SOME COMPLEMENTED FUNCTION SPACES IN $C(X)$

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et $X$ and $Z$ be compact Hausdorff spaces, and let $P$ be a linear subspace of $C(X)$ which is isometrically isomorphic to $C(Z)$. In this paper conditions, some necessary and some sufficient, are presented which insure that $P$ is complemented in $C(X)$. For example if $X$ is metrizable, $P$ contains a strictly positive function, and the decomposition induced on $X$ by $P$ is lower semi-continuous then $P$ is complemented in $C(X)$.

D. Amir has shown that not all such spaces $P$ are complemented when $X$ is metrizable ([1], see also R. Arens, [4]). However, R. Arens [4] has constructed a class of subspaces of $C(X)$ which are complemented. In §2 we present classes of complemented subspaces which extend the class exhibited by R. Arens [Theorem 4, Lemma 5, Theorem 8]. A comparison of these results preceds Theorem 8.

Suppose that $X$ is the Stone-Čech compactification of a locally compact completely regular space $Y$, $Z$ is a compactification of $Y$ which has first countable remainder, and $P$ is the natural embedding of $C(Z)$ in $C(X)$. In §3 we show that if $P$ is complemented in $C(X)$, then $Y$ is pseudo-compact. This theorem was proved by J. Conway [6] for the case in which $Z$ is the one point compactification of $Y$.

By introducing the concept of weakly separating in §2, we are paralleling the concept of a Choquet boundary. Related results and definitions are found in [22].

1. If $A$ and $B$ are subsets of a topological space, $\text{cl } A$ will denote the closure of $A$, and $A-B$ will denote the set of points which are in $A$ but not in $B$. If $E$ is a normed linear space, $S(E)$ and $E^*$ denote the unit ball in $E$ and the dual of $E$ respectively. If $K$ is a convex subset of a topological vector space, $\text{ext } K$ will represent the set of extreme points of $K$. If $g$ and $h$ are functions such that the range of $g$ is contained in the domain of $h$, the composite of $g$ and $h$ will be written $h \circ g$. Finally, if $X$ is a topological space and $x$ is in $X$, the point evaluation functional associated with $x$ is the linear functional $x^*$ defined on $C(X)$ by $x^*(f) = f(x)$ for each $f$ in $C(X)$. In this paper $C(X)$ will denote the Banach space of all bounded real-valued continuous functions on $X$ normed with the supremum norm.

2. Let $P$ be a subspace of a normed linear space $E$. We define $D(P) = \{ b \in S(E^*) : b \text{ restricted to } P \text{ is in } \text{ext } S(P^*) \}$. We say that $P$ is weakly separating (with respect to $E$) if $P$ separates the points
of \( D(P) \) intersect \( \text{ext } S(E^*) \), that is, if \( g \) and \( h \) are distinct points in this intersection, then there is a \( p \) in \( P \) such that \( g(p) \neq h(p) \). Although we have stated the definition for an arbitrary normed linear space, we are mainly interested in the space \( E = C(X) \), where \( X \) is a compact Hausdorff space. It follows readily from the definition that a subspace \( P \) of \( C(X) \) is weakly separating if for any two distinct point evaluation functionals \( x' \) and \( y' \) whose restrictions to \( P \) have norm one, there is a \( p \) in \( P \) such that \( |p(x)| \neq |p(y)| \). In particular, a subspace of \( C(X) \) which contains the constants and separates the points of \( X \), or a closed ideal in \( C(X) \) is weakly separating.

**Lemma 1.** Let \( P \) be a subspace of \( E \). The following are equivalent:

(i) \( P \) separates the members of \( D(P) \)

(ii) \( P \) separates the members of \( D(P) \) intersect \( \text{ext } S(E^*) \)

(iii) \( \text{ext } S(E^*) \) contains \( D(P) \).

**Proof.** (iii) implies (i). If \( P \) does not separate the elements of \( D(P) \), then there must exist distinct elements \( g \) and \( h \) in \( D(P) \) such that the restriction of \( g - h \) to \( P \) is the zero functional. It follows that \( b = (1/2)(g + h) \) agrees with \( g \) and \( h \) on \( P \). Hence \( b \) is in \( D(P) \) but not in \( \text{ext } S(E^*) \).

(ii) implies (iii). Now suppose that \( P \) separates the elements of \( D(P) \) intersect \( \text{ext } S(E^*) \). Let \( b \) be a point in \( D(P) \). We are to prove that \( b \) is in \( \text{ext } S(E^*) \). Let \( K = \{ k \in S(E^*) : k \text{ agrees with } b \text{ on } P \} \). Clearly \( K \) is a convex set containing \( b \). Also \( K \) is closed, and hence compact, in the weak* topology on \( E^* \). By the Krien-Milman theorem, \( K \) has extreme points. We will show that \( \text{ext } K \) is contained in \( \text{ext } S(E^*) \). Suppose \( k = (1/2)(g + h) \) where \( k \) is in \( \text{ext } K \) and \( g \) and \( h \) are in \( S(E^*) \). Thus for each \( p \) in \( P \), \( 1/2h(p) + 1/2g(p) = k(p) = b(p) \). The restrictions of \( g \) and \( h \) to \( P \) both belong to \( S(P^*) \), and the restriction of \( b \) is in \( \text{ext } S(P^*) \). Therefore \( g \) and \( h \) agree with \( b \) on \( P \) and both must belong to \( K \). Since \( k \) was assumed to be an extreme point of \( K \), we have \( g = h = k \). We conclude that \( \text{ext } S(E^*) \) contains \( \text{ext } K \). If \( b \) is the only point in \( K \), then \( b \) must be in \( \text{ext } S(E^*) \). Otherwise \( K \) must contain two distinct extreme points. Clearly \( P \) can not separate these two points of \( D(P) \) intersect \( \text{ext } S(E^*) \). This proves that (ii) implies (iii).

Since the fact that (i) implies (ii) is obvious, the proof is complete.

**Lemma 2.** If \( P \) is weakly separating in \( E \), then the weak topology on \( D(P) \) induced by \( P \) is equivalent to the weak topology induced by \( E \).
Proof. Clearly, the weak topology induced by $P$ is coarser than the one induced by $E$. To prove the converse, suppose that $g_i$ is a net of functionals in $D(P)$ which converge with respect to the weak topology induced by $P$ to a functional $g$ which is also in $D(P)$. If $g_i$ does not converge to $g$ with respect to the weak topology induced by $E$, there will exist a subnet which never intersects some neighborhood (in topology induced by $E$) of $g$. Since by Alaoglu’s theorem $S(E^*)$ is compact, we may assume the existence of a further subnet $g_j$ which converges to a functional $h$ distinct from $g$. Since $g_j$ is a subset of $g$, $h$ must agree with $g$ on $P$. Since the norm of $h$ is less than or equal to one, $h$ is in $D(P)$. Since $P$ does not distinguish between $g$ and $h$, the previous lemma contradicts the hypothesis that $P$ is weakly separating. The lemma is proved.

In the following let $X$ be a compact Hausdorff space.

**Lemma 3.** Let $P$ be a weakly separating subspace of $C(X)$. The following are equivalent:

(i) There is a projection of norm one of $C(X)$ onto $P$,

(ii) $P$ is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space $Z$,

(iii) There exist a closed subset $Y$ of $X$ such that $P$ is isometrically isomorphic to $C(Y)$ via the restriction mapping.

Furthermore, if $P$ is weakly separating there can exist at most one projection of norm one of $C(X)$ onto $P$.

*Proof.* (i) implies (iii). Let $L$ be a projection of norm one of $C(X)$ onto $P$. If $x'$ is an evaluation functional in $D(P)$, then $x' \circ L$ is a functional in $S(C(X)^*)$ which agrees with $x'$ on $P$. Since $P$ is weakly separating in $C(X)$, $x' \circ L = x'$. Hence for each $f$ in $C(X)$, $Lf$ agrees with $f$ on $\{x \in X : x' \text{ is in } D(P)\}$, and therefore on the closure $Y$ of this set. With a simple application of the Tietze Extension Theorem, we see that the restriction map carries $P$ onto $C(Y)$. Furthermore, this restriction mapping does not decrease the norm of points in $P$. For by Lemma 1 every functional in $D(P)$ can be expressed as either an evaluation functional of a point in $Y$ or as the negative of such a functional, and for $p$ in $P$, $\|p\| = \sup \{h(p) : h$ in $D(P)\}$. We have shown that the restriction mapping is an isometric isomorphism of $P$ onto $C(Y)$.

(ii) implies (i). Let $Z$ be a compact Hausdorff space, and let $L$ be an isometric isomorphism of $P$ onto $C(Z)$. Let $L'$ denote the adjoint of $L$. Since $L$ is an isometric isomorphism, $L'$ is an isometric isomorphism of $C(Z)^*$ onto $P^*$. Furthermore, $L'$ restricted to ext $S(C(Z)^*)$ is a homeomorphism onto ext $S(P^*)$ with the weak topologies induced by $C(Z)$ and $P$ respectively. Now for $x$ in ext $S(P^*)$, let
$H(x)$ be the unique element in $\text{ext } S(C(X)^*)$ which agrees with $x$ on $P$. For $z$ in $Z$ let $E(z)$ denote the evaluation functional of $z$. Now for $f$ in $C(X)$ consider the function $f \circ H \circ L' \circ E(\cdot)$ defined on $Z$. By Lemma 2 this function is continuous. The map $Q$ which carries $f$ in $C(X)$ onto $L^{-1}(f \circ H \circ L' \circ E(\cdot))$ is a mapping of norm one of $C(X)$ into $P$. Furthermore, if $p$ is in $P$, then $p \circ H \circ L' \circ E(z) = Lp(z)$, for all $z$ in $Z$. Thus $p \circ H \circ L' \circ E(\cdot) = Lp$, and $Q$ is a projection of $C(X)$ onto $P$.

It is evident that (iii) implies (ii).

To prove the second part of the lemma, suppose that $H$ and $L$ are two projections from $C(X)$ onto $P$, both of which have norm one. Let $Y$ be the subset of $X$ constructed in the proof that (i) implies (iii). For any $f$ in $C(X)$, we have shown that $Lf$, $Hf$ and $f$ all agree on $Y$. It of course follows that $(H - L)(f)$ vanishes on $Y$. However, we have shown that the restriction mapping carries $P$ isometrically onto $C(Y)$. Therefore, $(H - L)(f)$ must be the zero function, and $Hf = Lf$ for all $f$ in $C(X)$. This completes the proof.

We will say that a subspace $P$ of $C(X)$ has a \textit{weakly separating quotient} if it has the property that for any two distinct points $x$ and $y$ in $X$ such that $p(x) = -p(y)$ for every $p$ in $P$, then the evaluation functional of $x$ (or equivalently the evaluational functional of $y$) restricted to $P$ is not an extreme point of $S(P^*)$.

\textbf{Remark.} Each of the following properties on a subspace $P$ of $C(X)$ imply that $P$ has a weakly separating quotient:

(i) $P$ is weakly separating in $C(X)$,

(ii) $P$ contains a function which is strictly positive,

(iii) for each $p$ in $P$, $|p|$ is also in $P$.

A proof for the above remark is straightforward. In particular, any closed ideal in $C(X)$, or any subspace of $C(X)$ which contains the constants has a weakly separating quotient.

In order to state the next theorem we make a few more definitions. Let $X$ be a Hausdorff space and let $M$ be a partition of $X$ into closed subsets. For $x$ in $X$ let $M(x)$ denote the member of $M$ which contains $x$. Corresponding to the standard definitions we say that $M$ is \textit{lower semi-continuous} if $\{x \in X: M(x) \text{ intersect } U \text{ is non-empty} \}$ is an open set in $X$ for every open set $U$ in $X$.

If $P$ is a linear space of bounded, continuous functions, then the $P$-\textit{partition} of $X$ is the partition associated with the following equivalence relation $R$. A couple $(x, y)$ is in $R$ if and only if $p(x) = p(y)$ for every $p$ in $P$. Now let $K(P) = \bigcup \{K \text{ contained in } X: K \text{ is a member of the } P\text{-partition of } X, \text{ and } K \text{ contains more than one point of } X\}$. We will say that $P$ has a \textit{lower semi-continuous quotient} if the restriction of the $P$-partition to $\text{cl } K(P)$ is lower semicontinuous.

In the following let $X$ denote a compact Hausdorff space, and let
Let $P$ be a linear subspace of $(C(X))$ which has a weakly separating quotient.

**Theorem 4.** If there is a projection of norm one of $C(X)$ onto $P$, then $P$ is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space $Z$. Conversely, suppose that $X$ is metrizable, and that $P$ has a lower semi-continuous quotient. If $P$ is isometrically isomorphic to $C(Z)$, for some compact Hausdorff space $Z$, then there is a projection of $C(X)$ onto $P$ which has norm less than or equal to three.

**Proof.** Let $M$ denote the $P$-partition of $X$. Let $X/M$ have the quotient topology, and let $M(·)$ denote the natural mapping of $X$ onto $X/M$. We observe that $X/M$ is a compact Hausdorff space. Now let $Q$ denote the linear subspace of $C(X)$ consisting of all functions that are constant on each closed subset of $X$ which is a member of $M$. One can verify that $P$ is contained in $Q$, and that the mapping which carries $q$ in $Q$ onto the function $q \circ M^{-1}(·)$ in $C(X/M)$ is an isometric isomorphism of $Q$ onto $C(X/M)$. The image $P'$ of $P$ under this mapping is a weakly separating subspace of $C(X/M)$ since $P$ has a weakly separating quotient. If there is a projection of norm one from $C(X)$ onto $P$, then there certainly is a projection of norm one from $C(X/M)$ onto $P$. By the preceding lemma, we conclude that $P'$, and hence $P$, is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space $Z$.

To prove the second part of the theorem, we assume that $X$ is metrizable, $P$ has a lower semi-continuous quotient, and that there is a compact Hausdorff space $Z$ such that $P$ is isometrically isomorphic to $C(Z)$. We maintain the same notation used directly above. Since $P'$ is weakly separating in $C(X/M)$, and $P$ is isometrically isomorphic to $C(Z)$, it follows from the preceding lemma that there is a projection of norm one from $Q$ onto $P$. To complete the proof it will suffice to show that there is a projection from $C(X)$ onto $Q$ which has norm less than or equal to three. We will prove a stronger result.

Let $Y$ be a metric space. Let $K$ be a partition of $Y$ such that every member of $K$ is a complete subset of $Y$. A member of $K$ will be called a plural set if it contains two distinct points of $Y$. Let the restriction $K'$ of $K$ to the subset of $Y$,

$$B = \text{cl} \cup \{A \text{ contained in } Y: A \text{ a plural set in } K\}$$

be lower semi-continuous. Assume also that $B/K'$ is paracompact. Let $Q$ denote the subspace of $C(Y)$ consisting of the functions which are constant on each member of $K$. We recall that by the notation we adopted, $C(Y)$ is the Banach space of all bounded continuous functions on $Y$. The following lemma establishes the theorem.
LEMMA 5. There is a projection of $C(Y)$ onto $Q$ which has norm less than or equal to three.

Proof. In the usual manner we can embed $B$ into the unit ball of $C(B)^*$. With the weak topology on $C(B)^*$ induced by $C(B)$, $C(B)^*$ is a locally convex space, $B$ is embedded onto a homeomorphic image of itself, say $B'$, and the closed convex hull of compact subsets of $B'$ are again compact. Let $s$ denote the composite of the quotient mapping of $B$ onto $B/K'$ with the homeomorphism, $h$, between $B$ and $B'$.

We now show that $s^{-1}$ is a lower semi-continuous function carrying points in $B/K'$ onto closed subsets of $B'$. Let $U$ be an open set in $B'$. Let

$$W = \{ y \in B/K': s^{-1}(y) \text{ intersect } U \text{ is not empty} \}.$$

To show that $s^{-1}$ is lower semi-continuous we must show that $W$ is open in $B/K'$. We note that $W = s(U)$. Now since $K'$ is lower semi-continuous and $h^{-1} \circ s^{-1} \circ s \circ h(\cdot)$ carries a point $b$ in $B$ onto the member of $K'$ which contains $b$, the set

$$V = \{ b \in B; h^{-1} \circ s^{-1} \circ s \circ h(b) \text{ intersect } h^{-1}(U) \text{ is not empty} \}$$

is open in $B$. Hence $h(V) = \{ b' \in B; h^{-1} \circ s^{-1} \circ s(b') \text{ intersect } h^{-1}(U) \text{ is not empty} \}$ is open in $B'$. Since this last set is $s^{-1} \circ s(U)$, $s^{-1} \circ s(U)$ is open. Since $B/K'$ has the quotient topology induced by $s$, this implies that $s(U)$—and hence $W$—is open in $B/K'$. Therefore $s^{-1}$ is lower semi-continuous.

Now since $B/K'$ is paracompact, and since there is a metric on $B'$ (which induces an equivalent topology for $B'$) for which the set $s^{-1}(y)$ is complete for each $y$ in $B/K'$, we have satisfied the hypothesis for a selection theorem proved by E. Michael [20]. This theorem proves the existence of a continuous function $t$ which carries $B/K'$ into $C(B)^*$, and has property that $t(y)$ is contained in the closed convex hull of $s^{-1}(y)$ for each $y$ in $B/K'$.

We now define a projection from $C(B)$ onto $Q'$ the subspace of functions in $C(B)$ which are constant on members of $K'$. For $f$ in $C(B)$, let $L_f$ denote the function such that for each $b$ in $B$,

$$(L_f)(b) = [t(s \circ h(b))(f)].$$

Since $t$ is continuous on $B/K'$, $L_f$ is a continuous function. Since $t(s \circ h(b))$ is in the closed convex hull of $s^{-1} \circ s \circ h(b)$, the norm of $t(s \circ h(b))$ does not exceed one. Thus the maximum of $L_f$ over $B$ does not exceed the maximum of $f$ over $B$. Finally, one can verify that if $q$ is in $Q'$, $L_q = q$, and that for each $f$ in $C(B)$, $L_f$ is in $Q'$. We have shown that $L$ is a projection of norm one of $C(B)$ onto $Q'$. 
Since $Y$ is a metric space, there is an operator $E$ of norm one from $C(B)$ into $C(Y)$ such that $R \circ E f = f$ for every $f$ in $C(B)$. Here $R$ denotes the operator which assigns to each function in $C(Y)$ its restriction to $B$ (R. Arens [3], also Dugundji [8]). Following a construction due to Arens [4], we define an operator $J$ by $Jf = f + E(LRf - Rf)$. The proof of the lemma is completed by verifying that $J$ is a projection of $C(Y)$ onto $Q$ which has norm no greater than three.

In the following corollaries let $X$ denote a compact Hausdorff space.

**Corollary 6.** Let $P$ be a finite dimensional subspace of $C(X)$ which has a weakly separating quotient. There is a projection of norm one from $C(X)$ onto $P$ if and only if $P$ has a basis $\{p_i\}_{i=1}^n$ such that $\| \sum_{i=1}^n c_i p_i \| = \max |c_i|$.

**Corollary 7.** $C(X)$ contains a weakly separating subspace of co-dimension $n$ which has a projection of norm one if and only if $X$ contains $n$ isolated points.

**Proof.** To prove the necessity of the condition, let $L$ be a projection of norm one of $C(X)$ onto a weakly separating subspace $P$ of co-dimension $n$ in $C(X)$. Define $Y = \text{cl} \{ x \in X : x' \circ L = x' \}$. We will show that $X - Y$ contains precisely $n$ points. Since $X - Y$ is open, these points will be isolated. We observe that the range, $Q$, of $I - L$ has dimension $n$, and that if $q$ is in $Q$, then $q$ vanishes on $Y$. Since the functions in $Q$ take all their nonzero values on $X - Y$, $X - Y$ must contain at least $n$ points. If $X - Y$ contained $n + 1$ points, there would exist $n + 1$ open sets $U_i$ in $X - Y$, and corresponding functions $f_i$ of norm one which vanish off $U_i$. These functions span an $n + 1$ dimensional subspace of $C(X)$; hence there is a nonzero function $f$ in this span that is also in $P$. But $f$ vanishes on $Y$. By Lemma 3, the restriction map is an isometry of $P$ onto $C(Y)$. Hence we arrive at the contradiction that $f$ is the zero function.

If $X$ contains $n$ isolated points, the space of all functions in $C(X)$ which vanish on these $n$ points is a weakly separating subspace of $C(X)$ (since this space is an ideal) of co-dimension $n$ in $C(X)$. It is also clear there is a projection of norm one from $C(X)$ onto this subspace. The proof is completed.

**Remark.** R. Arens [4] has constructed an example of two compact metric spaces $X$ and $Z$ such that $C(X)$ contains an isometric isomorphic copy of $C(Z)$ which has a weakly separating quotient, but which is not complemented in $C(X)$. Hence the assumption that $P$ has a lower semi-continuous quotient cannot be simply omitted from the theorem,
The preceding theorem and lemma should be compared to Theorem 2.2 in (R. Arens [4]). Using the notation preceding the lemma, Professor Arens proved that under the following conditions there will exist a projection of norm less than or equal to three of $C(Y)$ onto $Q$:

(i) $K$ is a partition of $Y$ into closed subsets
(ii) $Y$ and $Y/K$ are metrizable
(iii) the quotient map of $Y$ onto $Y/K$ is upper semi-continuous
(iv) if $\{x_i\}$ is a sequence in $Y$ such that each $x_i$ belongs to a distinct plural set in $K$, then a member of $K$ which contains a limit point of $\{x_i\}$ is a singleton.

Apropos to property (ii), A. H. Stone has proved ([23]) that a metrizable space is paracompact. Property (iv) above implies that $K'$ is lower semi-continuous. In the special case that $Y$ is a complete metric space, the preceding lemma contains the above theorem of Arens. If $Y$ is compact, the previous theorem includes both of these results.

In the following, let $Y$ be a metrizable space, and $K$ a partition of $Y$ satisfying properties (i), (iii), and (iv) above. For each $K_i$ in $K$ let $P_i$ be a complemented subspace of $C(K_i)$ which contains the constants. Let $L_i$ denote a projection of $C(K_i)$ onto $P_i$. We assume that $m = \sup \{||L_i||\} < \infty$. Finally, let $Q$ denote the subspace of $C(Y)$ consisting of all functions $q$ such that the restriction of $q$ to $K_i$ is a function in $P_i$.

**Theorem 8.** There is a projection of $C(Y)$ onto $Q$ which has norm less than or equal to $2 + m$.

**Proof.** For a set $Z$ let $B(Z)$ denote the space of bounded functions on $Z$. Let $D = \cup \{K_i$ contained in $Y$: $K_i$ is a plural set in $K\}$. Let $R$ and $R_i$ denote the restriction map of $B(Y)$ onto $B(\text{cl} \ D)$ and of $B(Y)$ onto $B(K_i)$ respectively ($K_i$ in $K$). Let $E$ denote a linear mapping of $C(\text{cl} \ D)$ into $C(Y)$ such that $E$ has norm one, and $R_i \circ E$ is the identity mapping on $C(\text{cl} \ D)$. Let $H$ be the linear mapping of $C(Y)$ into $B(\text{cl} \ D)$ such that $R_i \circ H = L_i \circ R_i$ for all $K_i$ in $K$. Let $I$ denote the identity on $C(Y)$, and let $L = I + E \circ R(H - I)$. The proof consists of establishing that $L$ is the desired projection. The variation of a function $f$ defined on a set $Z$ is $\text{var}(f) = \max_{z \in Z} f(z) - \min_{z \in Z} f(z)$.

We proceed by proving four assertions, the last of which establishes the theorem.

**Assertion 1.** If $x_i$ is in $K_i$, $K_i$ is in $K$, $y$ is not in $D$ and $x_i$ con-

Professor Arens has communicated that the assumption that the quotient mapping be upper semi-continuous had been inadvertently omitted from the statement of his theorem.
verges to \( y \), then \( \text{var} (R_i f) \) converges to zero for each \( f \) in \( C(Y) \).

**Assertion 2.** \[ \| L_i \circ R_i f - R_i f \| \leq 1/2(1 + m) \text{var} (R_i f) . \]

**Assertion 3.** If \( f \) is in \( C(Y) \), \( Hf \) is in \( C(\text{cl} \ D) \).

**Assertion 4.** The operator \( L \) is a projection from \( C(Y) \) onto \( Q \) of norm at most \( 2 + m \).

If Assertion 1 is false it will be possible be find points \( z_i \) in \( K_i \) and a function \( f \) in \( C(Y) \) such that for some \( r \) greater than zero,
\[ f(x_i) - f(z_i) \] is greater than \( r \). Since \( f \) is continuous, we may assume that there is a neighborhood \( N \) of \( y \) such that \( z_i \) does not belong to \( N \). Put \( Z = \{ z_i \} \). Since the quotient map \( q \) of \( Y \) onto \( Y/K \) is, by hypothesis, closed \( g(\text{cl} \ Z) \) is closed in \( Y/K \). But \( q(x_i) = q(z_i) \) is in \( q(\text{cl} Z) \), and \( q(x_i) \) converges to \( q(y) \) by the continuity of \( q \). Thus \( q(y) = \{ y \} \) is in \( q(\text{cl} Z) \), and \( \{ y \} = q(z) \) for some \( z \) in \( \text{cl} \ Z \). But \( \text{cl} \ Z \) is contained in \( Y - N \) so \( z \neq y \). This contradicts the assumption that \( y \) is not in \( D \).

To prove the second assertion, let \( c = 1/2 \text{var} (R_i f) \). Since 1 is in \( P_i \), \( L_i \circ R_i 1 = 1 \). Hence

\[ \| L_i \circ R_i f - R_i f \| = \| L_i \circ R_i (f - c) - R_i (f - c) \| \leq \| L_i - I \| \]
\[ \cdot \| R_i (f - c) \| \leq (m + 1)(1/2) \text{var} (R_i f) . \]

To prove Assertion 3 let \( y \) be a point in \( \text{cl} \ D \). We distinguish two cases. Case 1, \( y \) is in \( D \). Let \( y \) be in the plural set \( K_i \) of the partition \( K \). From the assumption of property (iv) it follows that there is an open set \( U \) containing \( K_i \) which meets no other plural set in \( K \). Now let \( f \) be in \( C(Y) \) and let \( N \) be a neighborhood of \( Hf(y) \). Let \( V \) be a neighborhood of \( y \) such that \( (L_i \circ R_i f)(V \cap K_i) \) is contained in \( N \). Put \( W = V \cap U \) and let \( x \) be an arbitrary point in \( W \) intersect \( \text{cl} \ D \). Then \( x \) is in \( U_i \) and \( x \) is in the closed set \( K_i \). This shows that \( W \cap \text{cl} \ D \) is contained in \( K_i \cap V \). Hence on \( W \cap \text{cl} \ D \), \( Hf = L_i \circ R_i f \). Thus \( Hf(W \cap \text{cl} \ D) \) is contained in \( L_i \circ R_i f(K_i \cap V) \) which in turn is contained in \( N \).

Case 2, \( y \) is not in \( D \). In this case \( \{ y \} \) is in \( K_i \), and \( Hf(y) = f(y) \), since each \( P_i \) contains the constant functions. Let \( x_i \) converge to \( y \). Then

\[ | Hf(x_i) - Hf(y) | \leq | Hf(x_i) - f(x_i) | + | f(x_i) - f(y) | . \]

It is clear that \( f(x_i) \) converges to \( f(y) \). For the other term we use Assertions 1 and 2 above to write, with \( x_i \) in \( K_i \) (and \( K_i \) in \( K \)),
\[ |Hf(x_i) - f(x_i)| \leq |L_i \circ R_i f(x_i) - R_i f(x_i)| \]
\[ \leq (1/2)(m + 1) \var (R_i f) . \]

Since this last term converges to zero, \( Hf \) is continuous at \( y \).

To prove Assertion 4, we first observe that linearity and bound for \( L \) are obvious. If \( f \) is in \( C(Y) \) we must show that \( Lf \) is in \( Q \). Indeed,

\[ R \circ L = R + R \circ H - R = R \circ H . \]

Hence

\[ R_i \circ L = R_i \circ R \circ L = R_i \circ R \circ H = L_i \circ R_i , \]

for each plural set \( K_i \) in \( K \). Thus \( R_i \circ Lf \) is in \( P_i \) for each plural set \( K_i \) in \( K \). If \( K_i \) is a member of \( K \) which is not a plural set then, \( R_i \circ Lf \) is in \( P_i \) trivially since \( P_i \) contains the constants.

Now we must show that if \( f \) is in \( Q \) then \( Lf = f \). Since \( R_i f \) is in \( P_i \) for all \( K_i \) in \( K \), \( R_i \circ Hf = L_i \circ R_i f = R_i f \). Thus \( R \circ Hf = Rf \), and \( Lf = f + E(Rf - Rf) = f \). This completes the proof of the theorem.

REMARK. The assumption that \( Y \) is metrizable was used only to guarantee the existence of the linear mapping \( E \). If we drop the hypothesis that \( Y \) is metrizable and assume outright the existence of a bounded linear mapping \( E \) from \( C(\text{cl} D) \) into \( C(Y) \) such that \( R \circ E \) is the identity on \( C(\text{cl} D) \), then the same proof establishes the existence of a projection from \( C(Y) \) onto \( Q \) which has norm less than or equal to \( 1 + (m + 1) \| E \| . \)

COROLLARY 9. Let \( Y, K, K_i, P_i, \) and \( Q \) be as in the theorem. If each \( P_i \) has dimension less than \( n \), then there is a projection of norm at most \( n + 1 \) from \( C(Y) \) onto \( Q \).

3. Let \( X \) be a locally compact, Hausdorff space. A compactification of \( X \) is a compact Hausdorff space that contains \( X \) (a homeomorphic image of \( X \)) as a dense subspace. The Stone-Čech compactification of \( X \) will be denoted by \( \beta X \), and the one-point compactification will be denoted by \( pX \).

If \( K \) is an arbitrary compactification of \( X \), the linear mapping which carries a function in \( C(K) \) onto the unique function in \( C(\beta X) \) which agrees with it on \( X \), is an isometric isomorphism of \( C(K) \) into \( C(\beta X) \). We will therefore assume that \( C(\beta X) \) contains \( C(K) \).

If \( Y \) is a closed subset of a compact Hausdorff space \( K \), \( I_r \) will denote the ideal of functions in \( C(K) \) which vanish on \( Y \). Let \( N \) denote the non-negative integers with the discrete topology. If \( K \) is
a compactification of $X$, the remainder of $K$ (with respect to $X$) is the topological space $K - X$ equipped with the relative topology from $K$. In accordance with the usual terminology let $(m) = C(\beta N)$, $(c) = C(pN)$, and $(c_0) = I_{pN-N} = I_{\beta N-N}$, where the ideals are interpreted as subspaces of $C(pN)$ and $C(\beta N)$ respectively.

**Theorem 10.** Let $K$ be a compactification of $X$ which has a first countable remainder. If there is a bounded linear mapping of $C(\beta X)$ into $C(K)$ which acts as the identity on $I_{\beta X-X}$, then $X$ is pseudocompact.

We first will prove the following lemma.

**Lemma 11.** Let $M$ be a compactification of $N$ which has a first countable remainder. There does not exist a bounded linear mapping of $(m)$ onto any subspace of $C(M)$ which contains $(c_0)$.

**Proof of lemma.** Since $N$ is both locally compact and the union of a countable family of compact sets, $M - N$ is a compact set which is the intersection of a countable family $U$ of open sets in $M$. Let $x$ be a point in $M - N$. Let $V$ be a countable family of open sets in $M$ whose intersections with $M - N$ form a basis for the neighborhood system for $x$ in $M - N$. Let $W$ be the countable family of open sets in $M$ of the form $u$ intersect $v$, where $u$ is in $U$ and $v$ is in $V$. It is easy to see that the intersection of the members of $W$ is the singleton containing $x$. A compactness argument shows that $W$ is in fact a basis for the neighborhood system for $x$ in $M$. Since $N$ is first countable we have established that $M$ is first countable. Hence $M$ is sequentially compact.

There is a sequence of points in $N$, say $J$, which converges to some point $k$ in $M$. Now suppose $B$ is a subspace of $C(M)$ which contains $(c_0)$. The restriction of functions in $B$ to $J$ union $\{k\}$ carries $B$ onto a Banach space which is either isometrically isomorphic to $(c)$ or to $(c_0)$. In the former case since $(c_0)$ is complemented in $(c)$, there is a bounded linear mapping of $B$ onto $(c_0)$. In either case if there is a bounded linear mapping of $(m)$ onto $B$, there is a bounded linear mapping, $L$, of $(m)$ onto $(c_0)$. But no such mapping can exist. For since $(c_0)$ is a separable Banach space and $\beta N$ is extremally disconnected, $L$ must be weakly compact (Grothendieck [14], p. 168, Cor. 1). Now an application of the open mapping theorem implies the false assertion that $(c_0)$ is reflexive. This completes the proof of the lemma.

**Proof of theorem.** If $X$ is not pseudocompact there is countable family of disjoint open sets $V_i$ in $X$ such that $\text{cl} \cup \{V_i\} = \cup \{\text{cl} V_i\}$. For each $i$ let $U_i$ be an open set such that $\text{cl} U_i \subseteq V_i$, let $u_i$ be in $U_i$.
and let \( f_i \) be a continuous function which vanishes off \( U_i \) and attains its norm of one at \( u_i \). For a bounded sequence \( x = (x_1, x_2, \ldots) \) in \((m)\), let \( Ax \) be the unique function in \( C(\beta X) \) which agrees with \( \sum_{i=1}^{\infty} x_i f_i \) on \( X \). The mapping \( A \) is an isometric isomorphism of \((m)\) onto the range of \( A \). Let \( L \) be the hypothesized mapping of the theorem, and let \( J \) carry a function in \( C(\beta X) \) onto its restriction to \( \text{cl} \{ u_i \} \). Since \( \text{cl} \{ u_i \} - \{ u_i \} \) is contained in \( K - X \), \( \text{cl} \{ u_i \} \) is homeomorphic to a compactification \( M \) of \( N \) which has first countable remainder. Let \( G \) be the isometric isomorphism of \( C(\text{cl} \{ u_i \}) \) onto \( C(M) \) induced by this homeomorphism. The proof is completed by verifying that \( G \circ J \circ L \circ A \) is a bounded linear mapping of \((m)\) onto a subspace of \( C(M) \) which contains \((c_0)\).

The case in which \( K \) is the one-point compactification of \( X \) was first proved by J. Conway ([6]). Examples to show that pseudocompactness of \( X \) is not sufficient to guarantee the existence of a projection from \( C(\beta X) \) onto \( I_{\beta X - X} \) have been constructed by J. Conway ([6]) and by A. Pelczynski and V. N. Sudakov ([21]).

**Corollary 12.** Let \( X \) be an extremally disconnected, compact, Hausdorff space, and let \( P \) be a subspace of \( C(X) \) which contains the constants and separates the points of \( X \). If \( P \) is isometrically isomorphic to \( C(Z) \) for some compact Hausdorff space \( Z \), then the Šilov boundary of \( P \) is an extremally disconnected subset of \( X \) which has a pseudo-compact complement.

**Proof.** Under the hypothesis of the corollary, the Šilov boundary of \( P \) is the set \( Y \) of Lemma 3. To show that \( Y \) is extremally disconnected, we intend to apply a theorem due to Nachbin (Trans. AMS, 68 (1950), 28-46, 1950), Goodner ([13]), Kelley ([11]) and others. A Banach space \( B \) is called injective if every Banach space which contains an isometric isomorphic copy \( B' \) of \( B \), admits a projection of norm one onto \( B' \). The theorem we wish to apply states that a Banach space is injective if and only if it is isometrically isomorphic to \( C(Z) \), for a compact, extremally disconnected, Hausdorff space \( Z \). Now \( C(X) \) is injective and from Lemma 3 there is a projection of norm one from \( C(X) \) onto \( P \). From this it can be shown that \( C(Y) \) is injective, and hence \( Y \) is extremally disconnected.

From Lemma 3 it follows that \( I_Y \) is complemented in \( C(X) \). Let \( G = X - Y \). Since \( \text{cl} G \) is open in \( X \), \( I_{\text{cl} G - G} \) is complemented in \( C(\text{cl} G) \). Since \( \text{cl} G \) is extremally disconnected, it is the Stone-Čech compactification of \( G \) ([10], p. 69, Prob. 6M2). By the theorem, \( G \) is pseudocompact (in this case \( K \) is the one-point compactification of \( G \)), and the corollary is proved.
COROLLARY 13. If $X$ is a locally compact space such that $\beta X$ has a first countable remainder, then $X$ is pseudocompact.

REMARK. Relevant to the last corollary, we observe that if $Z$ is any compact Hausdorff space, there is a pseudocompact, locally compact space $X$ such that $\beta X - X$ is homeomorphic to $Z$. For let $y$ be a nonisolated point in $\beta N$ and let $X = (\beta N - \{y\}) \times Z$. From results in ([11]) and ([10], 6M3) we have that $X$ is pseudocompact, and $\beta X = \beta N \times Z$.

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Duane W. Bailey, *On symmetry in certain group algebras* .......................... 413
Lawrence Peter Belluce and Surender Kumar Jain, *Prime rings with a one-sided ideal satisfying a polynomial identity* ........................................ 421
L. Carlitz, *A note on certain biorthogonal polynomials* .......................... 425
Charles O. Christenson and Richard Paul Osborne, *Pointlike subsets of a manifold* .......................................................... 431
Russell James Egbert, *Products and quotients of probabilistic metric spaces* .......................................................... 437
Moses Glasner, Richard Emanuel Katz and Mitsuru Nakai, *Bisection into small annuli* .......................................................... 457
Karl Edwin Gustafson, *A note on left multiplication of semigroup generators* .......................................................... 463
I. Martin (Irving) Isaacs and Donald Steven Passman, *A characterization of groups in terms of the degrees of their characters. II* .............. 467
Howard Wilson Lambert and Richard Benjamin Sher, *Point-like 0-dimensional decompositions of S3* ............................................... 511
Oscar Tivis Nelson, *Subdirect decompositions of lattices of width two* .... 519
Ralph Tyrrell Rockafellar, *Integrals which are convex functionals* .......... 525
James McLean Sloss, *Reflection laws of systems of second order elliptic differential equations in two independent variables with constant coefficients* ............................................... 541
Bui An Ton, *Nonlinear elliptic convolution equations of Wiener-Hopf type in a bounded region* ............................................... 577
Daniel Eliot Wulbert, *Some complemented function spaces in C(X)* ...... 589
Zvi Ziegler, *On the characterization of measures of the cone dual to a generalized convexity cone* ............................................... 603