BISECTION INTO SMALL ANNULI

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In a Riemannian manifold the modulus of a relatively compact set with border consisting of two sets of components is introduced to measure its magnitude from the viewpoint of harmonic functions. The existence of a subdivision into two sets each having modulus arbitrarily close to one is established.

1. Let $M$ be a Riemannian manifold, i.e. a connected orientable $C^\infty$ $n$-manifold that carries a metric tensor $g_{ij}$. Consider a bordered compact region $E \subset M$ whose border is the union of two nonempty disjoint sets $\alpha$ and $\beta$ of components. We shall call the configuration $(E, \alpha, \beta)$ an annulus.

Let $h$ be the harmonic function on $E$ with continuous boundary values $0$ on $\alpha$ and $\log \mu > 0$ on $\beta$ such that

\[
\int_{\alpha} \ast dh = 2\pi .
\]

The number $\mu > 1$ is called the modulus of the annulus $(E, \alpha, \beta)$ and we set

\[
\mu = \text{mod}(E, \alpha, \beta) .
\]

Let $w$ be the harmonic measure of $\beta$ with respect to $E$, i.e. the harmonic function on $E$ with continuous boundary values $0$ on $\alpha$ and $1$ on $\beta$. By using Green’s formula we obtain

\[
\log \mu = \frac{2\pi}{D_E(w)} ,
\]

where $D_E(w)$ denotes the Dirichlet integral $\int_E dw \wedge \ast dw$ of $w$ over $E$.

An illustration of these concepts is obtained by taking the annulus $E = \{x \mid r \leq |x| \leq R\}$ in $n$-dimensional ($n \geq 3$) Euclidean space. The harmonic measure of $|x| = R$ with respect to $E$ is

\[
w = |x|^{2-n} - r^{2-n}
\]

and the modulus of $(E, |x| = r, |x| = R)$ is given by

\[
\log \mu = \pi^{1-(n/2)}(2 - n) \Gamma\left(\frac{n}{2}\right)(R^{2-n} - r^{2-n}) .
\]

Note that $\mu > 1$, in a sense, measures the relative thickness of $E$ and that $\mu \to 1$ as $R - r \to 0$.  

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Our result gains interest if we generalize the notion of annulus slightly. Let \( (E_j, \alpha_j, \beta_j) \) \((j = 1, \ldots, m)\) be annuli such that \( E_i \cap E_j = \emptyset \) for \( i \neq j \). Set \( E = \bigcup_{j=1}^{m} E_j, \alpha = \bigcup_{j=1}^{m} \alpha_j, \beta = \bigcup_{j=1}^{m} \beta_j \). Then we shall also call the configuration \((E, \alpha, \beta)\) an annulus. The modulus \( \mu = \mod (E, \alpha, \beta) \) and the harmonic measure of \( E \) with respect to \( \beta \) are defined exactly as for a connected annulus. Moreover, formula (2) is valid and consequently we have

\[
\frac{1}{\log \mu} = \sum_{j=1}^{m} \frac{1}{\log \mu_j},
\]

where \( \mu_j = \mod (E_j, \alpha_j, \beta_j) \).

2. Let \( M \) be a noncompact Riemannian manifold throughout this number. A function which is positive and harmonic on \( M \) except for a fundamental singularity is called a Green's function if it majorizes no nonconstant positive harmonic functions on \( M \). If a Green's functions exists, then \( M \) is called hyperbolic; otherwise it is called parabolic.

An increasing sequence \((\Omega_n)\) of bordered compact regions is called an exhaustion of \( M \) if \( \bigcup \Omega_n = M \). Note that the configuration \((\Omega_{n+1} - \overline{\Omega}_n, \partial \Omega_n, \partial \Omega_{n+1})\) is an annulus and denote its modulus by \( \mu_n \).

The parabolicity of a noncompact Riemannian manifold \( M \) is characterized by the following

**Modular Criterion.** There exists an exhaustion \((\Omega_n)\) of \( M \) with \( \prod \mu_n = \infty \) if and only if \( M \) is parabolic.

In the 2-dimensional case this criterion has been established by Sario [5] and Noshiro [4] and their work can easily be generalized to arbitrary Riemannian manifolds (cf. Smith [7], Glasner [2]).

One naturally asks whether a convergent modular product has any bearing on the hyperbolicity of a manifold. The main result of this paper is that any annulus can be separated into two annuli each having modulus less than \( 1 + \varepsilon \). This clearly answers the question in the negative and also settles Problem 3 in Sario [6].

3. Suppose the annulus \((E, \alpha, \beta)\) has components \((E_j, \alpha_j, \beta_j)\) \((j = 1, \ldots, m)\). Let \( \gamma_j \) be a hypersurface in \( E_j \) such that \( E_j - \gamma_j = E_j' \cup E_j'', E_j' \cap E_j'' = \emptyset \), and \((E_j', \alpha_j, \gamma_j)\) and \((E_j'', \gamma_j, \beta_j)\) are annuli. Set \( \gamma = \bigcup_{j=1}^{m} \gamma_j \). We shall call \( \gamma \) a bisecting surface of \((E, \alpha, \beta)\). Also set \( E' = \bigcup_{j=1}^{m} E_j' \) and \( E'' = \bigcup_{j=1}^{m} E_j'' \). We are now able to state the

**Theorem.** Given an annulus \((E, \alpha, \beta)\) and \( \varepsilon > 0 \) there exists a bisecting surface \( \gamma \) of \((E, \alpha, \beta)\) such that
This was established by Sario [5] for doubly connected plane regions using Koebe’s distortion theorem. All proofs for the 2-dimensional case known to the authors use either a distortion theorem, in essence, or an estimate (cf. Akaza-Kuroda [1]) obtained by means of Möbius transformations (Nakai-Sario [3]) which cannot be generalized to higher dimensions. Therefore, one is led to estimate the Dirichlet integral of the harmonic measure directly and the proof presented here seems to even give a more elementary proof for the 2-dimensional case.

4. Denote by $C(a, b) = C_{x_0}(a, b)$ the Euclidean cylinder

$$\sum_{j=1}^{n-1} (x_i - x_0)^2 < a^2, \quad x_0^n < x^n < x_0^n + b,$$

where $a, b > 0$ and $x_0 = (x_0^1, \ldots, x_0^n)$ is a fixed point. Let $\overline{\mathcal{G}}(a, b)$ be the class of $C^1$ functions $f$ on $C(a, b)$ with continuous boundary values 0 on $\overline{C(a, b)} \cap \{x^n = x_0^n\}$ and 1 on $\overline{C(a, b)} \cap \{x^n = x_0^n + b\}$. Also denote by $D^e$ the Dirichlet integral with respect to the Euclidean metric. We set $s$ equal to the surface area of $\sum_{i=1}^{n-1} (x_i)^2 = 1$, $x^n = 0$ and state the

**Lemma.** For every $f \in \overline{\mathcal{G}}(a, b)$,

$$D^e_{C_{x_0}(a, b)}(f) \geq \frac{sa^{n-1}}{b}$$

and equality holds for $f(x) = b^{-1}(x^n - x_0^n)$.

Clearly (6) is valid with equality for $f_0$. To prove (6) for an arbitrary $f$ we may assume $f \in C^1$ in a neighborhood of $\overline{C(a, b)}$. By Green’s formula we have

$$D^e_{C_{x_0}(a, b)}(f - f_0, f_0) = \int_{\partial C_{x_0}(a, b)} (f - f_0) \frac{\partial f_0}{\partial n} ds = 0,$$

since $f - f_0 = 0$ on the upper and lower boundary of the cylinder and $(\partial f_0/\partial n) = 0$ on the side of the cylinder. Consequently Schwarz’s inequality yields

$$D^e_{C_{x_0}(a, b)}(f) \cdot D^e_{C_{x_0}(a, b)}(f) \geq (D^e_{C_{x_0}(a, b)}(f, f_0))^2 = (D^e_{C_{x_0}(a, b)}(f_0))^2,$$

which completes the proof.

5. We are ready to prove the main result. Take a point $x_0 \in \alpha$ and a point $y_0 \in \beta$. Let $x^1, \ldots, x^n$ be a local coordinate system at
\( x_0 = (x_0^1, \ldots, x_0^n) \) valid in a neighborhood \( U \) of \( x_0 \) such that \( U \cap \alpha \) is given by \( x^n = x_0^n \) and \( x^n \) increases as \( x \) moves from \( \alpha \) to \( E \). Similarly, let \( y^1, \ldots, y^n \) be a local coordinate system at \( y_0 = (y_0^1, \ldots, y_0^n) \) valid in a neighborhood \( V \) of \( y_0 \) such that \( V \cap \beta \) is given by \( y^n = y_0^n \) and \( y^n \) increases as \( y \) moves from \( \beta \) to \( E \). Choose a constant \( c > 0 \) so small that

\[
\sqrt{g} \mid U \cup V > \sqrt{c}
\]

and also

\[
\frac{\partial^i}{\partial x^i} \mid U \cup V \geq \sqrt{c} \sum_{i=1}^n (\xi_i)^2
\]

for every vector \((\xi_1, \ldots, \xi_n)\). Now choose \( a > 0 \) sufficiently small to insure that \( \sum_{i=1}^n (x^i - x_0^i)^2 < a^2 \) with \( x^n = x_0^n \) and \( \sum_{i=1}^n (y^i - y_0^i)^2 < a^2 \) with \( y^n = y_0^n \) are contained in \( U \cap \alpha \) and \( V \cap \beta \), respectively. Finally choose \( b > 0 \) so that

\[
0 < b < \frac{\text{csa}^{n-1} \log (1 + \varepsilon)}{2\pi} ,
\]

\[
C_x(a, b) = \{x^n = x_0^n\} \subset E, \quad C_y(a, b) = \{y^n = y_0^n\} \subset E
\]

and

\[
C_x(a, b) \cap C_y(a, b) = \emptyset.
\]

Now take a bisecting surface \( \gamma \) of \((E, \alpha, \beta)\) subject to the requirements

\[
\gamma \cap (C_x(a, b) \cup C_y(a, b)) = \emptyset
\]

and

\[
\gamma \supset [C_x(a, b) \cap \{x^n = x_0^n + b\}] \cup C_y(a, b) \cap \{y^n = y_0^n + b\}]
\]

Let \( w' \) (resp. \( w'' \)) be the harmonic measure of \( \gamma \) (resp. \( \beta \)) with respect to \( E' \) (resp. \( E'' \)). Since \( E' \supset C_x(a, b) \), by using (7) and (8) we obtain

\[
D_{E'}(w') > D_{C_x(a, b)}(w') \geq cD_{C_x(a, b)}(w') .
\]

Hence by using (6) and (9) we have

\[
\frac{2\pi}{D_{E'}(w')} < \log (1 + \varepsilon)
\]

and in view of (2) we conclude that

\[
\text{mod} (E', \alpha, \gamma) < 1 + \varepsilon .
\]

A similar consideration for \( E'' \) establishes (4).
References


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