \subseteqq \alpha(x)^r$. Thus $\alpha(x)^r \in L^{\Delta}(R)$ and so $\alpha(x) \in R^{\Delta}$. Hence $\alpha(x) = 0$.

The following lemma is trivial in the case $R$ contains a central element. Without a central element the proof is more involved.

LEMMA 3. *Let $R$ be a prime ring with a polynomial identity. Then $\mathrm{Hom}_R(R, R)$ has a polynomial identity, if $R^{\Delta} = 0$.*

*Proof.* From Lemma 1 we know that $\hat{R}$ has a polynomial identity. Consider $\hat{R}$ realized as $\bigcup_{A \in L(R)} \mathrm{Hom}_R(A, R)/\equiv$. For $\alpha \in \mathrm{Hom}_R(R, R)$ let $\bar{\alpha}$ denote the equivalence class in $\hat{R}$ determined by $\alpha$. The mapping $\alpha \rightarrow \bar{\alpha}$ is a homomorphism of $\mathrm{Hom}_R(R, R)$ into $\hat{R}$. If $\bar{\alpha} = \bar{\beta}$ then for some $A \in L^{\Delta}(R)$ $\alpha(x) = \beta(x)$, $x \in A$. Thus $(\alpha - \beta)(A) = 0$. By Lemma 2 we see that $\alpha = \beta$. Thus $\alpha \longrightarrow \bar{\alpha}$ is an injection onto a subring of $\hat{R}$ and so $\mathrm{Hom}_R(R, R)$ has a polynomial identity.

The following theorem provides a sufficient condition on the right ideal $I$ having a polynomial identity to ensure the whole ring has a polynomial identity.

THEOREM 1. *Let $R$ be a prime ring having a right ideal $I \neq 0$, $I$ satisfying a polynomial identity and $I_l = 0$. Then $R$ satisfies a polynomial identity.*

*Proof.* By assumption $I_l$, the left annihilator of $I$, is 0. Hence $I$ is a prime ring itself. Considering $I$ as a left $I$-module we have by the obvious dual of Lemma 3 that $\mathrm{Hom}_I(I, I)$, (the left $I$-endomorphisms), has a polynomial identity. For $x \in R$ the mapping $x \longrightarrow r_x$, right multiplication by $x$, is an anti-isomorphism of $R$ into $\mathrm{Hom}_I(I, I)$. Thus $R$ itself satisfies a polynomial identity.

THEOREM 2. *Let $R$ be a right quotient simple ring, $I \neq 0$ a right ideal of $R$ satisfying a polynomial identity. Then $R$ satisfies a polynomial identity.*

*Proof.* From Goldie [3] we have that $I$ contains a uniform right ideal, thus we may assume $I$ is uniform. Since $R^{\Delta} = 0$ it follows that $\{x \mid x \in I,\ x^r \in L^{\Delta}(R)\} = 0$, hence from [6] we have that $K = \mathrm{Hom}_R(I, I)$ is an integral domain. Moreover it is known ([3]) that $\hat{K} \cong D$, $D$ a division ring, where $\hat{R} \cong D_n$. To complete the proof it suffices to show that $D$ has a polynomial identity; the latter will hold provided $K$ has a polynomial identity. To this end consider the homomorphism $a \longrightarrow l_a$, left multiplication by $a$, of $I$ into $K$. Let $J$ denote the image of this map. $J = 0$ implies $I^2 = 0$ which is impossible; hence $J$ is a nonzero subring of $K$ satisfying a polynomial identity. Let $\alpha \in K$ and let $l_a \in J$. Let $x \in I$. Then $\alpha l_a(x) = \alpha(ax) = \alpha(a)\cdot x = l_{\alpha(a)}(x)$. Thus $\alpha l_a = l_{\alpha(a)} \in J$. Hence $J$ is a left ideal of $K$. Since $K$ is an integral domain we have by an obvious dual to Theorem 1 that $K$ has a polynomial identity.

We now obtain, easily, the following.

THEOREM 3. *Let $R$ be a prime ring having a nonzero right ideal which satisfies a polynomial identity. Then, a necessary and sufficient condition that $\hat{R}$ satisfy a polynomial identity is that $R^{\Delta} = 0$ and $\hat{R}$ have at most a finite number of orthogonal idempotents.*

*Proof.* Necessity is clear. Conversely, then, since $\hat{R}$ is regular with at most finitely many orthogonal idempotents it follows from [7] that $\hat{R}$ has the descending chain condition (d.c.c.) on right ideals. $\hat{R}$ is prime, thus $\hat{R} \cong D_n$ for some division ring $D$. Since $L^s(R) \cong L^s(\hat{R})$ we see that $L^s(R)$ has d.c.c. Thus from [4] we see that $\hat{R}$ is a classical right quotient ring, hence Theorem 2 applies.

The following example (communicated orally to S. K. Jain by A. S. Amitsur) shows that the extension of an identity from a right ideal to the entire ring is not always possible. Let $F$ be a field and let $F_\infty$ be the ring of all infinite matrices of finite rank. Let $a = (A_{ij})$ be a matrix such that $a_{11} \neq 0$ and $a_{ij} = 0$ for $i, j \neq 1$. Let $I = aF_\infty$. Then $I$ satisfies the identity $(xy - yx)^2 = 0$ but $F_\infty$ satisfies no identity at all.

4. REMARKS. In the case that $R$ is primitive with a right ideal $I \neq 0$ having a polynomial identity then it is sufficient to assume that $R$ has at most a finite number of orthogonal idempotents to ensure that $R$ also have a polynomial identity.

There are other conditions one may impose upon $R$ and $I$ besides those given here, e.g. if $R$ has at most finitely many orthogonal idempotents and $I$ is a maximal right ideal or if $R^\Delta = 0$ and $I \in L^\Delta(R)$.

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