

# Pacific Journal of Mathematics

**A NOTE ON LEFT MULTIPLICATION OF SEMIGROUP  
GENERATORS**

KARL EDWIN GUSTAFSON

## A NOTE ON LEFT MULTIPLICATION OF SEMIGROUP GENERATORS

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**It is shown in this note that if  $A$  is the infinitesimal generator of a strongly continuous semigroup of contraction operators in any Banach space  $X$ , then so is  $BA$  for a broad class of bounded operators  $B$ ; the only requirement on  $B$  is that it transforms "in the right direction".**

In the recent paper [1] the following interesting result was obtained.

**THEOREM 1 (Dorroh).** *Let  $X$  be the Banach space of bounded functions on a set  $S$  under the supremum norm, let  $A$  be the infinitesimal generator of a contraction semigroup in  $X$ , and let  $B$  be the operator given by multiplication by  $p$ ,  $pX \subseteq X$ , where  $p$  is a positive function defined on  $S$ , bounded above, and bounded below above zero. Then  $BA$  is also the infinitesimal generator of a contraction semigroup in  $X$ .*

This leads naturally to the general question of preservation of the generator property under left multiplication; the purpose of this note is to present Theorem 2 below, which shows that for any Banach space, a large class of operators  $B$  are acceptable. In the following, the word "generator" will always mean generator of contraction semigroup.

In this note we will consider only left multiplication by everywhere defined bounded operators  $B$ . It is easily seen (e.g., [2, Corollary 3]) that  $A$  generates a contraction semigroup if and only if  $cA$  does,  $c > 0$ . Also by [4, Th. 2.1], if  $A$  is bounded,  $BA$  is a generator if and only if  $BA$  is dissipative; in this case clearly right multiplication also yields a generator. See [4, 5] for dissipativeness; we use dissipativeness in the sense [4], and recall that if  $BA$  is a generator, then  $BA$  is dissipative in all semi-inner products on  $X$ .

**THEOREM 2.** *Let  $X$  be any Banach space,  $A$  the infinitesimal generator of a contraction semigroup in  $X$ , and  $B$  a bounded operator in  $X$  such that  $\|\varepsilon B - I\| < 1$  for some  $\varepsilon > 0$ . Then  $BA$  generates a contraction semigroup in  $X$  if and only if  $BA$  is dissipative, (i.e.,  $\operatorname{Re} [BAx, x] \leq 0$ , all  $x \in D(A)$ ,  $[u, v]$  a semi-inner product (see [4])).*

*Proof.* We note that  $R(B) = X$  when  $\|\varepsilon B - I\| < 1$  for some  $\varepsilon > 0$ ; to show that  $BA$  is a generator it suffices to show that  $\varepsilon BA$  is a generator for some positive  $\varepsilon$ . From the relation  $\|\varepsilon B - I\| < 1 \leq \|(I - \varepsilon BA)^{-1}\|^{-1}$  we have by [2, Lemma 1] that:

$$\beta(I - \varepsilon BA) = \beta((I - \varepsilon BA) + (\varepsilon B - I)) \equiv \beta(\varepsilon B(I - A)) = \beta(\varepsilon B) = 0,$$

where  $\beta(T) = \dim X/\text{Cl}(R(T))$  is the deficiency index of an operator  $T$ . A closed implies  $\varepsilon BA$  closed (and therefore  $I - \varepsilon BA$  closed), since  $\varepsilon BA = A + (\varepsilon B - I)A$  and  $\|\varepsilon B - I\| < 1$ ;  $BA$  dissipative implies that  $I - \varepsilon BA$  possesses a continuous inverse, so that we therefore have  $R(I - \varepsilon BA)$  closed, and thus  $BA$  the generator of a contraction semigroup. This result also follows quickly from [2, Theorem 2].

In the above we made use of basic index theory as may be found in [3] and the well-known characterizations of generators as may be found in [3, 4, 5], for example. The index theory notation here is a convenience only; the argument can be presented without it.

**COROLLARY 3.** *Theorem 1 stated above.*

*Proof.* As shown in [1],  $pA$  is dissipative with respect to the semi-inner product used there, and clearly  $0 < m \leq p(s) \leq M$  implies that  $|\varepsilon p - 1| < 1 - \varepsilon m$  for small enough  $\varepsilon$ .

**COROLLARY 4.** *Let  $B$  be of the form  $cI + C$ ,  $\|C\| < c$ ,  $CA$  dissipative. Then  $BA$  is a generator if  $A$  is.*

*Proof.* Clearly  $c^{-1}B$  satisfies the conditions of Theorem 2; note  $\|\varepsilon B - I\| < 1$  for some  $\varepsilon > 0$  if and only if  $B$  is of the form  $cI + C$ ,  $\|C\| < c$ .

*Remarks.* The condition  $BA$  dissipative in Theorem 2, necessary for  $BA$  to be a generator, requires (in general) that  $B$  be in a "positive" rather than a dissipative direction. For example, if  $A, B$ , and  $BA$  are self-adjoint operators on a Hilbert space, then  $A$  is a generator if and only if  $A$  is negative, and then  $BA$  is a generator if  $B$  is positive.

The condition  $\|\varepsilon B - I\| < 1$  in Theorem 2 is easily seen to be equivalent to the condition:  $B$  strongly accretive, i.e.,  $\exists m = m(B)$  such that  $\text{Re}[Bx, x] \geq m > 0$  for  $\|x\| = 1$ , where  $[u, v]$  is the semi-inner product being used (see [4]). It is a sharp condition since equality  $\|\varepsilon B - I\| = 1$  cannot be permitted in general, as seen from the example  $B = 0, A$  unbounded, for then  $BA$  is not closed.

The effect of Theorem 2 is that, after the application of index

theory therein, one sees that the essential question concerning when  $BA$  is a generator is the question of when  $BA$  is dissipative. Three situations which can then occur are: (i) as in [1], for special operators  $B$ , one can find a semi-inner product for which  $BA$  is dissipative; (ii)  $A$  commutes with  $B$  (see [3]), for which one can easily obtain results such as  $A$  self-adjoint, dissipative, and  $B$  accretive imply  $BA$  dissipative; (iii) general (noncommuting)  $A$  and  $B$ . For case (iii) one can obtain the following interesting result (proof given in forthcoming paper by the author, Math. Zeitschrift). Let  $-A$  and  $B$  be strongly accretive operators on a Banach space. If

$$\min_{\varepsilon} \|\varepsilon B - I\| \leq m(-A) \cdot \|A\|^{-1},$$

then  $BA$  is dissipative. In particular, let  $A$  and  $B$  be self-adjoint operator: then  $(\|B\| - m(B)) \cdot (\|B\| + m(B))^{-1} \leq m(-A) \cdot \|A\|^{-1}$  is sufficient. Moreover these conditions can be sharpened by introducing the concept of the cosine of an operator. For certain operators the condition for  $BA$  to be dissipative can then be written as  $\sin B \leq \cos A$ .

The author appreciates useful expository suggestions from the referee. Extensions of these results to unbounded right and left multiplication will appear in a forthcoming paper by the author and G. Lumer.

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