POINT-LIKE 0-DIMENSIONAL DECOMPOSITIONS OF $S^3$

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This paper is concerned with upper semicontinuous decompositions of the 3-sphere which have the property that the closure of the sum of the nondegenerate elements projects onto a set which is 0-dimensional in the decomposition space. It is shown that such a decomposition is definable by cubes with handles if it is point-like. This fact is then used to obtain some properties of point-like decompositions of the 3-sphere which imply that the decomposition space is a topological 3-sphere. It is also shown that decompositions of the 3-sphere which are definable by cubes with one hole must be point-like if the decomposition space is a 3-sphere.

In this paper we consider upper semicontinuous decompositions of $S^3$, the Euclidean 3-sphere. In particular, we shall restrict ourselves to those decompositions $G$ of $S^3$ which have the property that the union of the nondegenerate elements of $G$ projects onto a set whose closure is 0-dimensional in the decomposition space of $G$. We shall refer to such decompositions as 0-dimensional decompositions of $S^3$. Numerous examples of such decompositions appear in the literature. (One should note that some of the examples and results to which we refer are in $E^3$, Euclidean 3-space, but the corresponding examples and results for $S^3$ will be obvious in each case.)

In §3, a technique of McMillan [10] is used to show that point-like 0-dimensional decompositions of $S^3$ are definable by cubes with handles. Armentrout [2] has shown this in the case where the decomposition space is homeomorphic with $S^3$. The proof of this theorem shows that compact proper subsets of $S^3$ with point-like components are definable by cubes with handles.

In §4 we give some properties of point-like 0-dimensional decompositions of $S^3$ which imply that the decomposition space is homeomorphic with $S^3$. These properties were suggested by Bing in §7 of [6].

It is not known whether monotone 0-dimensional decompositions of $S^3$ which yield $S^3$ must have point-like elements. Partial results in this direction have been obtained by Armentrout [2], Bean [5], and Martin [9]. Bing, in §4 of [6], has presented an example of a decomposition of $S^3$ which yields $S^3$ even though it is not a point-like decomposition, but this example is not 0-dimensional. In §5 we show that a 0-dimensional decomposition of $S^3$ that yields $S^3$ must have point-like elements if it is definable by cubes with one hole.
2. Definitions and notation. Let $G$ be an upper semicontinuous decomposition of $S^3$, the 3-sphere. We denote the decomposition space of $G$ by $S^3/G$, the union of the nondegenerate elements of $G$ by $H_G$, and the projection map from $S^3$ onto $S^3/G$ by $P$.

The decomposition $G$ is said to be monotone if each element of $G$ is a continuum. If $\text{cl}\, P(H_G)$ is 0-dimensional in $S^3/G$, then $G$ is a 0-dimensional decomposition of $S^3$. If each element of $G$ has a complement in $S^3$ which is homeomorphic with $E^3$, Euclidean 3-space, then $G$ is a point-like decomposition of $S^3$.

The sequence $M_1, M_2, M_3, \cdots$ is a defining sequence for $G$ if and only if $M_1, M_2, M_3, \cdots$ is a sequence of compact 3-manifolds with boundary in $S^3$ such that (1) for each positive integer $i$, $M_{i+1} \subset \text{Int} \, M_i$, and (2) $g$ is a nondegenerate element of $G$ if and only if $g$ is a nondegenerate component of $\bigcap_{i=1}^\infty M_i$. Here, as in the remainder of the paper, subsets of $S^3$ which are manifolds will be assumed to be polyhedral subsets of $S^3$. It is well known that if $G$ is a 0-dimensional decomposition of $S^3$, a defining sequence exists for $G$. If a defining sequence $M_1, M_2, M_3, \cdots$ exists for $G$ such that for each positive integer $i$, each component of $M_i$ is a cube with handles, $G$ is said to be definable by cubes with handles. If a defining sequence $M_1, M_2, M_3, \cdots$ exists for $G$ such that for each positive integer $i$, each component of $M_i$ is a cube with one hole, $G$ is said to be definable by cubes with one hole.

3. Some consequences of a result of McMillan. The following lemma is a special case of Lemma 1 of [11]. Its proof follows from the very useful technique used by McMillan to prove Theorem 1 of [10].

**Lemma 1.** (McMillan). In $S^3$, let $M'$ be a compact polyhedral 3-manifold with boundary such that $\text{Bd} M'$ is connected, and let $M$ be a compact polyhedral 3-manifold with boundary such that $M \subset \text{Int} \, M'$, and each loop in $M$ can be shrunk to a point in $\text{Int} \, M'$. Then there is a cube with handles $C$ such that $M \subset \text{Int} C \subset C \subset \text{Int} \, M'$.

**Lemma 2.** If $G$ is a point-like 0-dimensional decomposition of $S^3$, then there is a defining sequence $M_1, M_2, M_3, \cdots$ for $G$ such that for each positive integer $i$, each component of $M_i$ has a connected boundary.

**Proof.** Let $M'_1, M'_2, M'_3, \cdots$ be a defining sequence for $G$, let $n$ be a positive integer, and let $K$ be a component of $M'_n$. Let $g$ be a component of $\bigcap_{i=1}^\infty M'_i$ which lies in $K$ and let $U$ be an open subset of $K$ containing $g$ such that $\text{cl} \, U \cap \text{Bd} K = \emptyset$. Since $g$ is point-like, there is a 3-cell $C$ such that $g \subset \text{Int} \, C \subset C \subset U$. There is an integer $j$ such that $L$, the component of $M'_j$ containing $g$, lies in $\text{Int} \, C$. Since
C separates no points of $BdK$ in $K$, $L$ separates no points of $BdK$ in $K$.

Using compactness of $\bigcap_{i=1}^{\infty} M_i$, one obtains a finite collection $L_1, \ldots, L_k$ of mutually exclusive defining elements whose interiors cover $(\bigcap_{i=1}^{\infty} M_i) \cap K$ and so that no $L_i$ separates points of $BdK$ in $K$. It follows easily that $\bigcup_{i=1}^{\infty} L_i$ separates no points of $BdK$ in $K$. By suitable relabeling, we suppose then, that if $i$ is a positive integer and $K$ is a component of $M_i'$, $K \cap M_{i+1}'$ does not separate points of $BdK$ in $K$. We construct disjoint arcs in $K-M_{i+1}'$ connecting the boundary components of $K$ and "drill-out" these arcs to replace $K$ by a compact 3-manifold with connected boundary. Doing this for each component of each $M_i'$, we obtain a defining sequence $M_1, M_2, M_3, \ldots$ as required by the conclusion of the lemma.

**Theorem 1.** If $G$ is a point-like 0-dimensional decomposition of $S^3$, then $G$ is definable by cubes with handles.

**Proof.** Using Lemma 2, there is a defining sequence $M_1', M_2', M_3', \ldots$ for $G$ such that each component of each $M_i'$ has a connected boundary. Let $n$ be a positive integer and $N$ a component of $M_n'$. Since $G$ is point-like, there is no loss of generality in supposing that each loop in $M_{n+1}' \cap N$ can be shrunk to a point in $\text{Int} \ N$. From Lemma 1, there is a cube with handles, $C$, such that $(M_{n+1}' \cap N) \subset \text{Int} \ C \subset C \subset \text{Int} \ N$. Hence, there is a sequence $M_1, M_2, M_3, \ldots$ of compact 3-manifolds with boundary such that (1) for each positive integer $i$, $M_{i+1} \subset \text{Int} \ M_i \subset M_i \subset \text{Int} \ M'_i$, and (2) each component of $M_i$ is a cube with handles. The sequence $M_1, M_2, M_3, \ldots$ is a defining sequence for $G$ and so $G$ is definable by cubes with handles.

The proof of the next theorem follows from the proof of Theorem 1.

**Theorem 2.** If $M$ is a closed subset of $S^3$ such that each component of $M$ is point-like, then there exists a sequence $M_1, M_2, M_3, \ldots$ of compact 3-manifolds with boundary such that (1) for each positive integer $i$, $M_{i+1} \subset \text{Int} \ M_i \subset M_i \subset \text{Int} \ M'_i$, and (2) each component of $M_i$ is a cube with handles, and (3) $M = \bigcap_{i=1}^{\infty} M_i$.

The concept of equivalent decompositions of $S^3$ was introduced in [4] and the following theorem follows immediately from Theorem 1 of this paper and Theorem 8 of [4].

**Theorem 3.** If $G$ is a point-like 0-dimensional decomposition of $S^3$, then $G$ is equivalent to a point-like 0-dimensional decomposition of $S^3$ each of whose nondegenerate elements is a 1-dimensional continuum.
In the remaining two sections, we shall utilize some of the above results to investigate certain properties of 0-dimensional decompositions of $S^3$.

4. Properties of point-like 0-dimensional decompositions of $S^3$. In this section we give two properties, each of which is both necessary and sufficient to imply $S^3/G$ is homeomorphic to $S^3$.

A space $X$ will be said to have the Dehn's Lemma property if and only if the following condition holds: If $D$ is a disk and $f$ is a mapping of $D$ into $X$ such that on some neighborhood of $f(BdD)$, $f^{-1}$ is a function, and $U$ is neighborhood of the set of singular points of $f(D)$, then there is a disk $D'$ in $f(D) \cup U$ such that $BdD' = f(BdD)$.

A space $X$ will be said to have the map separation property if and only if the following condition holds: If $D$ is a disk and $f_1$, $f_2$, $f_3$ are maps of $D$ into $X$ such that (1) for each $i$, on some neighborhood of $f_i(BdD)$, $f_i^{-1}$ is a function, (2) if $i \neq j$, $f_i(BdD) \cap f_j(D) = \emptyset$, and (3) $U$ is a neighborhood of $f_1(D) \cup \cdots \cup f_n(D)$, then there exist maps $f'_1$, $\cdots$, $f'_n$ of $D$ into $X$ such that (1) for each $i$, $f'_i(BdD) = f_i(BdD)$, (2) $f'_1(D) \cup \cdots \cup f'_n(D) \subset U$, and (3) if $i \neq j$, $f'_i(D) \cap f'_j(D) = \emptyset$.

It is a well known (and useful) fact that $S^3$ has the Dehn's Lemma property and the map separation property.

**Theorem 4.** If $G$ is a point-like 0-dimensional decomposition of $S^3$, then $S^3/G$ is homeomorphic with $S^3$ if and only if $S^3/G$ has the Dehn's Lemma property.

**Proof.** The "if" portion of the theorem is the only part that requires proof. Let $U$ be an open set containing $cl\, H_0$ and $\varepsilon > 0$. We shall construct a homeomorphism $h_i: S^3 \to S^3$ such that if $x \in S^3 - U$, $h_i(x) = x$ and if $g \in G$, $\text{diam}\, h_i(g) < \varepsilon$. It will follow from Theorem 3 of [2] that $S^3/G$ is homeomorphic with $S^3$.

By Theorem 1, $G$ is definable by cubes with handles. Hence, there exist disjoint cubes with handles $C_1, \cdots, C_n$ such that $cl\, H_0 \subset \bigcup_{i=1}^{n} \text{Int}\, C_i \subset \bigcup_{i=1}^{n} C_i \subset U$. Let $W_1, \cdots, W_n$ be pairwise disjoint neighborhoods of $C_1, \cdots, C_n$ respectively such that $\bigcup_{i=1}^{n} W_i \subset U$. Since $C_1$ is a cube with (possibly 0) handles, there is a homeomorphism $h_0$ of $S^3$ onto $S^3$ such that $h_0(x) = x$ for $x \in S^3 - W_1$ and $h_0(C_1)$ can be written as the union of a finite number of cubes such that (1) each cube has diameter less than $\varepsilon/2$, (2) no three cubes have a point in common, and (3) the intersection of any two cubes is empty or a disk on the boundary of each. The homeomorphism $h_0$ can be thought of as pulling $C_1$ towards a 1-dimensional spine of $C_1$. Let $D_1, D_2, \cdots, D_k$ be the inverse images under $h_0$ of the disks obtained by intersecting the various cubes making up $h_0(C_1)$. We note that if a continuum in
C_i intersects at most one D_i, then its image under h_0 has diameter less than \( \varepsilon \). For each \( i = 1, \ldots, k \), let \( D'_i \) be a subdisk of \( D_i \) such that \( D'_i \subset \text{Int} \ D_i \) and \( D_i \cap \text{cl} \ H_g = \text{Int} \ D'_i \cap \text{cl} \ H_g \). Let \( D \) be a disk in \( S^3 \) such that Bd \( D \cap (\bigcup_{i=1}^k C_i) = \emptyset \) and \( \bigcup_{i=1}^k D_i = D \cap (\bigcup_{i=1}^k C_i) = D \cap C_i \). Denote the punctured disk cl \( (D - \bigcup_{i=1}^k D'_i) \) by \( D' \). Now \( P_1 = P \cap D \) is a map of \( D \) into \( S^3/G \) and \( P_1^{-1} \) is a homeomorphism on a neighborhood of \( P_1(\text{Bd} \ D) \). The singular set of \( P_1(D) \) is contained in \( P_1(\bigcup_{i=1}^k \text{Int} \ D'_i) \). Let \( V \) be an open set in \( S^3/G \) containing the singular set of \( P_1(D) \) and such that \( P_1^{-1}(V) \subset (\text{Int} \ C_i) - D' \). By hypothesis there exists a disk \( E \) in \( P_1(D) \cap V \) bounded by \( P_1(\text{Bd} \ D_1) \). Let \( E_1, \ldots, E_k \) be the subdisks of \( E \) bounded by \( P_1(\text{Bd} \ D'_1), \ldots, P_1(\text{Bd} \ D'_k) \) respectively, and let \( U_1, \ldots, U_k \) be open sets whose closures lie in \( P(\text{Int} \ C_i) \) such that for each \( i = 1, \ldots, k \), \( E_i \subset U_i \), and if \( i \neq j \), \( \text{cl} \ U_i \cap \text{cl} \ U_j = \emptyset \). By the proof of Theorem 2.1 of [12], each \( \text{Bd} \ D'_i \) can be shrunk to a point in \( P^{-1}(U_i) \). Each map can be "glued" to the annulus \( \text{cl} \ (D_i - D'_i) \) to obtain a map from \( D_i \) into \( D_i \cup P^{-1}(U_i) \) with no singularities on \( D_i - P^{-1}(\text{cl} \ U_i) \). We now apply Dehn's Lemma in \( S^3 \) to these maps to obtain disjoint disks \( F_1, \ldots, F_k \) such that (1) for each \( i \), \( \text{Bd} \ D_i = \text{Bd} \ F_i \), (2) \( \text{Int} \ F_i \subset \text{Int} C_i \), and (3) if \( g \in G \), \( g \) intersects no more than one of the disks \( F_1, \ldots, F_k \). Let \( h_i' \) be a homeomorphism of \( S^3 \) onto itself fixed on \( S^3 - \text{Int} C_i \) such that for each \( i \), \( h_i'(F_i) = D_i \). Let \( h_i = h_i h_i' \). Note that if \( g \in G \) and \( g \subset C_i \), \( \text{diam} \ h_i(g) < \varepsilon \). Let \( h_2, \ldots, h_n \) be homeomorphisms such as \( h_1 \) for the sets \( C_2, \ldots, C_n \). We define \( h_i : S^3 \to S^3 \) by \( h_i(x) = h_i h_2 \cdots h_n(x) \).

Remark. If \( G \) is the upper semicontinuous decomposition of \( S^3 \) whose only nondegenerate element is a polyhedral 2-sphere, then \( S^3/G \) has the Dehn's Lemma property but \( S^3/G \) is not homeomorphic with \( S^3 \).

The essential ideas of the proof of the following theorem are so like those of the proof of Theorem 4 that we shall not include the proof here.

**Theorem 5.** If \( G \) is a point-like 0-dimensional decomposition of \( S^3 \), then \( S^3/G \) is homeomorphic with \( S^3 \) if and only if \( S^3/G \) has the map separation property.

5. Decompositions of \( S^3 \) which yield \( S^3 \). Let \( S, T \) be polyhedral solid tori such that \( S \subset \text{Int} \ T \) and let \( J \) be a polygonal center curve of \( S \). Following a definition of Schubert [13] which was used in [7], we let \( N(S, T) \) be the \( \min_{D}(N(J \cap D)) \); where \( D \) is a polyhedral meridional disk of \( T \) and \( N(J \cap D) \) is the number of points in \( J \cap D \).

**Theorem 6.** If \( G \) is definable by cubes with one hole and \( S^3/G \)
is homeomorphic to $S^3$, then $G$ is point-like.

Proof. Let $M_1, M_2, \ldots$, be the defining sequence for $G$ and let $T_0$ be a component of some $M_n$. By hypothesis, $T_0$ is a cube with one hole. Let $g$ be a component of $\bigcap_{i=1}^{\infty} M_i$ contained in $T_0$. We first show that there is a defining stage $M_{n+m}$ such that each loop in the component of $M_{n+m}$ containing $g$ can be shrunk to a point in $T_0$.

For $i = 1, 2, 3, \ldots$, let $T_i$ be the component of $M_{n+i}$ that contains $g$. Then each $T_i$ is a cube with one hole, $T_{i+1} \subset \text{Int} T_i$, and $\bigcap_{i=1}^{\infty} T_i = g$. Suppose that there is a positive integer $s$ such that each $T_j$, $j \geq s$, is a solid torus. If the center curve of each $T_{j+1}$ cannot be shrunk to a point in $T_j$, then $g$ has nontrivial Čech cohomology, and it follows from Corollary 2 of [8] that $S^3/G$ is not homeomorphic to $S^3$, contradicting our hypothesis. Hence there is an $m$ such that the center curve of $T_m$ can be shrunk to a point in $T_0$, and hence each loop in $T_m$ can be shrunk to a point in $T_0$.

Suppose then that infinitely many of the $T_i$ are not solid tori. We may suppose for convenience that each $T_i$ is not a solid torus. By [1], each $T_i' = S^3 - \text{Int} T_i$ is a solid torus. We now have three cases.

Case I. Suppose there is an $m$ such that $N(T_{m-1}', T_m') = 0$. This implies that there is a meridional disk $D$ of $T_m'$ such that $D \cap T_{m-1}' = \emptyset$. Then there is a cube $K$ in $T_m'$ such that $T_m' \subset \text{Int} K$. It then follows that each loop in $T_m(= S^3 - \text{Int} T_m')$ can be shrunk to a point in $T_0$.

We now show that the remaining two cases cannot occur.

Case II. Suppose that there is a positive integer $s$ such that $N(T_j', T_{j+1}') = 1$ for $j \geq s$. Since $P(\bigcap_{i=1}^{\infty} M_i)$ is 0-dimensional there is a positive integer $t$ and a cube $K$ such that $P(T_{s+t}) \subset \text{Int} K \subset K \subset P(\text{Int} T_s)$. Let $D_{s+t}$ be a meridional disk of $T_{s+t}'$. Using Dehn's Lemma we may adjust $P(D_{s+t}')$ in $P(\text{Int} T_{s+t})$ so that it is polyhedral, and it follows that $P(T_{s+t}')$ is a solid torus with the adjusted $P(D_{s+t}')$ as a meridional disk. Let $J$ be a longitudinal simple closed curve of $T_{s+t}'$ such that $J \subset \text{Bd} T_{s+t}'$ and $J$ intersects $\text{Bd} D_{s+t}'$ at just one point. Let $A$ be an annulus with boundary components $A_1$ and $A_2$. By [13], $N(T_s', T_{s+t}') = 1$. Hence there is a mapping $f$ of $A$ into $T_{s+t}'$ such that $f \mid A_i$ is a homeomorphism, $f(A_1) = J$, and $f(A_2) \subset T_s'$. Now $P(f(A_2))$ can be shrunk to a point missing $K$ since it is contained in $S^3 - K$; hence $P(f(A_2))$ can be shrunk to a point in $P(T_{s+t}')$. But this implies that the longitudinal simple closed curve $P(J)$ of $P(T_{s+t}')$ can be shrunk to a point in $P(T_{s+t}')$. Hence Case II cannot occur.
Case III. Now assume there is a positive integer $s$ such that $N(T'_j, T'_{j+1}) > 1$ for $j \geq s$. Since each $T'_j$ is knotted in $S^3$, we may use an argument similar to that used in [7] to conclude that Case III cannot occur.

These three cases now imply that there is a defining stage $M_{n+m}$ such that each loop in the component of $M_{n+m}$ containing $g$ can be shrunk to a point in $T_0$. Since $T_0 \cap (\bigcup_{i=1}^n M_i)$ is compact, there is a defining stage $M_p(p \geq n+m)$ such that each loop in $T_0 \cap M_p$ can be shrunk to a point in $T_0$. By Lemma 1 there is a cube with handles $C$ such that $T_0 \cap M_p \subset \text{Int } C \subset C \subset \text{Int } T_0$. It then follows that $G$ is definable by cubes with handles. By Bean's result [5], $G$ is a point-like decomposition, and the proof of Theorem 6 is complete.

**Corollary.** Let $f$ be a mapping of $S^3$ onto $S^3$ and let $H = \text{cl } \{x : x \in S^3 \text{ and } f^{-1}(x) \text{ is nondegenerate}\}$. If $H$ is a 0-dimensional set which is definable by cubes with one hole, then for each $x \in S^3$, $S^3 - f^{-1}(x)$ is homeomorphic to $E^3$.

*Proof.* Let $G = \{f^{-1}(x) : x \in S^3\}$. It is not hard to show that $G$ is an upper semicontinuous decomposition of $S^3$ and that $S^3/G$ is homeomorphic to $S^3$. Since $H$ is definable by cubes with one hole, it follows that $G$ is definable by cubes with one hole. By Theorem 6, $G$ is a point-like decomposition of $S^3$; hence if $x \in S^3$, then $S^3 - f^{-1}(x)$ is homeomorphic to $E^3$.

**References**


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