POINTLIKE SUBSETS OF A MANIFOLD

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Morton Brown introduced the concept of a cellular subset of $S^n$. As a consequence of the generalized Schoenflies Theorem it is easy to show that a subset of $S^n$ is pointlike if and only if it is cellular. In this paper the obvious generalization of the definitions of pointlike and cellular sets are made and their relationship in a manifold is considered. It is easy to show that a cellular subset of a manifold is pointlike. While it is not true that a pointlike subset of a manifold is cellular, it is shown that a pointlike subset of a compact $n$-manifold lies in a contractible $n$-manifold with $(n - 1)$-sphere boundary. As a consequence of this it is shown that $K$ is a pointlike subset of a compact $n$-manifold $(n \neq 4)$ if and only if $K$ is cellular. The case $n = 4$ is still unsolved.

DEFINITIONS. An $n$-manifold is a connected separable locally Euclidean metric space. A connected separable metric space in which every point has a neighborhood whose closure is an $n$-cell is an $n$-manifold with boundary. Note that a manifold is a manifold with boundary but not conversely. A compact connected subset $K$ of an $n$-manifold $M$ is pointlike if $M \sim K$ is homeomorphic with $M \sim \{p\}$ where $p \in M$. A subset $K$ of an $n$-manifold $M$ is cellular if there is a sequence of $n$-cells $C_1, C_2, \ldots$ such that $C_{i+1} \subset \text{Int} \ C_i$ and $K = \cap C_i$. An $(n - 1)$-sphere $S^{n-1}$ that separates an $n$-manifold $M$ into components $A$ and $B$ is collared on the side containing $A$ if there is an embedding $h: S^{n-1} \times [0, 1] \to \overline{A}$ such that $h(x, 0) = x$. An $(n - 1)$-sphere $S^{n-1}$ in an $n$-manifold $M$ is bicollared if there is an embedding $h: S^{n-1} \times [0, 1] \to M$ such that $h(x, 1/2) = x$. A pseudo-sphere is a compact manifold that is a homotopy sphere. A compact contractible $n$-manifold with boundary is called a pseudo-cell. The Poincare Conjecture—known to be true for $n \neq 3, 4$ [7]—says that a pseudo-sphere is a sphere.

PRELIMINARY THEOREMS. The following theorem follows from the corresponding theorem for $E^n$ which is proved by the same methods as used in [4].

**Theorem 1.** A cellular subset of a manifold is pointlike.

One might think that a pointlike subset of a manifold is cellular. That this is not the case is shown by the following example.

**Example 1.** Let $M$ be $E^n$ minus the integers on the positive $x$-axis, and minus 1-spheres of radius $1/4$ centered at the negative
integers on the $x$-axis. The 1-sphere of radius 1/4 and center at 0 is pointlike but not cellular. A similar construction using linked 1-spheres gives an example of a pointlike subset of a manifold containing a loop that is homotopically nontrivial in the manifold. A cellular subset of a manifold is not necessarily contractible, for example the crumpled cube bounded by the Alexander Horned sphere is not simply connected even though it is cellular.

**Lemma 2.** Let $K$ be a pointlike subset of a compact manifold $M$ with boundary. Let $h': M \sim K \to M \sim \{p\}$ be a homeomorphism. Then $h'$ can be extended to a continuous map $h: M \to M$ such that $h^{-1}(p) = K$.

**Proof.** Define $h$ by

$$h(x) = \begin{cases} h'(x) & \text{for } x \in M \sim K, \\ p & \text{for } x \in K. \end{cases}$$

Let $U$ be an open neighborhood of $p$. Then $\sim U$ is compact; hence, $h^{-1}(\sim U)$ is compact so $M \sim h^{-1}(\sim U)$ is open. Clearly this set contains $K$. Thus $h$ is continuous.

**Lemma 3.** If $K$ is a pointlike subset of a compact $n$-manifold $M$ with boundary and $K$ lies in an open $n$-cell, then $K$ is cellular.

**Proof.** We shall show that if $U$ is a neighborhood of $K$ then there is an $n$-cell $C$ such that $K \subset \text{Int } C \subset U$. Using this a simple inductive argument completes the proof. Let $h: M \to M$ be the continuous map given by the previous lemma. Then $h(U)$ is a neighborhood of $p$. Let $C'$ be an $n$-cell with bicollared boundary in $h(U)$ containing $p$ in its interior. Then $h^{-1}(C') = C$ is a cell by the Generalized Schoenflies theorem.

By obvious modifications of the proof in [8], the Jordan-Brouwer Theorem can be shown to hold in a pseudo-$n$-sphere. Let $K$ be the closure of one of the complementary domains of $S^{n-1}$. If an $n$-cell is sewn to $K$ the result is another pseudo-sphere. Applications of the Van Kampen Theorem, the Mayer-Vietoris Sequence and the Hurewicz Isomorphism show that $K$ is $(n-2)$-connected. Theorem 6.6.5 and Theorem 6.2.20 of [8] show that $K$ is contractible.

**Lemma 4** (Pseudo Schoenflies Lemma). A bicollared $(n-1)$-sphere $S^{n-1}$ in a pseudo-sphere $M^n$ is the common boundary of two pseudo-cells.

**Main Result.**
**THEOREM 5.** If $K$ is a pointlike subset of a compact manifold $M^n$ and $K$ is an $(n - 1)$-sphere collared on the side containing $K$, then $K$ is a pseudo-cell.

**Proof.** Assume $n \geq 3$. Denote by $L$ the set $((M^n \sim K) \cup \text{collar of } K)$. Then $L$ and $K$ are closed and their union is $M^n$ while their intersection is simply connected. By the Van Kampen Theorem $\pi_1(M^n) = \pi_1(L) * \pi_1(K)$, where $*$ denotes the free product. Borsuk [2] has shown that every compact manifold is dominated by a polyhedron, that is there is a finite polyhedron $P$ and continuous maps $f: P \rightarrow M^n$ and $g: M^n \rightarrow P$ such that $f \circ g$ is a homotopic to $1_{M^n}$. It follows that $\pi_1(M^n)$ is a finitely presented group. Since $K$ is pointlike, $\pi_1(M^n \sim K) = \pi_1(L) = \pi_1(M^n \sim \{p\}) = \pi_1(M^n)$. We have $\pi_1(K) = \pi_1(K) * \pi_1(L) = \pi_1(K) * \pi_1(M^n)$. By Grusko's theorem [6], $\pi_1(K)$ is trivial.

To show that $\pi_q(K)$ is trivial for $q \leq n$ we show that $H_q(K)$ is trivial for $q \leq n - 2$, then we use duality to get $H_q(K) = 0$ for $q \leq n$. Since $K$ and $L$ form an excisive couple we may apply the Mayer-Vietoris Sequence to get

$$H_q(K \cap L) \rightarrow H_q(K) \oplus H_q(L) \rightarrow H_q(K \cup L) \rightarrow H_{q-1}(K \cap L),$$

$$1 \leq q \leq n - 2.$$ 

Since $K \cap L$ is an $n$-annulus this sequence becomes

$$0 \rightarrow H_q(K) \oplus H_q(L) \rightarrow H_q(K \cup L) \rightarrow 0,$$

which implies that $H_q(K) \oplus H_q(L) \approx H_q(K \cup L)$. Since $K$ is pointlike, $H_q(K \cup L) \approx H_q(L)$. Since there is a dominating polyhedron for $M^n$, $H_q(M^n)$ is a finitely generated group. It follows that $H_q(K)$ is trivial. By the Hurewicz Isomorphism Theorem, $\pi_q(K) = 0$ for $1 \leq q \leq n - 2$. Let $S$ be the compact manifold obtained by sewing a cell to the boundary of $K$. Then by duality, $S$ is a homotopy sphere. By Lemma 4, $K$ is contractible.

If $n = 2$ then $K$ can be shown to shown to be a 2-cell by the classification theorem for compact 2-manifolds with contours for boundary.

**COROLLARY 6.** Let $K$ be a pointlike subset of a compact manifold $M$, then $K$ lies in a pseudo-cell with sphere boundary.

**Proof:** Let $h: M \rightarrow M$ be the continuous map given by Lemma 2. Let $C'$ be a cell containing $p$ and having a bicollared boundary. Then $C'$ is pointlike so $h^{-1}(C') = C$ is a pointlike subset of $M$ with bicollared sphere boundary. The previous theorem shows that $C$ is a pseudo-cell.

**COROLLARY 7.** In a compact manifold in which every pseudo-cell with sphere boundary is a cell, a pointlike subset is cellular.
LEMMA 8. If $K$ is a pointlike subset of a compact manifold $M$, then there are infinitely many disjoint homeomorphic copies of $K$ in $M$.

Proof. Let $p \in M \sim K$ and let $h: M \sim K \rightarrow M \sim \{p\}$ be a homeomorphism. Let $h^{-1}(K) = K_1 \subset M \sim K$. Let $g_i$ be a homeomorphism of $M$ onto itself such that $g_i(p) = p_i \in K \cup K_1$ and $g_i = 1$ on $K$. Let $h_1 = g_1 \circ h$. Then $h_1^{-1}(K_1) = K_2$ is homeomorphic with $K$ and

$$h_1^{-1}(K_1) \cap (K_2 \cap K) = \emptyset.$$  

Continuing in this fashion we get $K, K_1, K_2, \ldots$.

The complement of two disjoint pointlike subsets of a manifold $M$ need not be homeomorphic with the complement of two points in $M$; for example two linked 1-spheres in the 3-manifold of Example 1.

THEOREM 9. A pointlike subset of a compact $n$-manifold ($n \neq 4$) is cellular.

Proof. By Corollary 6, the pointlike set lies in a pseudo-cell $P$ with sphere boundary. Sew a cell to $P$ along their boundaries to get a homotopy sphere $S^n$. Since the Poincare Conjecture has been proved [7] for $n \geq 5$, $S^n$ must be a sphere. The generalized Schoenflies Theorem [3] shows that $P$ is a cell. An application of Lemma 3 completes the proof when $n \geq 5$. If $K$ is a pointlike subset of a compact manifold $M$, then there are countably many disjoint homeomorphic copies of $K$ in $M$. Thus if $K$ is a pointlike subset of $M$ that is not cellular, then $M$ must contain countably many disjoint pseudo-cells that are not cells. If $n = 3$, $M$ is triangulable so an application of Bing’s Side Approximation Theorem [1] allows us to assume that each pseudo-cell has a polyhedral sphere boundary. Kneser [5] has shown that such a decomposition can contain only finitely many such sets that are not cells.

We note that we have a generalization of the Generalized Schoenflies theorem: If $S^{n-1}$ is a bicollared ($n-1$)-sphere that separates a compact $n$-manifold $M$ and one of the components of $M - S^{n-1}$ is pointlike, then that component is a pseudo-cell.

One should observe that the proof the Theorem 5 shows: If $K$ is a pointlike subset of an $n$-manifold $M$, $\pi_{m}(M)$ is finitely generated for $1 \leq m \leq n$, and $\hat{K}$ is an $(n-1)$-sphere collared on the side containing $K$, then $K$ is a pseudo-cell.

Using arguments like those used in the proof of Theorem 5, one can show that a compact $n$-manifold ($n \neq 4$) can be written as the connected sum of at most finitely many nontrivial summands.

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Question. If we drill countably many disjoint cells out of $S^4$ and sew in pseudo-cells, is the resulting space ever a manifold?

BIBLIOGRAPHY


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