

Pacific Journal of Mathematics

A NOTE ON CERTAIN BIORTHOGONAL POLYNOMIALS

L. CARLITZ

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Konhauser has introduced two polynomial sets $\{Y_n^c(x; k)\}$, $\{Z_n^c(x; k)\}$ that are biorthogonal with respect to the weight function $e^{-x}x^c$ over the interval $(0, \infty)$. An explicit expression was obtained for $Z_n^c(x; k)$ but not for $Y_n^c(x; k)$. An explicit polynomial expression for $Y_n^c(x; k)$ is given in the present paper.

1. Konhauser [2] has discussed two sets of polynomials $Y_n^c(x; k)$, $Z_n^c(x; k)$, $n = 0, 1, \dots$, $k = 1, 2, 3, \dots$, $c > -1$; $Y_n^c(x; k)$ is a polynomial in x while $Z_n^c(x; k)$ is a polynomial in x^k . Moreover

$$(1) \quad \int_0^\infty e^{-x}x^c Y_n^c(x; k) x^{ki} dx = \begin{cases} 0 & (0 \leq i < n) \\ \neq 0 & (i = n) \end{cases}$$

and

$$(2) \quad \int_0^\infty e^{-x}x^c Z_n^c(x; k) x^i dx = \begin{cases} 0 & (0 \leq i < n) \\ \neq 0 & (i = n) . \end{cases}$$

For $k = 1$, conditions (1) and (2) reduce to the orthogonality conditions satisfied by the Laguerre polynomials $L_n^c(x)$.

It follows from (1) and (2) that

$$(3) \quad \int_0^\infty e^{-x}x^c Y_i^c(x; k) Z_j^c(x; k) dx = \begin{cases} 0 & (i \neq j) \\ \neq 0 & (i = j) . \end{cases}$$

The polynomial sets $\{Y_n^c(x; k)\}$, $\{Z_n^c(x; k)\}$ are accordingly said to be biorthogonal with respect to the weight function $e^{-x}x^c$ over the interval $(0, \infty)$.

Konhauser showed that

$$(4) \quad Z_n^c(x; k) = \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + c + 1)}$$

As for $Y_n^c(x; k)$, he showed that

$$(5) \quad \begin{aligned} Y_n^c(x; k) &= \frac{k}{2i} \int_C \frac{e^{-xt}(t+1)^{c+kn}}{[(t+1)^k - 1]^{n+1}} dt \\ &= \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left\{ \frac{e^{-xt}(t+1)^{c+kn} t^{n+1}}{[(t+1)^{k+1} - 1]^{n+1}} \right\}_{t=0} . \end{aligned}$$

In the integral in (5), C may be taken as a small circle about the origin in the t -plane.

In the present note we give a generating function and an explicit polynomial expression for the polynomial $Y_n^c(x; k)$. Moreover we show that $Y_n^c(x; k)$ can be identified with a polynomial studied recently by S. K. Chatterjea [1].

2. We apply the Lagrange expansion in the form [4, p. 125]

$$(6) \quad \frac{f(t)}{1 - w\phi'(t)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \left\{ \frac{d^n}{dt^n} [f(t)(\phi(t))^n] \right\}_{t=0},$$

where

$$w = \frac{t}{\phi(t)}, \quad \phi(t) = a_0 + a_1 t + \dots \quad (a_0 \neq 0).$$

Take

$$f(t) = \frac{e^{-xt}(t+1)^{ct}}{(t+1)^k - 1}, \quad \phi(t) = \frac{(t+1)^{kt}}{(t+1)^k - 1}.$$

Then we have

$$1 - w\phi'(t) = \frac{kt}{(t+1)(t+1)^k - 1},$$

so that

$$\frac{f(t)}{1 - w\phi'(t)} = e^{-xt}(t+1)^{c+1}.$$

Thus, by (5) and (6), we have

$$(7) \quad e^{-xt}(t+1)^{c+1} = \sum_{n=0}^{\infty} Y_n^c(x; k) \left(\frac{t}{\phi(t)} \right)^n.$$

If we put

$$w = \frac{t}{\phi(t)} = \frac{(t+1)^k - 1}{(t+1)^k} = 1 - (t+1)^{-k},$$

then

$$t = (1 - w)^{-1/k} - 1$$

and (7) becomes

$$(8) \quad (1 - w)^{-(c+1)/k} \exp\{-x[(1 - w)^{-1/k} - 1]\} = \sum_{n=0}^{\infty} Y_n^c(x; k) w^n.$$

In the next place, we have

$$\begin{aligned}
 & (1 - w)^{-(c+1)/k} \exp \{-x[(1 - w)^{-1/k} - 1]\} \\
 &= (1 - w)^{-(c+1)/k} \sum_{r=0}^{\infty} \frac{x^r}{r!} [(1 - w)^{-1/k} - 1]^r \\
 &= \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} (1 - w)^{-(s+c+1)/k} \\
 &= \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \sum_{n=0}^{\infty} \frac{((s + c + 1)/k)_n}{n!} w^n \\
 &= \sum_{n=0}^{\infty} \frac{w^n}{n!} \sum_{r=0}^n \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s + c + 1}{k}\right)_n,
 \end{aligned}$$

where

$$(a)_n = a(a + 1) \cdots (a + n - 1), \quad (a)_0 = 1.$$

It therefore follows from (8) that

$$(9) \quad Y_n^c(x; k) = \frac{1}{n!} \sum_{r=0}^n \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s + c + 1}{k}\right)_n.$$

3. Chatterjea [1] has defined the polynomial

$$(10) \quad T_{k,n}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^{xk} D^n(e^{\alpha+n} e^{-xk})$$

with $k = 1, 2, 3, \dots$. The case $\alpha = 0$ had been discussed by Palas [3]. Chatterjea showed that (10) implies

$$(11) \quad T_{k,n}^{(\alpha)}(x) = \sum_{r=0}^{\infty} \frac{x^{kr}}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \binom{\alpha + n + ks}{n}.$$

He also obtained operational formulas and a generating function for $T_{k,n}^{(\alpha)}(x)$. The assumption that k is a positive integer is not used in deriving (11).

If we replace k by k^{-1} and α by $k^{-1}\alpha$, (10) becomes

$$T_{k^{-1},k}^{(-1\alpha)}(x) = \sum_{r=0}^n \frac{x^{kr}}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \binom{k^{-1}(\alpha + s) + n}{n}.$$

On the other hand, since

$$\frac{1}{n!} \left(\frac{s + c + 1}{k}\right)_n = \binom{k^{-1}(s + c + 1) + n - 1}{n},$$

(9) gives

$$Y_n^{c+k-1}(x^k; k) = \sum_{r=0}^n \frac{x^{kr}}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \binom{k^{-1}(s + c) + n}{n}.$$

It follows at once that

$$(12) \quad Y_n^{c+k-1}(x^k; k) = T_{k^{-1},n}^{(k^{-1}c)}(x),$$

or, if we prefer,

$$(13) \quad Y_n^{k\alpha+k-1}(x^k; k) = T_{k-1, n}^{(\alpha-1)}(x) .$$

4. It may be of interest to point out that a formula equivalent to (9) can be obtained without the use of the Lagrange expansion. In the integral representation (5), put

$$t = (1 + u)^{1/k} - 1 .$$

Then (5) becomes

$$Y_n^c(x; k) = \frac{1}{2\pi i} \int_C \frac{\exp\{-x[(1-u)^{1/k}-1]\}(1+u)^{k^{-1}(c+1)+n-1}}{u^{n+1}} du ,$$

where C denotes a small circle about the origin in the u -plane. The numerator of the integral is equal to

$$\begin{aligned} & \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} (1+u)^{k^{-1}(c+s+1)+n-1} \\ &= \sum_{m=0}^{\infty} u^m \sum_{r=0}^m \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \binom{k^{-1}(c+s+1)+n-1}{m} . \end{aligned}$$

Taking $m = n$, we therefore get

$$(14) \quad Y_n^c(x; k) = \sum_{r=0}^n \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \binom{k^{-1}(c+s+1)+n-1}{n} .$$

Since

$$\binom{c+n-1}{n} = \frac{(c)_n}{n!} ,$$

it is evident that (14) is identical with (9).

5. Making use of the explicit formulas (4) and (9), we can give a rather brief proof of (3). Indeed we have

$$\begin{aligned} J_{n,m} &= \int_0^{\infty} e^{-x} x^c Z_n^c(x; k) Y_m^c(x; k) dx \\ &= \frac{\Gamma(kn+c+1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{\Gamma(kj+c+1)} \\ &\quad \cdot \frac{1}{m!} \sum_{r=0}^m \frac{1}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s+c+1}{k}\right)_m \cdot \int_0^{\infty} e^{-x} x^{kj+c+r} dx \\ &= \frac{\Gamma(kn+c+1)}{n! m!} \sum_{j=0}^n (-1)^j \binom{n}{j} \\ &\quad \cdot \sum_{r=0}^m \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s+c+1}{k}\right)_m \binom{kj+c+r}{r} . \end{aligned}$$

If $f(x)$ is a polynomial of degree m , it is familiar that

$$f(x) = \sum_{r=0}^m \binom{x}{r} \Delta^r f(0),$$

where

$$\Delta^r f(0) = \sum_{s=0}^r (-1)^{r-s} \binom{r}{s} f(s).$$

In particular, for

$$f(x) = \left(\frac{x + c + 1}{k} \right)_m,$$

we have

$$\begin{aligned} \left(\frac{x + c + 1}{k} \right)_m &= \sum_{r=0}^m \binom{x}{r} \sum_{s=0}^r (-1)^{r-s} \binom{r}{s} \left(\frac{s + c + 1}{k} \right)_m \\ &= \sum_{r=0}^m \binom{x + r - 1}{r} \sum_{s=r}^n (-1)^s \binom{r}{s} \left(\frac{s + c + 1}{k} \right)_m. \end{aligned}$$

For $x = -kj - c - 1$ this reduces to

$$(-j)_m = \sum_{r=0}^m \binom{kj + c + r}{r} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s + c + 1}{k} \right)_m.$$

Thus

$$\begin{aligned} J_{n,m} &= \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(-j)_m}{m!} \\ &= (-1)^m \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j}{m}. \end{aligned}$$

Since

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j}{m} = \binom{n}{m} \sum_{j=m}^n (-1)^j \binom{n-m}{j-m} = (-1)^m \binom{n}{m} (1-1)^{n-m}$$

it is evident that

$$(15) \quad J_{n,m} = \frac{\Gamma(kn + c + 1)}{n!} \delta_{nm}$$

in agreement with (3). In particular

$$J_{n,n} = \frac{\Gamma(kn + c + 1)}{n!}$$

as proved in [2].

A little more generally, we have

$$\begin{aligned}
 J'_{n,m} &= \int_0^\infty e^{-x} x_c Z_n^c(x; k) Y_n^{c'}(x; k) dx \\
 &= \frac{\Gamma(kn + c + 1)}{n! m!} \sum_{j=0}^n (-1)^j \binom{n}{j} \left(-j - \frac{c - c'}{k}\right)_m \\
 &= (-1)^m \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j + a}{m},
 \end{aligned}$$

where $a = (c - c')/k$. It follows that

$$(16) \quad J'_{n,m} = \begin{cases} 0 & (n > m), \\ (-1)^{n+m} \frac{\Gamma(kn + c + 1)}{n!} \binom{a}{m - n} & (n \leq m). \end{cases}$$

Clearly (16) includes (15).

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