A RADICAL FOR LATTICE-ORDERED RINGS

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The main result of this paper states that for a lattice-ordered ring (l-ring) $A$ with no nonzero nilpotent $l$-ideals the following are equivalent: (i) $A$ is an $f$-ring; (ii) $A$ is a subdirect union of totally-ordered rings with no nonzero divisors of zero; (iii) $x^+x^- = 0$ for all $x \in A$; (iv) $x^+ax^- = 0$ for all $x, a \in A$; and (v) $a(b \lor c) = ab \lor ac$ and $(b \lor c)a = ba \lor ca$ for all $a, b, c \in A$ with $a \geq 0$. In particular, the equivalence of (i) and (iii) implies that an $l$-ring which has an identity that is a weak order unit and which has no nonzero nilpotent $l$-ideals is necessarily an $f$-ring.

The basic tool in our considerations is the notion of prime $l$-ideal. Specifically, call a proper $l$-ideal $P$ of an $l$-ring $A$ prime if $I \subseteq P$ or $J \subseteq P$ whenever $I$ and $J$ are $l$-ideals of $A$ with $IJ \subseteq P$. Various conditions are obtained on $A$, each of which forces $A$ modulo every prime $l$-ideal to be totally-ordered with no nonzero divisors of zero. Moreover the relationship between the join of all the nilpotent $l$-ideals of $A$ and the intersection of all the prime $l$-ideals of $A$ is investigated in order to obtain the theorem mentioned above.

The $P$-radical of an $l$-ring $A$ is the intersection of all the prime $l$-ideals of $A$. In § 2 the general theory of the $P$-radical is considered. The results here are analogous to ring theoretic results found in McCoy [4] and Jacobson [2] (Chapter VIII).

In § 3 the general theory of the $P$-radical which is more or less independent of the order structure is tied together with the order. Specifically we investigate the relationship between the $P$-radical and the join of all of the nilpotent $l$-ideals for various classes of $l$-rings. § 4 contains a proof of the theorem mentioned above.

2. Prime $l$-ideals and the $P$-radical. The results of this section are analogous to ring theoretic results found in McCoy [4] and Jacobson [2] (Chapter VIII). Consequently after proving a few of the results in detail, we sketch proofs indicating the idiosynrasies they take on in $l$-rings and note the analogous result in McCoy or Jacobson.

The reader is referred to Birkhoff and Pierce [1] and Johnson [3] for the general theory of $l$-rings. Our notation is the same as Johnson [3]. Also, the word $l$-ideal, unmodified means proper $l$-ideal.

DEFINITION 2.1. (i) An $l$-ideal $P$ of an $l$-ring $A$ is prime if $I \subseteq P$ or $J \subseteq P$ whenever $I$ and $J$ are $l$-ideals of $A$ with $IJ \subseteq P$.

(ii) A nonzero $l$-ring $A$ is prime if $\{0\}$ is a prime $l$-ideal.
(iii) A nonzero $l$-ring $A$ is an $l$-domain if $A^+\setminus\{0\}$ is closed under multiplication.

**Remark.** If $I$ and $J$ are $l$-ideals of an $l$-ring $A$, then $IJ$ denotes the ring theoretic product of the ideals $I$ and $J$. Note that $IJ$ is not, in general, an $l$-ideal. We can "make $IJ$ into an $l$-ideal" by forming $\langle IJ \rangle$, the smallest $l$-ideal containing $IJ$. Birkhoff and Pierce [1] have denoted this by $I \cdot J$ and called it the $l$-product of $I$ and $J$. As we shall have occasion to use the notation $\langle S \rangle$ for the $l$-ideal generated by a subset $S$ of an $l$-ring $A$, we use the notation $\langle IJ \rangle$ for the $l$-product of two $l$-ideals $I$ and $J$. Note that if $I$, $J$, and $P$ are $l$-ideals of $A$, then $IJ \subseteq P$ if and only if $\langle IJ \rangle \subseteq P$; and hence the definition of prime $l$-ideal is independent of the choice of $IJ$ or $\langle IJ \rangle$.

To set the situation we note that a prime $l$-ideal need not be prime as a ring ideal. In fact, a prime $l$-ideal of an archimedean commutative $l$-ring in which the square of every element is positive need not be prime as a ring ideal (See 2.3 below.). However, Johnson [3] has shown.

**Theorem 2.2.** Let $A$ be an $f$-ring, and let $P$ be an $l$-ideal of $A$. Then the following are equivalent:

(i) $A/P$ is totally-ordered with no nonzero divisors of zero;

(ii) $P$ is prime as a ring ideal; and

(iii) $P$ is a prime $l$-ideal.

In § 4 we generalize 2.2 to several classes of $l$-rings each of which properly contains the class of $f$-rings.

**Example 2.3.** A prime $l$-ideal of an archimedean commutative $l$-ring in which the square of every element is positive which is not prime as a ring ideal.

Let $S$ be the semigroup consisting of two elements $a$ and $b$ with multiplication $ab = ba = a^2 = b^2 = a$, and let $R(S)$ denote the semigroup ring on $S$ with real coefficients. Make $R(S)$ into an archimedean commutative $l$-ring by decreeing that $\alpha a + \beta b \geq 0$ if $\alpha \geq 0$ and $\beta \geq 0$ where $\alpha$ and $\beta$ are real numbers. Then the square of every element of $R(S)$ is positive since $(\alpha a + \beta b)^2 = (\alpha + \beta)^2 a$. Now, $\{0\}$ is not prime as a ring ideal since $(a-b)^2 = 0$. However, it is easy to see that $R(S)$ is an $l$-domain, and hence $\{0\}$ is a prime $l$-ideal by the next result.

2.4. If $P$ is $l$-ideal of an $l$-ring $A$ such that $A^+\setminus P$ is closed

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1 An $f$-ring is an $l$-ring in which $a \land b = 0$ and $c \geq 0$ imply $ca \land b = 0$ and $ac \land b = 0$. In [1] Birkhoff and Pierce showed that the class of $f$-rings is identical with the class of subdirect unions of totally-ordered rings.
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under multiplication, then $P$ is a prime $l$-ideal. The converse holds is $A$ is commutative.

Proof. First suppose that $I$ and $J$ are $l$-ideals of $A$ with $IJ \subseteq P$. If $I$ is not contained in $P$, then there is a non-zero positive element $a \in I \setminus P$. Let $b$ be a positive element of $J$. Then $ab \in IJ \subseteq P$, so that $b \in P$ since $a \in P$. It follows that $J \subseteq P$.

Now suppose that $A$ is commutative, $P$ is a prime $l$-ideal of $A$, and $a_i, a_z \in A^+$ with $a_ia_z \in P$. Then $\langle a_ia_z \rangle \subseteq P$. Let $z_i \in \langle a_i \rangle$, $i = 1, 2$. Then $|z_i| \leq n_i a_i + r_z a_i$ $(i = 1, 2)$ for suitable $r_z \in A^+$ and suitable nonnegative integers $n_i$. Thus

$$|z_1 z_2| \leq |z_1| |z_2| \leq (n_1 a_1 + r_1 a_1)(n_2 a_2 + r_2 a_2)$$

which belongs to $P$ since $A$ is commutative and $\langle a_ia_z \rangle \subseteq P$. It follows that $\langle a_1 \times a_z \rangle \subseteq P$; and hence either $a_i \in P$ or $a_z \in P$.

The following characterization of prime $l$-ideals will be used repeatedly in the sequel.

2.5. An $l$-ideal $P$ of an $l$-ring $A$ is prime if and only if $a, b \in A^+$ and $aA^+b \subseteq P$ imply $a \in P$ or $b \in P$.

Proof. Necessity. From $aA^+b \subseteq P$ it follows that

$$\langle A^+aA^+ \times A^+bA^+ \rangle \subseteq P.$$

Thus either $\langle A^+aA^+ \rangle \subseteq P$ or $\langle A^+bA^+ \rangle \subseteq P$. Suppose that $\langle A^+aA^+ \rangle \subseteq P$. Then $\langle a \rangle^3 \subseteq P$, and hence $\langle \langle a \times a \rangle \times a \rangle \subseteq P$. Thus either $\langle a \rangle^3 \subseteq P$ or $\langle a \rangle \subseteq P$. In either case we have that $a \in P$.

Sufficiency. If $I$ and $J$ are $l$-ideals of $A$ which are not contained in $P$, then there is an $a \in I \setminus P$ and a $b \in J \setminus P$. If $IJ \subseteq P$, then $aA^+b \subseteq IJ \subseteq P$; so that $a \in P$ or $b \in P$. Since this contradicts the choice of $a$ and $b$, $IJ$ is not contained in $P$; and we are done.

Note that 2.5 says that an $l$-ideal $P$ of an $l$-ring $A$ is prime if and only if $A^+\setminus P$ is an $m$-system in the sense of

DEFINITION 2.6. A nonempty subset $M$ of an $l$-ring $A$ is an $m$-system if each element of $M$ is positive and if for $a, b \in M$ there is an $x \in A^+$ such that $axb \in M$.

Note that nonempty subset $S$ of $A^+$ which is closed under multiplication is an $m$-system since $aab \in S$ whenever $a, b \in S$.

The next result, as did the proceeding, has its analogue in [4].

2.7. Let $M$ be an $m$-system of an $l$-ring $A$, and let $I$ be an $l$-
ideal of \( A \) that does not meet \( M \). Then \( I \) is contained in a prime
\( l \)-ideal that does not meet \( M \).

\textbf{Proof.} The existence of an \( l \)-ideal \( P \) of \( A \) which is maximal with
respect to the property of not meeting \( I \) is guaranteed by Zorn’s
Lemma. We show that \( P \) is prime. The proof of this is an in [4]
(Lemma 4) once one knows that the \( l \)-ideal generated by \( P \) and a
positive element \( a \) of \( A \) not in \( P \) is \( \{ z \in A : |z| \leq p + na + ra + sa + tav \}
where \( r, s, t, v \in A^+, p \in P^+, \) and \( n \) is a nonnegative integer).

\textbf{Definition 2.8.} The \( P \)-radical, \( P(A) \), of an \( l \)-ring \( A \) is the
intersection of all of the prime \( l \)-ideals of \( A \).

Recall that the \( l \)-radical of an \( l \)-ring \( A \) is the set \( N(A) = \{ a \in A:
there is a positive integer \( n = n(a) \) such that
\[
 x_0 | a | x_1 | a | x_2 \cdots x_{n-1} | a | x_n = 0
\]
for all \( x_0, x_1, x_2, \ldots, x_n \in A \} ([1], p. 45.) If \( A \) is comutative, then
\( N(A) = \{ a \in A : |a| \) is nilpotent} ([1], Corollary 1, p. 45). Moreover,
for an arbitrary \( l \)-ring \( A \), \( N(A) \) is the join of all of the nilpotent \( l \)-
ideals of \( A \) ([1], Th. 5).

Now suppose that \( a \in A \) is not nilpotent. Then since \( 0 < |a^n| \leq |a|^n \) for all \( n \), \( |a| \) is not nilpotent. Thus, by 2.7, there is a prime
\( l \)-ideal \( P \) of \( A \) not meeting the \( m \)-system \( \{|a|, |a|^2, \ldots, |a|^n, \ldots\} \). It
follows that \( a \) does not belong to \( P(A) \), and hence every element of
\( P(A) \) is nilpotent. Now note that every prime \( l \)-ideal of \( A \) contains
every nilpotent \( l \)-ideal of \( A \), and hence we have

\textbf{2.9.} The \( P \)-radical of an \( l \)-ring \( A \) is a nil \( l \)-ideal of \( A \) containing
the \( l \)-radical of \( A \).

The proof of the next result is as in [4] (Theorem 5).

\textbf{2.10.} If \( A \) is an \( l \)-ring, then \( P(A/P(A)) \) is zero.

The next result is useful in relating the \( l \)-radical to the \( P \)-radical.

\textbf{2.11.} Let \( I \) be an \( l \)-ideal of an \( l \)-ring \( A \) such that \( N(A/I) \) is
zero, and let \( J \) be an \( l \)-ideal of \( A \) properly containing \( I \). Then there
is a prime \( l \)-ideal \( P \) of \( A \) containing \( I \) but but not containing \( J \).

\textbf{Proof.} (After Jacobson, [2], p. 196) Choose \( a_0 \in J^{-}/I \). Then since
\( N(A/I) \) is zero, \( A/I \) has no nonzero nilpotent \( l \)-ideals; and hence \( \langle a_0 \rangle^k \)
is not contained in \( I \) for any positive integer \( k \). Now, \( \langle A^+ a_0 A^+ \rangle^k \) is
not contained in \( I \) since \( \langle \alpha_0 \rangle^3 \subseteq \langle A^+a_0A^+ \rangle \) and \( \langle \alpha_0 \rangle^6 \) is not contained in \( I \). Now suppose that \( a_0b \alpha_0 \in I \) for all \( b \in A^+ \). Then for \( z \in \langle A^+a_0A^+ \rangle^6 \), there are \( x_i, y_i \in \langle A^+a_0A^+ \rangle \) and \( t_i, u_i, v_i, w_i \in A^+ \) such that

\[
|z| \leq \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} (t_ia_0u_i)(v_ia_0w_i).
\]

But \( a_0u_i v_i a_0 \in I^+ \), so that \( z \in I \). Consequently there is a \( b_0 \in A^+ \) such that \( a_i = a_0b_0a_0 \in J^+ \setminus I \). Similarly, there is a \( b_i \in A^+ \) such that \( a_i = a_i b_i a_i \in J^+ \setminus I \). Containing inductively, we obtain two sequences: \( \{a_i\}_{i=0}^{\infty} \subseteq J^+ \setminus I \) and \( \{b_i\}_{i=0}^{\infty} \subseteq A^+ \) such that \( a_n = a_{n-1}b_{n-1}a_{n-1} \in J^+ \setminus I \) for all \( n \geq 1 \). It follows that \( \{a_i\}_{i=0}^{\infty} \) is an \( m \)-system that does not meet \( I \). By 2.7 there is a prime \( l \)-ideal \( P \) of \( A \) containing \( I \) that does not meet \( \{a_i\}_{i=0}^{\infty} \). Since \( a_i \in J \) for \( i \geq 0 \), we know that \( J \) is not contained in \( P \); and hence \( P \) is as desired.

2.12. If \( A \) is an \( l \)-ring, then \( P(A) = \bigcap \{I: I \text{ is an } l \text{-ideal of } A \text{ and } N(A/I) \text{ is zero}\} \).

**Proof.** Let \( \mathcal{L}(A) = \bigcap \{I: I \text{ is an } l \text{-ideal of } A \text{ and } N(A/I) \text{ is zero}\} \). If \( P \) is a prime \( l \)-ideal of \( A \), then \( N(A/P) \subseteq P(A/P) = \{0\} \). Thus \( \mathcal{L}(A) \subseteq P(A) \).

Now let \( J/\mathcal{L}(A) \) be a nilpotent \( l \)-ideal of \( A/\mathcal{L}(A) \), and let \( I \) be an \( l \)-ideal of \( A \) such that \( N(A/I) \) is zero. Then \( J^n \subseteq \mathcal{L}(A) \) for some positive integer \( n \); and since \( \mathcal{L}(A) \subseteq I \), we know that \( \mathcal{L}(A) \subseteq I \). It follows that \( \langle I + J \rangle/I \) is a nilpotent \( l \)-ideal of \( A/I \). Since \( N(A/I) \) is zero, it follows that \( J \subseteq I \). Thus \( J \subseteq \mathcal{L}(A) \), so that \( N(A/\mathcal{L}(A)) \) is zero. Now if \( \mathcal{L}(A) \) is properly contained in \( P(A) \), then, by 2.11 there is a prime \( l \)-ideal containing \( \mathcal{L}(A) \) but not containing \( P(A) \). Since this contradicts the definition of \( P(A) \), \( \mathcal{L}(A) = P(A) \).

2.13. If \( A \) is an \( l \)-ring, the \( N(A/N(A)) \) is zero if and only if \( N(A) = P(A) \). Hence \( N(A) \) is zero if and only if \( P(A) \) is zero.

**Proof.** If \( N(A/N(A)) \) is zero, then \( P(A) = \bigcap \{I: I \text{ is an } l \text{-ideal of } A \text{ and } N(A/I) \text{ is zero}\} \subseteq N(A) \subseteq P(A) \).

If \( N(A) = P(A) \), then \( N(A/N(A)) = N(A/P(A)) \subseteq P(A/P(A)) \) which is zero.

The next result has its analogue in [4] (Theorem 6). It will be used in § 4 to obtain the theorem mentioned in the introduction.

2.14. An \( l \)-ring \( A \) has zero \( l \)-radical if and only if it is a subdirect union of prime \( l \)-rings.
Proof. The proof is immediate from 2.13.

The remaining results of this section will be useful in the next section where we determine various classes of l-rings for which the P-radical equals the l-radical.

2.15. If \( A \) is an l-ring, then \( P(A) = \{a \in A: \text{any m-system containing } |a| \text{ contains } 0 \} \).

Proof. Suppose that there is an m-system \( M \) containing \(|a|\) that does not contain 0. Then, by 2.7, there is a prime l-ideal \( P \) of \( A \) that does not meet \( M \). Thus \(|a|\) does not belong to \( P \), and it follows that \( a \) does not belong to \( P(A) \).

Conversely, let \( a \in A \) be such that any m-system containing \(|a|\) contains 0, and let \( P \) be a prime l-ideal of \( A \). If \( a \) does not belong to \( P \), then \( A^+ \setminus P \) is an m-system containing \(|a|\). Thus \( 0 \in A^+ \setminus P \) which is clearly impossible. Hence \( a \in P(A) \).

2.16. If \( A \) is an l-ring, then \( N(A) = \{a \in A: \text{there is a positive integer } n = n(a) \text{ such that } (x \mid a \mid)^n x = 0 \text{ for all } x \in A^+ \} \).

Proof. It is clear from the definition of \( N(A) \) that if \( a \in N(A) \), then there is a positive integer \( n \) such that \((x \mid a \mid)^n x = 0 \) for all \( x \in A^+ \).

Conversely, suppose that there is a positive integer \( n \) such that \((x \mid a \mid)^n x = 0 \) for all \( x \in A^+ \), and let \( x_0, x_1, \ldots, x_n \in A^+ \). Then, since \( x = x_0 \vee x_1 \vee \cdots \vee x_n \geq x_i \) for all \( i = 0, 1, \ldots, n \), it follows that \( 0 = (x \mid a \mid)^n x \geq x_0 \mid a \mid x_1 \cdots x_{n-1} \mid a \mid x_n \geq 0 \). Since every element of \( A \) is the difference of two positive elements, the result follows.

2.17. If \( I \) is a right (respectively, left) l-ideal of an l-ring \( A \), then \( P(I) = P(A) \cap I \).

Proof. Let \( a \in P(I) \) and let \( M \) be an m-system in \( A \) containing \(|a|\). We show that \( M \cap I \) is an m-system in \( I \). Let \( x, y \in M \cap I \). Then there is a \( z \in A^+ \) such \( xzy \in M \cap I \). Again there is a \( z_i \in A^+ \) such that \( xzyz_i xzy \in M \cap I \). But \( xzyz_i xz \in I^+ \) since \( I \) is a right (respectively, left) l-ideal; hence \( M \cap I \) is an m-system in \( I \). By 2.15, \( 0 \in M \cap I \) since \(|a| \in M \cap I \) and \( a \in P(I) \). Again, by 2.15, it follows that \( a \in P(A) \cap I \).

Conversely, let \( a \in P(A) \cap I \), and let \( M \) be an m-system in \( I \) containing \(|a|\). Then \( M \) is an m-system in \( A \) containing \(|a|\). By 2.15, \( M \) contains 0; and hence \( a \in P(I) \).

2.18. If \( I \) is a right (respectively, left) l-ideal of an l-ring \( A \),
then \( N(I) = N(A) \cap I \).

**Proof.** If \( a \in N(I) \), then, by 2.16, there is a positive integer \( n \) such that \((x | a)^n x = 0\) for all \( x \in I^+ \). But for \( y \in A^+ \) we know that \( y | a | y \in I^+ \), and hence \( 0 = (y | a | y | x)^n y = (y | a |)^n y \); so that
\[
y \in N(A) \cap I
\]
by 2.16. That \( N(A) \cap I \subseteq N(I) \) is clear from the definition of \( N(A) \).

3. The \( P \)-radical equals the \( l \)-radical. Birkhoff and Pierce ([1], p. 45, Example 8) have given an example of an \( l \)-ring \( A \) such that \( N(A/N(A)) \) is not zero. By 2.13, the \( l \)-radical of such an \( l \)-ring is properly contained in its \( P \)-radical. However, there are many \( l \)-rings for which the \( l \)-radical is equal to the \( P \)-radical. In this section we identify some of them and prove some results about \( l \)-rings in which the square of every element is positive.

**Theorem 3.1.** If \( A \) is an \( l \)-ring which is commutative, or satisfies either the ascending or descending chain condition on \( l \)-ideals, or is an \( f \)-ring, then \( N(A) = P(A) \).

**Proof.** Birkhoff and Pierce ([1], p. 46, Corollary 4; and [1], p. 63, Corollary 1) have shown that if an \( l \)-ring \( A \) is commutative, or satisfies either the ascending or descending chain condition on \( l \)-ideals, or is an \( f \)-ring, then \( N(A/N(A)) \) is zero. The result follows from 2.13.

**Corollary 3.2.** If \( A \) is an \( l \)-ring, and if \( P(A) \) is commutative, or satisfies either the ascending or descending chain condition on \( l \)-ideals, or is an \( f \)-ring, then \( N(A) = P(A) \).

**Proof.** Using 2.9, 2.17, 2.18, and 3.1, we have
\[
N(A) = N(A) \cap P(A) = N(P(A)) = P(P(A)) = P(A) \cap P(A) = P(A).
\]

In [1] Birkhoff and Pierce show that is \( A \) is an \( l \)-ring with an identity element \( 1 \) that is a weak order unit\(^2\), then every nilpotent of \( A \) is, in absolutive value, \( \leq 1 \). We generalize this result to

**Theorem 3.3.** Let \( A \) be an \( l \)-ring with an identity element \( 1 \), and suppose that the square of every element of \( A \) is positive. Then each nilpotent \( x \) of \( A \) is, in absolute value, \( \leq 1 \).

**Proof.** (We are indebted to the referee for this proof.) The
\(^2\) A positive element \( e \) of an \( l \)-ring \( A \) is a weak order unit if \( e \land x = 0 \) and \( x \in A \) imply \( x = 0 \).
proof is by induction on the nilpotency index $k$ of $x$. For $k = 1$ the result is trivial. For $k \geq 1$ nilpotency index of $x^2$ is less than $k$. Thus $x^2 = \lfloor x^2 \rfloor \leq 1$. Since $0 \leq (x - 1)^2 = x^2 - 2x + 1$ and $0 \leq (x + 1)^2 = x^2 + 2x + 1$, we have that $-(1 + x^2) \leq 2x \leq 1 + x^2$. Thus $2 \mid x = \lfloor 2x \rfloor \leq 1 + x^2 \leq 2$. 1, and hence $|x| \leq 1$.

**Corollary 3.4.** Let $A$ be an $l$-ring with an identity element 1, and suppose that the square of every element of $A$ is positive. Then $N(A) = P(A)$.

**Proof.** By 3.3, $B(A) = \{x \in A: |x| \leq n1\}$ for some positive integer $n$ contains all of the nilpotents of $A$, and hence it contains $P(A)$. Now, Birkhoff and Pierce [1] have shown (and it is easy to see) that $B(A)$ is a sub-$l$-ring of $A$ which is an $j$-ring. Consequently $P(A)$ is a sub-$j$-ring of $A$, so that by 3.2, $N(A) = P(A)$.

We now turn our attention to finding a sufficient condition for the $P$-radical of an $l$-ring $A$ in which the square of every element is positive to be equal to $\{x \in A; |x| \text{ is nilpotent}\}$.

**Lemma 3.5.** Let $A$ be an $l$-ring in which the square of every element is positive. Then for $a, b \in A^+$ with $a^2 = b^2 = 0$, we have that $ab = ba = 0$.

**Proof.** Since $ab, ba$, and $(a - b)^2$ are positive, we know that $0 \leq (a - b)^2 = -ba - ab \leq 0$. Thus $ab + ba = 0$, and the lemma follows.

**Lemma 3.6.** Let $A$ be a prime $l$-ring in which the square of every element is positive. Then $A$ is an $l$-domain if and only if $a, b \in A^+$ and $a^2 = b^2 = 0$ imply $ba = 0$.

**Proof.** Necessity is clear since if $A$ is an $l$-domain and $a, b \in A^+$ are such that $ab = 0$, then either $a = 0$ or $b = 0$.

Conversely, we first show that $A$ has no nonzero positive nilpotents of index 2. Suppose that $a \in A^+$ and $a^2 = 0$, and let $z \in A^+$. We will show that $aza = 0$. There are three cases.

1. $0 \leq za \leq az$. Then $0 \leq aza \leq a^2z = 0$, so that $aza = 0$.
2. $0 \leq az \leq za$. Then $0 \leqaza \leq za^2 = 0$, so that $aza = 0$.
3. $(za - az) \in A^+ \cup -(A^+)$. Then $(za - az)^+ > 0$ and $(za - az)^- > 0$. Now $0 \leq (za - az)^+(za - az)^- = (za - az)^+(az - za)^- \leq za^2z = 0$. Thus $(za - az)^+(za - az)^- = 0$, and hence $(za - az)^-(za - az)^+ = 0$ since $(za - az)^+ \cap (za - az)^- = 0$. Now $(za - az)^+y(za - az)^-$ is a positive nilpotent of index 2 for any $y \in A^+$; so that, by 3.5, $a(za - az)^+y(za -$
Since $A$ is a prime $l$-ring and $(za - az)^+ > 0$, we know that $a(za - az)^+ = 0$ by 2.5. Similarly, $a(za - az)^- = 0$. Consequently, we have that $0 = a'[(za - az)^+ - (za - az)^-] = a(za - az) = az$ for all $z \in A^+$. Again using 2.5, it follows that $a = 0$.

Now let $a, b \in A^+$ with $ab = 0$. Then for any $z \in A^+$, $bza$ is a nilpotent of index 2 and hence is 0. Thus, by 2.5, $a = 0$ or $b = 0$; and the proof is complete.

**Remark.** We do not know if every prime $l$-ring $A$ in which the square of every element is positive satisfies: $a, b \in A$, $a \wedge b = 0$, and $ab = 0$ imply $ba = 0$.

**Theorem 3.7.** Let $A$ be an $l$-ring in which the square of every element is positive, suppose that disjoint elements of $A$ commute, and suppose that $A$ has zero $l$-radical. Then $A$ is a subdirect union of $l$-domains in which all squares are positive and disjoint elements commute.

**Proof.** B 2.14, $A$ is a subdirect union of a family $\{A_\alpha; \alpha \in \Gamma\}$ of prime $l$-rings. Since both of the properties of disjoint elements commuting and all square being positive are preserved under homomorphisms, each $A_\alpha$ has these properties and hence is an $l$-domain by 3.6.

**Corollary 3.8.** Let $A$ be an $l$-ring in which the square of every element is positive, and suppose that disjoint elements of $A$ commute. Then $P(A) = \{x \in A: |x|$ is nilpotent$\}$. Moreover, if $A$ has an identity element 1, then $P(A) = \{x \in A: x$ is nilpotent\}.

**Proof.** Since $P(A/P(A))$ is zero, $A/P(A)$ is a subdirect union of $l$-domains by 3.7. It follows that $A/P(A)$ has no nonzero positive nilpotents, and hence all of the positive nilpotents of $A$ are in $P(A)$. The first part of the corollary now follows since $P(A)$ is a nil $l$-ideal.

Finally, if $A$ has a positive identity 1, then every nilpotent of $A$ is contained in the sub-$f$-ring $B(A) = \{x \in A: |x| \leq n1$ for some non-negative integer $n\}$ of $A$ by 3.3. But an element of an $f$-ring is nilpotent if and only if its absolute value is. Thus, by the first part, $P(A) = \{x \in A: x$ is nilpotent\}.

**Theorem 3.9.** Let $A$ be an archimedean $l$-ring in which the square of every element is positive. Then

(i) if $x \in A^+$ and $x^2 = 0$, then $xA = Ax = \{0\};$

(ii) every positive nilpotent of $A$ has index $\leq 3$;

(iii) $P(A)A^2 = A^2P(A) = P(A)^3 = \{0\};$

(iv) $N(A) = P(A) = \{x \in A: |x|$ is nilpotent$\};$
(v) if $A$ has no nonzero positive left or right annihilators, then $A$ has no nonzero positive nilpotents; and

(vi) if $A$ has an identity element 1, then $A$ has no nonzero nilpotents.

Proof. The proof is broken up into several steps.

(1) If $x \in A^+$ and $x^2 = 0$, then $xA = Ax = \{0\}$.

Proof. Let $y \in A^+$, and let $n$ be an integer. Then $0 \leq (nx - y)^2 = n^2x^2 - nxy - nyx + y^2$; and hence $n(xy + yx) \leq y^2$. Since $A$ is Archimedean, $xy + yx = 0$. Since $xy$ and $yx$ are positive, $xy = yx = 0$. Since every element of $A$ is the difference of two positive elements, $xA = Ax = \{0\}$.

(2) Every positive nilpotent of $A$ has index $\leq 3$.

Proof. Let $x$ be a positive nilpotent of index $n \geq 4$. Then $2n - 4 \geq n$, so that $(x^{n-2})^2 = 0$. Hence, by (1), $x^{n-1} = x(x^{n-2}) = 0$; and the result follows.

(3) Let $\eta(A) = \{x \in A: |x|$ is nilpotent$\}$. Then $N(A) = P(A) = \eta(A)$.

Proof. Let $x \in \eta(A)$. For $y \in A^+$ and $n$ an integer, we have that $0 \leq (n|x| - y)^2 = n^2|x|^2 - n|x|y - ny|x| + y^2; so that n(|x|y + y|x|) \leq n^2|x|^2 + y^2$. But $|x|^2 = 0$ by (2), so that $|x|^2$ is both a left and right annihilator of $A$ by (1). Hence for $z \in A^+$ we have that $(|x|yz + y|x|z) \leq y^2z$. Since $A$ is Archimedean, it follows that $|x|yz = y|x|z = 0$; and hence $|x|yz = y|x|z = 0$ for all $y, z \in A$. Since $y|x|z = 0$ for all $y, z \in A$, we have that $x \in N(A)$; and hence $N(A) \subseteq P(A) \subseteq \eta(A) \subseteq N(A)$.

Note that since $|x|yz = 0$ and $\eta(A) = P(A)$, we have that $P(A)A^z = P(A)^z = \{0\}$. Moreover, if the inequality $n(|x|y + y|x|) \leq n^2|x| + y^2$ is multiplied on the left by $z \in A^+$, then it follows that $A^zP(A) = \{0\}$. We have now completed the proofs of parts (i) through (iv).

Part (v) is an immediate consequence of part (i); and part (vi) follows from part (i) and (v) since if $A$ has an identity element, then $x$ is nilpotent if and only if $|x|$ is.

4. Subdirect unions of totally-ordered rings with no nonzero divisors of zero. In this section we prove the theorem mentioned in the introduction. It is a consequence of the following three propositions.

PROPOSITION 4.1. Let $A$ be an $l$-ring which satisfies the identity $x^+ax^- = 0$. Then an $l$-ideal $P$ of $A$ is prime if and only if $A/P$ is totally ordered with no nonzero divisors of zero.

Proof. If $A/P$ has no nonzero divisors of zero, then $P$ is a prime $l$-ideal by 2.4.
Conversely, we may suppose that $A$ is a prime $l$-ring since the identity $x^+ax^- = 0$ is preserved under homomorphisms. But if $x^+ax^- = 0$ for all $a \in A^+$, then either $x^+ = 0$ or $x^- = 0$ by 2.5. It follows that $A$ is totally-ordered. By 2.2, $A$ has no nonzero divisors of zero.

In the next proposition we shall call an $l$-ring in which $a(b \lor c) = ab \lor ac$ and $(b \lor c)a = ba \lor ca$ for $a \geq 0$ a distributive $l$-ring. Note that a distributive $l$-ring also satisfies $a(b \land c) = ab \land ac$ and $(b \land c)a = ba \land ca$ for $a \geq 0$.

**Proposition.** Let $A$ be a distributive $l$-ring. Then an $l$-ideal $P$ of $A$ is prime if and only if $A/P$ is totally-ordered with no nonzero divisors of zero.

**Proof.** Sufficiency is a restatement of 2.4. Conversely, let $P$ be a prime $l$-ideal of $A$. Since $A/P$ is a distributive $l$-ring, we may assume that $A$ is a prime $l$-ring. If $a \in A^+$ is either a left or right annihilator, then $aA^+a = \{0\}$; so that, since $A$ is a prime $l$-ring, $a = 0$ by 2.5. But ([1], Th. 14) a distributive $l$-ring with no nonzero left or right positive annihilators is an $f$-ring. Hence $A$ is totally-ordered with no nonzero divisors of zero by 2.2.

**Proposition 4.3.** Let $A$ be an $l$-ring which satisfies the identity $x^+x^- = 0$. Then an $l$-ideal $P$ of $A$ is prime if $A/P$ is totally-ordered with no nonzero divisors of zero.

**Proof.** Sufficiency is a restatement of 2.4. Conversely, we may assume that $A$ is a prime $l$-ring since the identity $x^+x^- = 0$ is preserved under homomorphisms. Then ([1], p. 59, Lemma 2) all squares of $A$ are positive. Also, disjoint elements of $A$ commute since $x^+x^- = 0$ for all $x \in A$. Thus, by 3.6, $A$ is an $l$-domain. Since $x^+x^- = 0$ for all $x \in A$, it follows that $A$ is totally-ordered; and hence $A$ has no nonzero divisors of zero by 2.2.

**Theorem 4.4.** Let $A$ be an $l$-ring with zero $l$-radical. Then the following are equivalent:

(i) $A$ is an $f$-ring;

(ii) $A$ is a subdirect union of totally-ordered rings with no nonzero divisors of zero;

(iii) $x^+ax^- = 0$ for all $x, a \in A$;

(iv) if $a, b, c \in A$ with $a \geq 0$, then $a(b \lor c) = ab \lor ac$ and $(b \lor c)a = ba \lor ca$; and

(v) $x^+x^- = 0$ for all $z \in A$.

**Proof.** The equivalence of (i) and (ii) was proved by Pierce ([1],
Also see Johnson [3](Theorem I. 4.8).

Since (iii), (iv), and (v) hold in any totally-ordered ring and are preserved under the formation of subdirect unions, it is clear that (i) implies (iii), (i) implies (iv), and (i) implies (v).

Now let $A$ be an $l$-ring with zero $l$-radical. Then, by 2.14, $A$ is subdirect union of a family $\{A_\alpha: \alpha \in \Gamma\}$ of prime $l$-rings. If $A$ satisfies (iii) [(iv), (v)], then each $A_\alpha$ satisfies (iii) [(iv), (v)] since (iii) [(iv), (v)] is preserved under homomorphisms. By Proposition 4.1[4.2, 4.3], each $A_\alpha$ is totally-ordered with no nonzero divisors of zero, and the proof is complete.

The following corollary of 4.4 answers affirmatively the question of Birkhoff and Pierce originally asked in [1].

**Corollary 4.5.** Let $A$ be an $l$-ring with an identity element 1, and suppose that $A$ has zero $l$-radical. Then $A$ is an $f$-ring if and only if 1 is a weak order unit.

**Proof.** Since ([1], Th. 15) 1 is a weak order unit if and only if $x^+x^- = 0$ for all $x \in A$, the corollary follows from the equivalence of (i) and (v) above.

Finally we note

**Corollary 4.6.** Let $A$ be an $l$-ring which satisfies either (iii), (iv), or (v) of 4.4. Then $P(A) = \{x \in A: x$ is nilpotent\}.

**Proof.** $A/P(A)$ is a subdirect union of totally-ordered rings with no nonzero divisors of zero. Hence all of the nilpotents of $A$ are in $P(A)$. Since $P(A)$ is a nil $l$-ideal, the corollary follows.

**References**


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