A NOTE ON LEFT MULTIPLICATION OF SEMIGROUP GENERATORS

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It is shown in this note that if $A$ is the infinitesimal generator of a strongly continuous semigroup of contraction operators in any Banach space $X$, then so is $BA$ for a broad class of bounded operators $B$; the only requirement on $B$ is that it transforms "in the right direction".

In the recent paper [1] the following interesting result was obtained.

**Theorem 1 (Dorroh).** Let $X$ be the Banach space of bounded functions on a set $S$ under the supremum norm, let $A$ be the infinitesimal generator of a contraction semigroup in $X$, and let $B$ be the operator given by multiplication by $p$, $pX \subseteq X$, where $p$ is a positive function defined on $S$, bounded above, and bounded below above zero. Then $BA$ is also the infinitesimal generator of a contraction semigroup in $X$.

This leads naturally to the general question of preservation of the generator property under left multiplication; the purpose of this note is to present Theorem 2 below, which shows that for any Banach space, a large class of operators $B$ are acceptable. In the following, the word "generator" will always mean generator of contraction semigroup.

In this note we will consider only left multiplication by everywhere defined bounded operators $B$. It is easily seen (e.g., [2, Corollary 3]) that $A$ generates a contraction semigroup if and only if $cA$ does, $c > 0$. Also by [4, Th. 2.1], if $A$ is bounded, $BA$ is a generator if and only if $BA$ is dissipative; in this case clearly right multiplication also yields a generator. See [4, 5] for dissipativeness; we use dissipativeness in the sense [4], and recall that if $BA$ is a generator, then $BA$ is dissipative in all semi-inner products on $X$.

**Theorem 2.** Let $X$ be any Banach space, $A$ the infinitesimal generator of a contraction semigroup in $X$, and $B$ a bounded operator in $X$ such that $\|eB - I\| < 1$ for some $\varepsilon > 0$. Then $BA$ generates a contraction semigroup in $X$ if and only if $BA$ is dissipative, (i.e., $\Re [BAx, x] \leq 0$, all $x \in D(A), [u, v]$ a semi-inner product (see [4])).
Proof. We note that $R(B) = X$ when $\|\varepsilon B - I\| < 1$ for some $\varepsilon > 0$; to show that $BA$ is a generator it suffices to show that $\varepsilon BA$ is a generator for some positive $\varepsilon$. From the relation $\|\varepsilon B - I\| < 1 \leq \|(I - \varepsilon BA)^{-1}\|^{-1}$ we have by [2, Lemma 1] that:

$$\beta(I - \varepsilon BA) = \beta((I - \varepsilon BA) + (\varepsilon B - I)) = \beta(\varepsilon B(I - A)) = \beta(\varepsilon B) = 0,$$

where $\beta(T) = \dim X/\text{Cl}(R(T))$ is the deficiency index of an operator $T$. A closed implies $\varepsilon BA$ closed (and therefore $I - \varepsilon BA$ closed), since $\varepsilon BA = A + (\varepsilon B - I)A$ and $\|\varepsilon B - I\| < 1$; $BA$ dissipative implies that $I - \varepsilon BA$ possesses a continuous inverse, so that we therefore have $R(I - \varepsilon BA)$ closed, and thus $BA$ the generator of a contraction semigroup. This result also follows quickly from [2, Theorem 2].

In the above we made use of basic index theory as may be found in [3] and the well-known characterizations of generators as may be found in [3, 4, 5], for example. The index theory notation here is a convenience only; the argument can be presented without it.

Corollary 3. Theorem 1 stated above.

Proof. As shown in [1], $pA$ is dissipative with respect to the semi-inner product used there, and clearly $0 < m \leq p(s) \leq M$ implies that $|\varepsilon p - 1| < 1 - \varepsilon m$ for small enough $\varepsilon$.

Corollary 4. Let $B$ be of the form $cI + C$, $\|C\| < c$, $CA$ dissipative. Then $BA$ is a generator if $A$ is.

Proof. Clearly $c^{-1}B$ satisfies the conditions of Theorem 2; note $\|\varepsilon B - I\| < 1$ for some $\varepsilon > 0$ if and only if $B$ is of the form $cI + C$, $\|C\| < c$.

Remarks. The condition $BA$ dissipative in Theorem 2, necessary for $BA$ to be a generator, requires (in general) that $B$ be in a “positive” rather than a dissipative direction. For example, if $A, B,$ and $BA$ are self-adjoint operators on a Hilbert space, then $A$ is a generator if and only if $A$ is negative, and then $BA$ is a generator if $B$ is positive.

The condition $||\varepsilon B - I|| < 1$ in Theorem 2 is easily seen to be equivalent to the condition: $B$ strongly accretive, i.e., $\exists m = m(B)$ such that $\Re \langle Bx, x \rangle \geq m > 0$ for $\|x\| = 1$, where $\langle u, v \rangle$ is the semi-inner product being used (see [4]). It is a sharp condition since equality $||\varepsilon B - I|| = 1$ cannot be permitted in general, as seen from the example $B = 0, A$ unbounded, for then $BA$ is not closed.

The effect of Theorem 2 is that, after the application of index
theory therein, one sees that the essential question concerning when $BA$ is a generator is the question of when $BA$ is dissipative. Three situations which can then occur are: (i) as in [1], for special operators $B$, one can find a semi-inner product for which $BA$ is dissipative; (ii) $A$ commutes with $B$ (see [3]), for which one can easily obtain results such as $A$ self-adjoint, dissipative, and $B$ accretive imply $BA$ dissipative; (iii) general (noncommuting) $A$ and $B$. For case (iii) one can obtain the following interesting result (proof given in forthcoming paper by the author, Math. Zeitschrift). Let $-A$ and $B$ be strongly accretive operators on a Banach space. If

$$\min_{\epsilon} \| \epsilon B - I \| \leq m(-A) \cdot \| A \|^{-1},$$

then $BA$ is dissipative. In particular, let $A$ and $B$ be self-adjoint operator: then $(\| B \| - m(B)) \cdot (\| B \| + m(B))^{-1} \leq m(-A) \cdot \| A \|^{-1}$ is sufficient. Moreover these conditions can be sharpened by introducing the concept of the cosine of an operator. For certain operators the condition for $BA$ to be dissipative can then be written as $\sin B \leq \cos A$.

The author appreciates useful expository suggestions from the referee. Extensions of these results to unbounded right and left multiplication will appear in a forthcoming paper by the author and G. Lumer.

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Duane W. Bailey, *On symmetry in certain group algebras* .................. 413
Lawrence Peter Belluce and Surender Kumar Jain, *Prime rings with a one-sided ideal satisfying a polynomial identity* ..................... 421
L. Carlitz, *A note on certain biorthogonal polynomials* ..................... 425
Charles O. Christenson and Richard Paul Osborne, *Pointlike subsets of a manifold* ..................................................... 431
Russell James Egbert, *Products and quotients of probabilistic metric spaces* .......................................................... 437
Moses Glasner, Richard Emanuel Katz and Mitsuru Nakai, *Bisection into small annuli* ...................................................... 457
Karl Edwin Gustafson, *A note on left multiplication of semigroup generators* .......................................................... 463
I. Martin (Irving) Isaacs and Donald Steven Passman, *A characterization of groups in terms of the degrees of their characters. II* .............. 467
Howard Wilson Lambert and Richard Benjamin Sher, *Point-like 0-dimensional decompositions of $S^3$* ........................................ 511
Oscar Tivis Nelson, *Subdirect decompositions of lattices of width two* .... 519
Ralph Tyrrell Rockafellar, *Integrals which are convex functionals* ........ 525
James McLean Sloss, *Reflection laws of systems of second order elliptic differential equations in two independent variables with constant coefficients* ..................................................... 541
Bui An Ton, *Nonlinear elliptic convolution equations of Wiener-Hopf type in a bounded region* ........................................... 577
Daniel Eliot Wulbert, *Some complemented function spaces in $C(X)$* .... 589
Zvi Ziegler, *On the characterization of measures of the cone dual to a generalized convexity cone* ....................................... 603