PRIME RINGS WITH A ONE-SIDED IDEAL SATISFYING A POLYNOMIAL IDENTITY

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It is known that the existence of a nonzero commutative one-sided ideal in a prime ring implies that the whole ring is commutative. Since rings satisfying a polynomial identity are natural generalizations of commutative rings the question arises as to what extent the above mentioned result can be extended to include these generalizations. That is, if $R$ is a prime ring and $I$ a nonzero one-sided ideal which satisfies a polynomial identity does $R$ satisfy a polynomial identity?

This paper initiates an investigation of this problem. A counter example, given later, will show that the answer to the above question may be negative, even when $R$ is a simple primitive ring with nonzero socle. The main theorem of this paper is Theorem 3 which states:

Let $R$ be a prime ring having a nonzero right ideal which satisfies a polynomial identity. Then, a necessary and sufficient condition that $R$ satisfy a polynomial identity is that $R$ have zero right singular ideal and $\hat{R}$, the right quotient ring of $R$, have at most finitely many orthogonal idempotents.

2. In the following given a ring $R$, $R^t(\Delta R)$ denotes the right (left) singular ideal of $R$. Thus $R^t = \{x \mid x \in R, x^* \in L^t(R)\}$ where $L^t(R)$ denotes the set of right ideals of $R$ that meet, in a nonzero fashion, all right ideals of $R$. Similarly for $\Delta R$ and $J_L(R)$.

If $Q$ is a ring such that $R$ is a subring of $Q$ and $qR \cap R \neq 0$ for each $q \in Q$ then $Q$ is called a right quotient ring for $R$. Moreover if $Q = \{ab^{-1} \mid a, b \in R, b \text{ regular}\}$ then $Q$ is called a classical right quotient ring. Following [2] we say that a ring $R$ is right quotient simple if and only if it has a classical right quotient ring $Q$ with $Q \cong D_n$, $D_n$ a ring of $n \times n$ matrices over a division ring $D$.

From [4] we know that if $R$ is a prime ring with $R^t = 0$ then $R$ has a unique maximal right quotient ring $\hat{R}$ where $\hat{R}$ is a prime regular ring. Moreover, letting $L(R)$ denote the lattice of right ideals of $R$, there is a mapping $s: A \rightarrow A^s$ of $L(R)$ which is a closure operation satisfying $0^s = 0$, $(A \cap B)^s = A^s \cap B^s$ and $(x^{-1} A)^s = x^{-1} A^s$. The set $L^s(R)$ of closed ideals of $R$ can be made into a lattice in a natural way and it is shown in [4] that $L^s(R) \cong L^s(\hat{R})$ under the mapping $A \rightarrow A \cap R$, $A \in L^s(\hat{R})$. We shall have occasion to use the following realization of $\hat{R}$. Let $E = \bigcup_{A \in L^t(R)} \text{Hom}_R(A, R)$. On $E$
define the relation, \( \alpha \equiv \beta \) if for some \( A \in L^\Delta(R) \), \( A \subseteq \text{Dom } \alpha \cap \text{Dom } \beta \) and \( \alpha(x) = \beta(x) \) for each \( x \in A \). It is shown in [5] that \( \equiv \) is an equivalence relation and that \( E/\equiv \) is a ring and in fact is \( \hat{R} \).

The above remarks apply similarly to a prime ring \( R \) for which \( R = 0 \).

3. In this section occur the basic results of this paper. We will have occasion to use the result of Posner [8] stating that if \( R \) is a prime ring with polynomial identity then \( \hat{R} \) is a classical two-sided quotient ring having the same multilinear identities as \( R \). That part of Posner's argument that shows if \( R \) has a polynomial identity then so does \( \hat{R} \) is a very complicated argument and we take this opportunity to present a simple alternative argument.

**Lemma 1.** Let \( R \) be a prime ring with polynomial identity. Then \( \hat{R} \) has a polynomial identity.

**Proof.** From Posner [8] we know that \( R \) has left and right quotient conditions and hence \( R \) is right quotient simple, with \( \hat{R} \cong D_n \).

By a theorem of Faith and Utumi [2] \( R \) contains an integral domain \( K \) with right quotient ring \( \hat{K} \cong \hat{D} \). Since \( K \) satisfies a polynomial identity we have by Amitsur [1] that \( \hat{K} \) also has a polynomial identity. Thus \( D_n \), and hence \( D_n \), is finite dimensional over its center; thus \( D_n \), so \( \hat{R} \), has a standard identity.

**Lemma 2.** Let \( R \) be a prime ring with \( R^t = 0 \), let \( A \in L^\Delta(R) \) and let \( \alpha \in \text{Hom}_R(R, R) \), \( R \) considered as a right \( R \)-module. If \( \alpha(A) = 0 \) then \( \alpha = 0 \).

**Proof.** Let \( x \in R \); then we have that \( x^{-1}A \in L^\Delta(R) \). If \( r \in x^{-1}A \) then \( xr \in A \) and thus \( \alpha(xr) = 0 \). Since \( \alpha \) is a right \( R \)-endomorphism, \( \alpha(xr) = \alpha(x) \cdot r \); it follows that \( \alpha(x) \cdot x^{-1}A = 0 \), hence \( x^{-1}A \subseteq \alpha(x)^t \). Thus \( \alpha(x)^t \in L^\Delta(R) \) and so \( \alpha(x) \in R^t \). Hence \( \alpha(x) = 0 \).

The following lemma is trivial in the case \( R \) contains a central element. Without a central element the proof is more involved.

**Lemma 3.** Let \( R \) be a prime ring with a polynomial identity. Then \( \text{Hom}_R(R, R) \) has a polynomial identity, if \( R^t = 0 \).

**Proof.** From Lemma 1 we know that \( \hat{R} \) has a polynomial identity. Consider \( \hat{R} \) realized as \( \bigcup_{A \in L(R)} \text{Hom}_R(A, R)/\equiv \). For \( \alpha \in \text{Hom}_R(R, R) \) let \( \bar{\alpha} \) denote the equivalence class in \( \hat{R} \) determined by \( \alpha \). The mapping \( \alpha \mapsto \bar{\alpha} \) is a homomorphism of \( \text{Hom}_R(R, R) \) into \( \hat{R} \). If \( \bar{\alpha} = \bar{\beta} \) then for
The following theorem provides a sufficient condition on the right ideal $I$ having a polynomial identity to ensure the whole ring has a polynomial identity.

**Theorem 1.** Let $R$ be a prime ring having a right ideal $I \neq 0$, $I$ satisfying a polynomial identity and $I_1 = 0$. Then $R$ satisfies a polynomial identity.

**Proof.** By assumption $I$, the left annihilator of $I$, is 0. Hence $I$ is a prime ring itself. Considering $I$ as a left $I$-module we have by the obvious dual of Lemma 3 that $\text{Hom}_I(I, I)$, (the left $I$-endomorphisms), has a polynomial identity. For $x \in R$ the mapping $x \rightarrow r_x$, right multiplication by $x$, is an anti-isomorphism of $R$ into $\text{Hom}_I(I, I)$. Thus $R$ itself satisfies a polynomial identity.

**Theorem 2.** Let $R$ be a right quotient simple ring, $I \neq 0$ a right ideal of $R$ satisfying a polynomial identity. Then $R$ satisfies a polynomial identity.

**Proof.** From Goldie [3] we have that $I$ contains a uniform right ideal, thus we may assume $I$ is uniform. Since $R^2 = 0$ it follows that 

$$\{x \mid x \in I, x^r \in L^4(R)\} = 0,$$

hence from [6] we have that $K = \text{Hom}_R(I, I)$ is an integral domain. Moreover it is known ([3]) that $\hat{R} \cong D$, $D$ a division ring, where $\hat{R} \cong D_x$. To complete the proof it suffices to show that $D$ has a polynomial identity; the latter will hold provided $K$ has a polynomial identity. To this end consider the homomorphism $a \rightarrow l_a$, left multiplication by $a$, of $I$ into $K$. Let $J$ denote the image of this map. $J = 0$ implies $I^2 = 0$ which is impossible; hence $J$ is a nonzero subring of $K$ satisfying a polynomial identity. Let $a \in K$ and let $l_a \in J$. Let $x \in I$. Then $al_a(x) = \alpha(ax) = \alpha(a) \cdot x = l_{a(a)}(x)$. Thus $al_a = l_{a(a)} \in J$. Hence $J$ is a left ideal of $K$. Since $K$ is an integral domain we have by an obvious dual to Theorem 1 that $K$ has a polynomial identity.

We now obtain, easily, the following.

**Theorem 3.** Let $R$ be a prime ring having a nonzero right ideal which satisfies a polynomial identity. Then, a necessary and sufficient condition that $\hat{R}$ satisfy a polynomial identity is that $R^2 = 0$ and $\hat{R}$ have at most a finite number of orthogonal idempotents.
Proof. Necessity is clear. Conversely, then, since $\hat{R}$ is regular with at most finitely many orthogonal idempotents, it follows from [7] that $\hat{R}$ has the descending chain condition (d.c.c.) on right ideals, $\hat{R}$ is prime, thus $\hat{R} \cong D_n$ for some division ring $D$. Since $L^*(\hat{R}) \cong L^*(R)$ we see that $L^*(R)$ has d.c.c. Thus from [4] we see that $\hat{R}$ is a classical right quotient ring, hence Theorem 2 applies.

The following example (communicated orally to S. K. Jain by A. S. Amitsur) shows that the extension of an identity from a right ideal to the entire ring is not always possible. Let $F$ be a field and let $F_\infty$ be the ring of all infinite matrices of finite rank. Let $a = (A_{ij})$ be a matrix such that $a_{11} \neq 0$ and $a_{ij} = 0$ for $i, j \neq 1$. Let $I = aF_\infty$. Then $I$ satisfies the identity $(xy - yx)^2 = 0$ but $F_\infty$ satisfies no identity at all.

4. Remarks. In the case that $R$ is primitive with a right ideal $I \neq 0$ having a polynomial identity then it is sufficient to assume that $R$ has at most a finite number of orthogonal idempotents to ensure that $R$ also have a polynomial identity.

There are other conditions one may impose upon $R$ and $I$ besides those given here, e.g. if $R$ has at most finitely many orthogonal idempotents and $I$ is a maximal right ideal or if $R^t = 0$ and $I \in L^t(R)$.

References


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Duane W. Bailey, *On symmetry in certain group algebras* ......................... 413
Lawrence Peter Belluce and Surender Kumar Jain, *Prime rings with a one-sided ideal satisfying a polynomial identity* ................................. 421
L. Carlitz, *A note on certain biorthogonal polynomials* ............................. 425
Charles O. Christenson and Richard Paul Osborne, *Pointlike subsets of a manifold* ................................................................. 431
Russell James Egbert, *Products and quotients of probabilistic metric spaces* ................................................................. 437
Moses Glasner, Richard Emanuel Katz and Mitsuru Nakai, *Bisection into small annuli* ................................................................. 457
Karl Edwin Gustafson, *A note on left multiplication of semigroup generators* ................................................................. 463
I. Martin (Irving) Isaacs and Donald Steven Passman, *A characterization of groups in terms of the degrees of their characters. II* ................................. 467
Howard Wilson Lambert and Richard Benjamin Sher, *Point-like 0-dimensional decompositions of $S^3$* ................................................................. 511
Oscar Tivis Nelson, *Subdirect decompositions of lattices of width two* ...... 519
Ralph Tyrrell Rockafellar, *Integrals which are convex functionals* ............... 525
James McLean Sloss, *Reflection laws of systems of second order elliptic differential equations in two independent variables with constant coefficients* ................................................................. 541
Bui An Ton, *Nonlinear elliptic convolution equations of Wiener-Hopf type in a bounded region* ................................................................. 577
Daniel Eliot Wulbert, *Some complemented function spaces in $C(X)$* ........... 589
Zvi Ziegler, *On the characterization of measures of the cone dual to a generalized convexity cone* ................................................................. 603