ON A CLASS OF CONVOLUTION TRANSFORMS

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In this paper the convolution transform

\[ f(x) = \int_{-\infty}^{\infty} G(x - t) \varphi(t) \, dt = (G^* \varphi)(x) \]

whose kernel \( G(t) \) is the Fourier transform of \([E(iy)]^{-1}\) where \( E(s) \) is defined by

\[ E(s) = e^{bs} \prod_{k=1}^{\infty} (1 - s/a_k) \exp (s \text{Re} a_k^{-1}) , \]

\[ \text{Re} b = b \quad \text{and} \quad \sum |a_k|^{-2} < \infty \]

will be studied. An inversion theory similar to that achieved when \( a_k \) of (1.2) are real will be obtained. The results will show that under certain rather weak conditions, an infinite subsequence \( a_{k(i)} \) of \( a_k \) can satisfy

\[ \min \{| \arg a_{k(i)} |, | \arg - a_{k(i)} | \} \geq \frac{\pi}{4} . \]

Classes of transforms will be introduced that allow the occurrence of \( \min \{| \arg a_k |, | \arg - a_k | \} \geq \pi/4 \) for all \( k \).

We hope this will partly answer a problem set by Dauns and Widder [1] in Remark 1, page 441.

The inversion operator \( P_m(D) \) is defined by

\[ P_m(D) = \exp \{(b - b_m)D\} \prod_{k=1}^{m} \left( 1 - \frac{D}{a_k} \right) \exp \left( \text{Re} \frac{1}{a_k} D \right) \]

where \( D = d/dx \), \( \exp (kD)f(x) = f(x + k) \) and \( \lim_{m \to \infty} b_m = 0 \).

The inversion formula will be

\[ \lim_{t \to \infty} P_m(t) f(x) = \varphi(x) . \]

This inversion formula was achieved under general conditions on \( \varphi(x) \) in the case \( a_k \) were real by I. I. Hirschman and D. V. Widder in a series of papers and in their book, “The convolution transform” [7]. Hirschman and Widder [6] also found a slightly changed version of (1.4) when \( \sum_{k=1}^{r} (\text{Im} a_k/\text{Re} a_k)^2 < \infty \). A. O. Garder [5] showed that if \( a_{2k-1} = -a_{2k} \) then \( \arg a_{2k} \) can tend to 0 or \( \pi \) slower than is required in [6]. Dauns and Widder [1] showed that if \( a_{2k-1} = -a_{2k}, 0 \leq \text{Re} a_{2k-1} \in \uparrow \) and \( |\arg a_{2k-1}| < (\pi/4) - \gamma \), where \( \gamma \) is independent of \( k \), then (1.4) can be achieved.

It will be noted that in [1] and [5] the \( a_k \)'s were in a special order. The order of the \( a_k \)'s, though having no influence on \( E(s) \),

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may be quite important when treating (1.4) as discussed with some examples in [2] and [4].

We shall define class $A(2)$ (that will depend also on the order of the $a_k$'s). The sequence $\{a_k\}$ belongs to class $A(2)$ if $\text{Re } a_k \neq 0$,

(1.5) \[
\sum_{k=1}^{\infty} \left( \text{Im} \left( \frac{(a_{2k-1}^{-1} + a_{2k}^{-1})^2}{|a_{2k-1}|^{-2} + |a_{2k}|^{-2}} \right) \right) < \infty ,
\]

(1.6) \[
(1 - \theta)(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) + 4 \text{Im } a_{2k-1}^{-1} \text{Im } a_{2k}^{-1} > 0
\]

for $k > k_0$ for some $\theta$, $0 < \theta < 1$ where $\theta$ is independent of $k$, and

(1.7) \[
\frac{(\text{Im} \left( a_{2k-1}^{-1} + a_{2k}^{-1} + a_{2k-2}^{-1} a_{2k}^{-1} \right))^2}{|a_{2k-1}|^2 + |a_{2k}|^2 + 4 \text{Im } a_{2k-1}^{-1} \text{Im } a_{2k}^{-1}} < 1 - \gamma
\]

for $k \geq k_1$ for some $\gamma$, $0 < \gamma < 1$ where $\gamma$ is independent of $k$.

A transform belongs to $A(2)$ if there is an order under which $\{a_n\} \in A(2)$. Class $A(2)$ includes the transforms of [1], [5] and [6].

**Lemma 1.1.** $\sum_{k=1}^{\infty} \left( \text{Im} \frac{a_k}{\text{Re } a_k} \right)^2 < \infty$ implies $\{a_k\} \in A(2)$ (and the order does not matter).

**Proof.** $\sum_{k=1}^{\infty} \left( \text{Im} \frac{a_k}{\text{Re } a_k} \right)^2 < \infty$ implies $\sum_{k=1}^{\infty} \left( \text{Im} \frac{a_k}{|a_k|} \right)^2 < \infty$ which implies $\sum_{k=1}^{\infty} \left( \text{Im} \frac{a_k^{-1}}{|a_k|} \right)^2 < \infty$ which implies (1.5). To prove that $a_k$ satisfies (1.6) and (1.7) is not difficult.

**Remark.** The inversion operator introduced by Hirschman and Widder [6] was slightly different from (1.4) but since

\[
\sum_{k=1}^{\infty} \left( (\text{Re } a_k)^{-1} - \text{Re } a_k^{-1} \right) = \sum_{k=1}^{\infty} \frac{(\text{Im } a_k)^2}{|a_k|^2 \text{Re } a_k} < \infty ,
\]

the difference is a change in $b$ and $b_m$ without changing $\lim_{m \to \infty} b_m = 0$.

**Lemma 1.2.** Let $a_{2k-1} = -a_{2k}$, let $\text{Re } a_{2k} > 0$ and $|\text{arg } a_{2k}| < (\pi/4) - \eta_1$ for $k > k_2$, where $\eta_1$ satisfies $0 < \eta_1 < \pi/4$ and $\eta_1$ is independent of $k$, then $\{a_k\} \in \text{class } A(2)$.

**Proof.** It is easy to see that the sum in (1.5) is equal to zero and the right side of (1.7) is equal to zero. $|\text{arg } a_{2k}| < (\pi/4) - \eta_1$ implies (1.6), with $\theta = 1 - 2(\text{Sin } ((\pi/4) - \eta_1))^2$, for $k > k_2$.

This shows that the transforms treated in [1] are included in class $A(2)$.

**Lemma 1.3.** Let $a_{2k-1} = \bar{a}_{2k}$ and let $\min \{|\text{arg } a_{2k}|, |\text{arg } - a_{2k}|\} < (\pi/4) - \eta_2$ for $k \geq k_2$ where $\eta_2$, $0 < \eta_2 < \pi/4$, is independent of $k$, then $\{a_k\} \in A(2)$.
Proof. It is easy to see that the sum in (1.5) and the right side of (1.7) are equal to zero. One can show that \( \min \{ |\arg a_{2k}|, |\arg -a_{2k}| \} < (\pi/4) - \eta_2 \) implies (1.6) with \( \theta = 1 - 2(\text{Sin}((\pi/4) - \eta_2))^2 \) for \( k \geq k_2 \).

Lemma 1.3 shows that the transforms treated by A. O. Garder [5] belong to class \( A(2) \). Some cases which do not belong to class \( A(2) \) will be treated, among them will be the case when \( a_{2k-1} = -a_{2k} \) and \( \min \{ |\arg a_{2k}|, |\arg -a_{2k}| \} = \pi/4 \) (see Remark 2, [1], p. 442) where estimates different from those achieved for class \( A(2) \) will be obtained.

For the definition of \( G(t) \)

\[
G(t) = \frac{1}{2\pi t} \int_{-\infty}^{\infty} [E(s)]^{-1} e^{\ast t} dt
\]

we have to assume that the integral on the right converges.

For the convergence of (1.8) we shall have to estimate \( E(iy) \) and to these estimates the various classes correspond.

2. Estimates for \( E_{2m}(s) \) when \( \{a_k\} \in \text{class } A(2) \). In previous papers (see [1] and [6] for example) it was found useful and important to estimate \( E_m(s) \) which is defined by

\[
E_m(s) = e^{\ast m s} \prod_{k=m+1}^{\infty} (1 - s/a_k) \exp(s \text{Re } a_k^{-1}).
\]

In order to estimate \( E_m(s) \) we shall estimate one term first.

**Lemma 2.1.** Let \( \{a_k\} \in \text{class } A(2) \) then for \( k \geq K \)

\[
| (1 - iy/a_{2k-1})(1 - iy/a_{2k}) |^2 \geq (1 + \alpha y^2/|a_{2k-1}|^2)(1 + \alpha y^2/|a_{2k}|^2)(1 - \alpha^{-2}[(\text{Im } a_{2k-1}^{-1})^2 + a_{2k}^{-2}])(1 + \alpha y^2/|a_{2k-1}|^2 + |a_{2k}|^2)) \)

where \( 0 < \alpha < 1 \) and \( \alpha \) is independent of \( k \). (\( \alpha \) does depend on \( \theta \) and \( \eta \) of the definition of class \( A(2) \)).

**Proof.** By a simple calculation we get

\[
I_k = | (1 - iy/a_{2k-1})(1 - iy/a_{2k}) |^2 = 1 + 2y \text{Im } (a_{2k-1}^{-1} + a_{2k}^{-1}) + y^4 |a_{2k-1}|^2 + |a_{2k}|^2 + 4 \text{Im } a_{2k-1}^{-1} \text{Im } a_{2k}^{-1}
\]

\[
= 2y^3 \text{Im } (a_{2k-1}^{-1}a_{2k}^{-1}(a_{2k-1}^{-1} + a_{2k}^{-1}) + y^4 |a_{2k-1}|^2 |a_{2k}|^2).
\]

We assume \( K \geq k_1 \) and therefore by (1.7) we get
\[(1 - \frac{\eta}{2})(|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \text{Im } a_{2k-1}^{-1} \text{Im } a_{2k}^{-1}) \cdot \left[ 1 - \frac{\eta}{2} \right] |a_{2k-1}^{-1} \cdot a_{2k}|^{-2} \]
\[
< (1 - \eta)\left( 1 - \eta + \frac{\eta^2}{4} \right) < 1 - \frac{\eta^2}{4}.
\]

It is easy to see that \(y^2(A + 2By + Cy^2) \geq 0\) whenever \(A > 0\), \(C > 0\) and \(B^2 < AC\). We substitute

\[
A = \left(1 - \frac{\eta}{2}\right)(|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \text{Im } a_{2k-1}^{-1} \text{Im } a_{2k}^{-1}),
\]
\[
B = \text{Im } \{a_{2k-1}^{-1} + a_{2k}^{-1}a_{2k}^{-1}\}, \quad \text{and}
\]
\[
C = \left(1 - \frac{\eta}{2}\right) |a_{2k-1}^{-1} a_{2k}|^{-2}.
\]

We use (1.6), (1.7) and the above calculation to show that, for \(k > \max(k_0, k_1)\), \(A > 0\), \(C > 0\) and \(B^2 > AC\). By omitting \(y^2(A + 2By + Cy^2)\) from the right side of the equation defining \(I_k\) we obtain

\[
I_k \geq 1 + 2y \text{Im } (a_{2k-1}^{-1} + a_{2k}^{-1})
\]
\[
+ \frac{\eta \theta}{2} y^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) + \frac{\eta y^4}{2} |a_{2k-1} a_{2k}|^{-2}
\]

by minimum consideration

\[
1 + 2y \text{Im } (a_{2k-1}^{-1} + a_{2k}^{-1})
\]
\[
+ \frac{\eta \theta}{4} y^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) \geq 1 - \frac{4}{\eta^2} \frac{(|a_{2k-1}^{-1} + a_{2k}^{-1}|^2)^2}{(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})}
\]

the last term tends to 1 for large \(k\) because of (1.5). Using (2.3), (2.4) and letting the coefficients of \(y^2\) and \(y^4\) be smaller, we obtain (2.2) with \(\alpha = \gamma \theta / 4\).

**Lemma 2.2.** Suppose \(\{a_k\} \in \text{class } A(2)\). Then for \(k > K\) there exist \(A\) and \(B, 0 < A < B < 1\) independent of \(k\) (but they depend on \(\eta\) and \(\theta\)) so that for any \(r, r < \min(|a_{2k-1}|, |a_{2k}|)\), we shall have:

(a) For \(|\sigma| \leq Ar\) and \(|y| \leq Br\)

\[
H_k(\sigma) \equiv \left(1 - (\sigma + iy)/a_{2k-1}\right)\left(1 - (\sigma + iy)/a_{2k}\right)^{-1} \exp\left(2\sigma \text{Re } (a_{2k-1}^{-1} + a_{2k}^{-1})\right)
\]
\[
\geq 1 - 2\alpha^{-1} \frac{\text{Im } (a_{2k-1}^{-1} + a_{2k}^{-1})}{|a_{2k-1}|^{-2} + |a_{2k}|^{-2}} - \frac{\eta^2}{4} \frac{(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})}{(|a_{2k-1}^{-1} + a_{2k}^{-1}|^2)^2}.
\]

(b) For \(|\sigma| \leq Ar\) and \(|y| \geq Br\)
\[ H_k(\sigma) \geq \left( 1 + \frac{\alpha}{4} y^2 |a_{2k-1}|^{-2} \right) \left( 1 + \frac{\alpha}{4} y^2 |a_{2k}|^{-2} \right) \times \left( 1 - \frac{2}{\alpha} \frac{(\text{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{|a_{2k-1}|^{-2} + |a_{2k}|^{-2}} \right), \]

where \( \alpha \) is that of Lemma 2.1.

**Proof.** By a simple calculation

\[
\left| \left( 1 - \frac{\sigma}{a_k} \right)^2 \right|^2 = \left| \left( 1 - \frac{\sigma}{a_k} \right)^2 \right|^2
- 2\sigma \text{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}) + [\sigma^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \text{Re} a_{2k-1}^{-1} \text{Re} a_{2k}^{-1})
+ \sigma^4 |a_{2k-1}|^{-2} + 2\sigma^2 y^2 |a_{2k-1} a_{2k}|^{-2} - 4\sigma y \text{Im} (a_{2k-1}^{-1} a_{2k}^{-1})
- 2(\sigma^2 + y^2)\sigma \text{Re} [(a_{2k-1}^{-1} + a_{2k}^{-1}) a_{2k} a_{2k}^{-1}]
+ 2\sigma y \text{Im} [(a_{2k-1}^{-1} + a_{2k}^{-1}) a_{2k}^{-1} a_{2k}^{-1}]] = I_k - 2\sigma \text{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}) + J_k.\]

For the estimation of \( J_k \) we shall recall that

\[(2.5) \quad |(a_{2k-1}^{-1} + a_{2k}^{-1}) a_{2k}^{-1} a_{2k}^{-1}| \leq 2(|a_{2k-1}|^{-3} + |a_{2k}|^{-3}) \]
and

\[(2.6) \quad |a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \text{Re} a_{2k-1}^{-1} \text{Re} a_{2k}^{-1} \geq -2 |\text{Re} a_{2k-1}^{-1} \text{Re} a_{2k}^{-1}| \]
\[ \geq - (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}). \]

To prove (a) assume \(|\sigma| \leq Ar, |y| \leq Br.\) Using (2.5) and (2.6) and dropping positive terms we obtain for \( A < B \)

\[ J_k \geq (-A^2 - 2AB)^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2})
+ (4(A^2 + B^2)A - 4A^2 B)^2 (|a_{2k-1}|^{-3} + |a_{2k}|^{-3})
\geq (-3B^2 - 12B)^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}). \]

Choosing \( A < B \) and (for instance) \( B = 3^{-2} \) and using Lemma 2.1 with \( y = 0 \) we obtain

\[ H_k(\sigma) \geq \left( 1 - \frac{\alpha^{-1}(\text{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{|a_{2k-1}|^{-2} + |a_{2k}|^{-2}} \right) - \frac{1}{9} r^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2})
- 2\sigma \text{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}) \exp (2\sigma \text{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}))
\geq 1 - 2\alpha^{-1}(\text{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2})
- \frac{1}{4} r^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) - 4\sigma^2 (\text{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2.\]

(The coefficients in the above estimation are not the best but they are convenient). To prove (b) (for which we are free to choose \( A, A < B \)) we recall that for \( A \leq \beta B, 0 < \beta < 1 \) and \(|\sigma| < Ar \) we
have
\[2\sigma^2 \text{Re } a_{2k-1} \text{Re } a^{-1}_{2k} \leq \beta^2 B^2 \tau^2 (|a_{2k-1}|^2 + |a_{2k}|^2),\]
\[2\sigma^2 y \text{Im } \{(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1} \cdot a_{2k}^{-1}\} \leq \sigma^2 y a_{2k-1} a_{2k}^{-1} - 2\sigma^2 (|a_{2k-1}|^2 + |a_{2k}|^2),\]
\[2y^2 \sigma \text{Re } \{(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1} \cdot a_{2k}^{-1}\} \leq \beta y a_{2k-1} a_{2k}^{-1} - 2\beta y^2 (|a_{2k-1}|^2 + |a_{2k}|^2)\]
and
\[2\sigma^2 \text{Re } \{(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1} \cdot a_{2k}^{-1}\} \leq \sigma^4 a_{2k-1} a_{2k}^{-1} - 2\sigma^2 (|a_{2k-1}|^2 + |a_{2k}|^2).\]

Choosing \(\beta\) so that \(5\beta^2 + 4\beta < \alpha/4\) and \(K\) so that
\[\alpha^{-1}(\text{Im } (a_{2k-1}^{-1} + a_{2k}^{-1}))^2 / (|a_{2k-1}|^2 + |a_{2k}|^2) \leq \frac{1}{4}\]
for \(k \geq K\) we obtain by the above estimations
\[H_k(\sigma) \leq \left(1 + \frac{\alpha}{2} y^2 |a_{2k-1}|^{-2}\right) \left(1 + \frac{\alpha}{2} |a_{2k}|^{-2}\right) \left(1 - \frac{1}{\alpha} (\text{Im } (a_{2k-1}^{-1} + a_{2k}^{-1}))^2\right),\]
\[\times \left[|a_{2k-1}|^{-2} + |a_{2k}|^{-2}\right]^{-1} - 2\sigma \text{Re } (a_{2k-1}^{-1} + a_{2k}^{-1}) \cdot \exp (2\sigma \text{Re } (a_{2k-1}^{-1} + a_{2k}^{-1})) \leq \left(1 + \frac{3\alpha}{8} y^2 |a_{2k-1}|^{-2}\right) \left(1 + \frac{3\alpha}{8} y^2 |a_{2k}|^{-2}\right) \left(1 - \frac{2}{\alpha} (\text{Im } (a_{2k-1}^{-1} + a_{2k}^{-1}))^2\right),\]
\[\times \left[|a_{2k-1}|^{-2} + |a_{2k}|^{-2}\right]^{-1} \left(1 - 4\sigma^2 (\text{Re } (a_{2k-1}^{-1} + a_{2k}^{-1}))^2\right)\]
Since \(\beta < \alpha \cdot 4^{-2}, \beta^2 < \alpha \cdot 4^{-4}\) we obtain (b) easily.

Define \(S_m\) and \(S_m^{(1)}\) (see [7], [2] and [4]) by
\[
S_m = \sum_{k = m+1}^{\infty} |a_k|^{-2} \tag{2.7}
\]
\[
S_m^{(1)} = S_m - \max_{k(1) < \cdots < k(1)} \sum_{i=1}^{l} |a_{k(i)}|^{-2} . \tag{2.8}
\]
Define also \(r_m\) by
\[
r_m = \min_{k > m} |a_k| \tag{2.9}
\]
One can easily see that \(S_m^{(0)} = S_m\) and \(S_m^{(1)} = S_m - r_m^{-2}\).

**Theorem 2.3.** Let \(\{a_k\} \in \text{class } A(2)\), then for \(m \geq K, |\sigma| \leq AS_{2m}^{-1/2}\), and \(b_{2m} = 0\) we have
\[
|E_{2m}(\sigma + iy)| \geq \sqrt{2}/2 . \tag{2.10}
\]
(A being that of Lemma 2.2.)
Proof. To prove (2.10) we use Lemma 2.2(a) whose conditions are satisfied since $S_{2m} > r_{2m}^{-2}, S_{2m}^{-1/2} < r_{2m} = \min_{k > 2m} |a_k|$. We also recall that for $A_n > 0$ and $\sum_{n=m+1}^{\infty} A_n < 1/2$ we have

$$\prod_{n=m+1}^{\infty} (1 - A_n) \geq 1 - \sum_{n=m+1}^{\infty} A_n \geq \frac{1}{2}.$$ 

Remembering that for large $m$

$$2\alpha^{-1} \sum_{k=m+1}^{\infty} (\text{Im} (a_{2k-1}^{-1} + a_{2k-1}^{-1}))^2 / (|a_{2k-1}^{-1}|^2 + |a_{2k}^{-1}|^2) < \frac{1}{8}$$

and

$$4\alpha^2 \sum_{k=m+1}^{\infty} (\text{Re} (a_{2k-1}^{-1} + a_{2k-1}^{-1}))^2 \leq 8A^2 S_{2m} \sum_{k=m+1}^{\infty} (|a_{2k-1}^{-1}|^2 + |a_{2k}^{-1}|^2) \leq 8A^2 < 8^{-1}$$

and using Lemma 2.2(a) we conclude the proof of (2.10) in the case where $|\sigma| \leq S_{2m}^{-1/2}$ and $|y| \leq BS_{2m}^{-1/2}$. Using Lemma 2.2(b), (2.10) in the case where $|\sigma| \leq AS_{2m}^{-1/2}, |y| \geq BS_{2m}^{-1/2}$ follows by an argumentation similar to that used in the first part. Then:

**Theorem 2.4.** Let $\{a_k\} \in A(2), b_{2m} = 0$, then for $m \geq k, |\sigma| \leq AS_{2m}^{-1/2}$ and $|y| \geq BS_{2m}^{-1/2}$ we have

$$|E_{2m}(\sigma + iy)| \geq \frac{3}{4} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{2n} \left(\frac{\alpha}{4}\right)^n \prod_{l=0}^{n-1} S_{2m}^{(l)}\right)^{1/2}.$$ (2.11)

Proof. Using (1.5) we can choose, by the method in the proof of Theorem 2.3, $m$ so that

$$\sum_{k=m+1}^{\infty} \left(1 - \frac{2}{\alpha}[(\text{Im} (a_{2k-1}^{-1} + a_{2k-1}^{-1}))^2 / (|a_{2k-1}^{-1}|^2 + |a_{2k}^{-1}|^2)]\right) \geq \frac{9}{16}$$

(9/16 can be replaced of course by any $1 - \varepsilon$).

$$\sum_{n=1}^{\infty} \frac{1}{n!} y^{2n} \left(\frac{\alpha}{2}\right)^n \prod_{l=0}^{n-1} S_{2m}^{(l)}$$

converges for all $y$ since $S_{2m} = S_{2m}^{(0)} > S_{2m}^{(1)} > \cdots > S_{2m}^{(n)}$. By Lemma 2.2 and (2.12) we have

$$|E_{2m}(\sigma + iy)| \geq \frac{3}{4} \left(1 + \sum_{n=1}^{\infty} y^{2n} \left(\frac{\alpha}{4}\right)^n \sum_{k_{(1)}>2m, k_{(i)}<k_{(i+1)}} |a_{k(1)} \cdots a_{k(n)}|^{-2}\right)^{1/2}.$$ But we have
I(n, m) = \sum_{2m < k(1) < \cdots < k(n + 1)} |a_{k(1)} \cdots a_{k(n)}|^2 = \frac{1}{n!} \sum_{k(i) > 2m} |a_{k(1)} \cdots a_{k(n - 1)}|^2 \cdot \Lambda_n \cdot \Gamma_{2 - J} \gamma_i \cdot f(c) > 2i \\
\therefore \sum_{k(i) > 2m} |a_{k(1)} \cdots a_{k(n - 1)}|^2 \\
\therefore \frac{1}{n!} S^{n-1}_{2m} \sum_{k(i) > 2m} |a_{k(1)} \cdots a_{k(n - 1)}|^2.

Since
\[ \sum_{k(i) > 2m} |a_{k(i)}|^2 = S^{(l)}_{2m} = S^{(l)}_{2m}, \]
by induction \( I(n, m) \geq 1/n! \). \( \Pi_{l=0}^{m-1} S^{(l)}_{2m} \), which concludes the proof of the theorem.

**Theorem 2.5.** Let \( \{a_k\} \in A(2) \), \( b_{2m} = 0 \), and \( \sigma \) satisfies \( \text{Re} a_k \neq \sigma \) for all \( k > n \), then for \( p, n = 0, 1, 2, \cdots \) there exist \( k(p, \sigma, n) \) and \( k_2(p, \sigma, n) \) so that

\[
|E_{2m}(\sigma + i\tau)|^2 \geq k_1(p, \sigma, n) + k_2(p, \sigma, n) \tau^{2p}.
\]

**Proof.** Since \( S_{2m} = o(1)m \longrightarrow \infty \) we can choose \( m \) so that \( AS_{2m}^{-1/2} \geq \sigma \) (for \( A \) of Theorems 2.3 and 2.4). Combining Theorems 2.3, 2.4 and the fact that \( |\prod_{m=2m+1}^{m} (1 - \sigma + i\tau/a_k))e^{\text{Re} a_k^{-1}}| \geq \delta \) whenever \( \text{Re} a_k \neq \sigma \), we obtain (2.13).

3. Estimates for \( E_{2m}(s) \) in special cases when \( \{a_k\} \in A(2) \). In this section we shall estimate \( E_{2m}(s) \) in case \( \{a_k\} \) does not necessarily belong to \( A(2) \) but \( a_{2k} = -a_{2k} \) or \( a_{2k-1} = \overline{a_{2k}} \) and some other conditions are satisfied.

First we prove some lemmas concerning the above mentioned cases.

**Lemma 3.1.** Let \( a \) be a complex number \( \text{Re} a \neq 0 \), then for all real \( y \) and \( q \geq 1 \)

\[
I(a) = \left| (1 - \frac{i y}{a}) + (1 - \frac{i y}{a}) \right|^2 = \left| (1 - \frac{i y}{a}) \right|^2 + \left| (1 - \frac{i y}{a}) \right|^2 = \left( 1 - \frac{1}{q} \right) y^4 |a|^{-4} + \left( 1 + y^4 |a|^{-4} \right) \text{Re} a^2 \geq 0.
\]

**Proof.** Simple calculation yields

\[
\left| (1 - \frac{i y}{a}) \right|^2 = 1 - q \left( \frac{\text{Re} a^2}{|a|^2} \right)^2 + \left( \sqrt{q} \left| \frac{\text{Re} a^2}{|a|^2} \right| + \frac{1}{\sqrt{q}} y^2 |a|^{-2} \right) + \left( 1 - \frac{1}{q} \right) y^4 |a|^{-4}
\]
from which (3.1) is immediate.

**Lemma 3.2.** Let $a$ be complex number, $\Re a \neq 0$, then

$$
\left| \left(1 - \frac{\sigma + iy}{a}\right)\left(1 + \frac{\sigma + iy}{a}\right) \right|^2
$$

(3.2)

$$
= I(a) + 2\sigma^2|a|^2 - 2(\Re a)^2|a|^{-4} + \sigma^4|a|^{-4}
+ 2\sigma^2y^2|a|^{-4} + 4\sigma y(\Im a^2)|a|^{-4},
$$

(3.3)

where $I(a)$ is defined in Lemma 3.1.

**Proof.** The proof is a corollary of the proof of Lemma 2.2 combined with Lemma 3.1.

**Lemma 3.3.** Let $\Re a \neq 0$, then for $K > 1$ there exists $A$ and $B$, independent of $a$, $0 < A < B < 1$ such that for $r < |\Re a|$ we have:

(a) For $|\sigma| \leq Ar$ and $|y| \leq Br$

$$
\left| \left(1 - \frac{\sigma + iy}{a}\right)\left(1 + \frac{\sigma + iy}{a}\right) \right|^2
\geq 1 - K^{-2}r^2|a|^{-2} - (\min (0, (\Re a^2)|a|^{-2}))^2.
$$

(3.4)

(b) For $|\sigma| \leq Ar_1 \leq Ar$, $|y| \geq Br$ and $\delta > 0$

$$
\left| \left(1 - \frac{\sigma + iy}{a}\right)\left(1 + \frac{\sigma + iy}{a}\right) \right|^2
\geq \left(1 + \frac{1}{4}y^4|a|^{-4}\right)(1 - 2(\min (0, (\Re a^2)|a|^{-2}))^2
- K^{-1}(r^2|a|^{-2} + r_1^2|a|^{-1-\delta} + |a|^{-2+2\delta})).
$$

(3.5)

**Proof.** To prove (3.4) we use (3.2) and (3.1) with $q = 1$ and obtain the result by choosing $B$ so that $6B^2 < K^{-1}$, and dropping some positive terms.

To prove (3.5) we use

$$
4\sigma y(\Im a^2)|a|^{-4} \geq -4|\sigma y||a|^{-2} \geq -\left(\frac{1}{\beta^2}y^2|a|^{-(3-\delta)} + 2\beta^2\sigma^2|a|^{-(1+\delta)}\right)
$$

and
Choosing $4/\beta^4 \leq 1/K$ or $\beta \geq \sqrt[4]{4K}$ and $A$ so that $2\beta^2 A^2 < K^{-1}$ or $A^2 < 1/4K \sqrt{K}$ or $A < 1/2K$ one can conclude the proof by using Lemma 3.1 (choosing there $q = 2$ in case $\text{Re } a^2 < 0$) and dropping some positive terms.

**Lemma 3.4.** Let $\text{Re } a \neq 0$, then for $K > 1$ there exist $A$ independent of $a, 0 < A < 1$, such that for $r < |\text{Re } a|$ and $|\sigma| \leq Ar$ we have

\[
\left| \left(1 - \frac{\sigma + iy}{a}\right) \left(1 - \frac{\sigma + iy}{\bar{a}}\right) \right|^2 \exp \left(4\sigma \text{Re } a/|a|^2\right)
\]

\[
\geq \left(1 + \frac{1}{4} y^4 |a|^{-4}\right)(1 - 2(\min (0, \text{Re } a^2/|a|^3))^2 - K^{-1} r^2 |a|^{-2}).
\]

**Proof.** Using (3.3) of Lemma 3.2, Lemma 3.1 with $q = 3/2$, the estimations

\[-4\sigma^3 |a|^{-4} \text{Re } a \geq -\sigma^4 |a|^{-4} - 4\sigma^2 (\text{Re } a)^2 |a|^{-4},\]

\[-4y^2 \sigma |a|^{-4} \text{Re } a \geq -\frac{1}{2^5} y^4 |a|^{-4} - 4\sigma^2 |a|^{-2}\]

and dropping some positive terms we obtain

\[
\left| \left(1 - \frac{\sigma + iy}{a}\right) \left(1 - \frac{\sigma + iy}{\bar{a}}\right) \right|^2 \geq 1 + \left(\frac{1}{3} - \frac{1}{2^5}\right) y^4 |a|^{-4}
\]

\[
-\frac{3}{2} (\min (0, \text{Re } a^2/|a|^3))^2 - 4^3 A^2 r^2 |a|^{-2} - 4\sigma (\text{Re } a)|a|^{-2}.
\]

Choosing $A$ so that $4^3 A^2 < 1/4K$, which implies

\[-4^3 A^2 r^2 |a|^{-2} > -\frac{1}{4K} r^2 |a|^{-2}, \quad 4 |\sigma| |a|^{-1} < \frac{1}{4},\]

and

\[\exp (4\sigma (\text{Re } a)|a|^{-2}) \geq 1 + 4\sigma (\text{Re } a)|a|^{-2} - 4^3 \sigma^2 |a|^{-2}\]

from which (3.6) follows.

We shall define now two classes of convolution transforms by the function $E(s)$ and the sequence $\{a_k\}$.

**Definition 3.1.** $\{a_k\} \in \text{class } B(2, \delta)$ if
(3.7) \[ E(s) = \prod_{k=1}^\infty (1 - s^2a_k^{-2}) , \]

(3.8) \[ \sum_{k=1}^\infty |a_k|^{-t(\text{Re} a_k^2)} < \infty \]

and

(3.9) \[ \sum_{k=1}^\infty |a_k|^{-1-\delta} < \infty , \quad \sum_{k=1}^\infty |a_k|^{-2+\delta} < \infty \quad \text{for some } \delta > 0 . \]

**Definition 3.2.** \( \{a_k\} \in B(2) \) if there is \( \delta > 0 \) so that \( \{a_k\} \in B(2, \delta) \).

**Definition 3.3.** \( \{a_k\} \in \text{class } C(2) \) if

(3.10) \[ E(s) = \prod_{k=1}^\infty (1 - sa_k^{-1})(1 - s\cdot\bar{a}_k^{-1}) , \]

if condition (3.8) is satisfied and \( \sum |a_k|^{-2} < \infty \).

**Remark.** \( S_{2m} = 2 \sum_{k=m+1}^\infty |a_k|^2 \) in case of class \( B(2) \) and \( C(2) \).

We have to introduce some more notations before being able to prove the estimation on \( E(s) \) for transforms of classes \( B(2) \) and \( C(2) \).

(3.11) \[ Q_m = \sum_{k=m+1}^\infty |a_k|^{-4} . \]

(3.12) \[ Q_m^{(j)} = Q_m - \max_{m < k(1) < \cdots < k(j)} \left\{ \sum_{r=1}^j |a_{k(r)}|^{-4} \right\} . \]

We shall state the estimations for classes \( B(2) \) and \( C(2) \) together and then outline the proofs.

**Theorem 3.5.** If \( \{a_k\} \in B(2, \delta) \), then for \( m \geq M \) and some \( A \) and \( B \) we have:

(a) \( |\sigma| \leq AS_{2m}^{-1/2} \), \( |y| \leq BS_{2m}^{-1/2} \) imply

(3.13) \[ |E_{2m}(s)| \geq 3/4 . \]

(b) \( |\sigma| \leq A(\sum_{k=m+1}^\infty |a_k|^{-1-\delta})^{-1/1+\delta} \) and \( |y| \geq BS_{2m}^{-1/2} \) imply

(3.14) \[ |E_{2m}(s)| \geq \frac{3}{4} \left( 1 + \sum_{n=1}^\infty \frac{1}{n!} y^{4n} \prod_{l=0}^{n-1} Q_m^{(l)} \right)^{1/2} . \]

**Theorem 3.6.** If \( \{a_k\} \in C(2) \) then for \( m \geq M \) there exists an \( A \)
so that for \( |\sigma| \leq AS_{2m}^{-1/2} \) (3.14) is valid.

**Proof of Theorems 3.5 and 3.6.** The proof follows the proof of Theorems 2.3 and 2.4 Using Lemmas 3.3 and 3.4 we have to choose
\( r = S_{\gamma m}^{-1/2} \) and \( r_1 = \left(2 \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{-1/(1+\delta)} \) \((r_1 \text{ necessary only in proving Theorem 3.5 from Lemma 3.3}). Obviously \( r_1 < \min_{k>m} |a_k|, r \leq \min_{k>m} |a_k| \). Also we have

\[
\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \geq \left(\sum_{k=m+1}^{\infty} |a_k|^{-2}\right) \left(\min_{k>m} |a_k|\right)^{-1-\delta}
\]

This implies

\[
r_1 = \left(2 \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{-1/(1+\delta)} \leq (S_{\gamma m})^{-1/2} \leq r.
\]

Choose \( m \) and \( K \) so that \( \sum_{k=m}^{\infty} (\min (0, \text{Re } a_k^2/|a_k|^2))^2 < \varepsilon_1, 1/K < \varepsilon_1 \) \((K \text{ of Lemmas 3.3 and 3.4})\) and, for proving Theorem 3.5, \( \sum_{k=m+1}^{\infty} |a_k|^{-2+1/3} < \varepsilon_1 \). The choice \( \varepsilon_1 \leq 1/16 \) will yield the number \( 3/4 \) in (3.13) \((\text{every } 1 - \gamma \text{ could be achieved by } \varepsilon_1 \text{ small enough})\) and the coefficient \( 3/4 \) in (3.14).

To complete the proof we have to show

\[
\prod_{k=m+1}^{\infty} \left(1 + \frac{1}{4} y^t |a_k|^{-t}\right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{y^n}{n-I} Q_m^{(1)} ,
\]

the proof of which follows stepwise that of Theorem 2.4.

The classes in this section are not included in \( A(2) \) since \((1.6) \) may fail to be valid. The estimates in this section are weaker in the case where the transforms are also \( A(2) \).

**Theorem 3.7.** Let \( \{a_k\} \in B(2) \) or \( C(2) \). Then for \( \sigma \) satisfying \( \text{Re } a_k \neq 0 \) for all \( k > n \), and for \( p, n = 0, 1, 2 \cdots \) there exist \( k_1(p, \sigma, n) \) and \( k_2(p, \sigma, n) \) such that when \( \sigma \neq \text{Re } a_k \)

\[
|E_{2n}(\sigma + i\tau)|^2 \geq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{\varepsilon p} .
\]

**Proof.** Deduced from Theorems 3.5 and 3.6 as Theorem 2.5 is deduced from Theorem 2.4 and 2.3.

4. Estimates for \( G_m(t) \). We define \( G_m(t) \), in the usual manner, by

\[
G_m(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [E_m(s)]^{-1-i\tau} ds, \quad G_0(t) = G(t) .
\]

We define also:
(4.2) \( \alpha(m) = \max \{ \Re a_k, -\infty \mid \Re a_k < 0 \text{ and } k > m \} \).

(4.3) \( \beta(m) = \min \{ \Re a_k, \infty \mid \Re a_k > 0 \text{ and } k > m \} \).

We recall that in the cases \( \{a_k\} \in A(2), \{a_k\} \in B(2) \) and \( \{a_k\} \in C(2) \) we have

\[
E_{2n}(\sigma + i\tau) \geq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p},
\]

for \( n, p = 0, 1, 2 \cdots \) and \( \alpha(2n) < \sigma < \beta(2n) \).

**Theorem 4.1.** Let \( E_n(s), P_n(D) \) and \( G_n(t) \) be defined by (2.1), (1.3) and (4.1); let (4.4) be satisfied for \( m(l) \), a subsequence of \( m \), then:

A. For any \( \sigma \) satisfying \( \sigma(m(l)) < \sigma < \beta(m(l)) \) we have

\[
G_{m(l)}(t) = P_{m(l)}(D)G(t) = \frac{1}{2\pi i} \int_{-\infty}^{\sigma+i\infty} [E_{m(l)}(s)]^{-1}e^{st}ds.
\]

B. Suppose in case \( \alpha(m(l)) \neq -\infty \) that \( a_{k(1,1)} = \cdots = a_{k(1,m_1+1)} \), \( a_{k(2,1)} = \cdots = a_{k(2,m_2+1)} \), \ldots, \( a_{k(r,1)} = \cdots = a_{k(r,m_r+1)} \) are all with indices greater than \( m(l) \) and \( \alpha(m(l)) = \Re a_{k(1,1)} = \Re a_{k(2,1)} = \cdots = \Re a_{k(r,1)} \), then

\[
\frac{d^v}{dt^v}G_{m(l)}(t) = \sum_{i=1}^{r} \frac{d^v}{dt^v}\{P_i(t)e^{ia_{k(i,1)}}\} + 0(e^{k(t)}) \quad t \to \infty
\]

where \( p_i(t) \) are polynomials of order \( m_i \) and \( k \) is any real number satisfying

\[
\max \{ \Re a_k, -\infty \mid k > m(l), \Re a_k < \alpha(m(l)) \} < k < \alpha(m(l)).
\]

C. Suppose \( \alpha(m(l)) = -\infty \), then

\[
\frac{d^v}{dt^v}G_{m(l)}(t) = 0(e^{kt}) \quad t \to \infty \quad \text{for any real } k, k < 0.
\]

D. Suppose in case \( \beta(m(l)) \neq \infty \) that \( a_{r(1,1)} = \cdots = a_{r(1,m_1+1)} \), \( \cdots a_{r(j,1)} = \cdots = a_{r(j,m_j+1)} \) are all with indices greater than \( m(l) \) and \( \beta(m(l)) = \Re a_{r(1,1)} = \cdots = \Re a_{r(j,1)} \), then

\[
\frac{d^v}{dt^v}G_{m(l)}(t) = \sum_{i=1}^{j} \frac{d^v}{dt^v}\{q_i(t)e^{ia_{r(i,1)}}\} + 0(e^{kt}) \quad t \to -\infty
\]

where \( q_i(t) \) are polynomials of order \( m_i \) and \( k \) is a real number satisfying \( \beta(m(l)) < k < \min \{ \Re a_k, \infty \mid k > m(l), \Re a_k > \beta(m(l)) \} \).

E. Suppose \( \beta(m(l)) = \infty \), then

\[
\frac{d^v}{dt^v}G_{m(l)}(t) = 0(e^{kt}) \quad t \to -\infty
\]
where $k$ is any real positive number.

F. For $\alpha(m(l)) < \Re s < \beta(m(l))$ we have

\begin{equation}
\frac{1}{E_{m(l)}(s)} = \int_{-\infty}^{\infty} e^{st} G_{m(l)}(t) dt
\end{equation}

which implies

\begin{equation}
1 = \int_{-\infty}^{\infty} G_{m(l)}(t) dt .
\end{equation}

Proof. The proof follows the method used in Hirschman and Widder’s book “The convolution transform” [6, p. 108]. Formula (4.4), that was proved for class $A(2), B(2)$ and $C(2)$, is used here instead of the theorems on $E_{m}(s)$ in [6].

The following result will estimate $G_{2m}(t)$ in the case when $m$ is large near the point $t = 0$ as well as when $|t| \to \infty$.

**Theorem 4.2.** Let $\{a_{k}\} \in A(2)$ and suppose that for some $n$ $S_{2n}^{(n+1)} L_{n} S_{2m}$ where $L_{n} > 0$ is independent of $m$, then there exist $M(n) > 0$ and $A > 0$ such that

\begin{equation}
|G_{2m}(t)| \leq M(n) S_{2m}^{-(n+1)/2} \exp(-A \cdot S_{2m}^{-(n+1)/2} |t|).
\end{equation}

Proof. By Theorem 4.1.A we have

\[ G_{2m}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\sigma t} E_{2m}(\sigma + iy)}{E_{2m}(\sigma + iy)} dy \]

and therefore

\[ G_{2m}^{(n)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\sigma + iy)^n e^{-\sigma t} \frac{1}{E_{2m}(\sigma + iy)} dy . \]

Remembering that $S_{2n}^{(n+1)} \geq L_{n} S_{2m}$ implies $S_{2m}^{(k)} \geq L_{n} S_{2m}$ for $k \leq n + 1$, and using Theorems 2.3 and 2.4 we obtain, choosing $\sigma = A S_{2m}^{-1/2}$ for the case $t > 0$,

\[ |G_{2m}^{(n)}(t)| \leq \frac{1}{2\pi} \exp(-A S_{2m}^{-1/2} t) \left\{ |\sigma| + |y| \right\}^n \frac{1}{E_{2m}(\sigma + iy)} \bigg| dy + \int_{|y| \geq B S_{2m}^{-1/2}} \frac{|\sigma| + |y| \right\}^n dy \bigg| \leq \exp(-A S_{2m}^{-1/2} t) \left\{ \frac{2}{2\pi} (A + B)^n S_{2m}^{-(n+1)/2} \right\}
\]

\[ + \sum_{k=0}^{n} \binom{n}{k} S_{2m}^{-k/2} \frac{2}{3\pi} \int_{B S_{2m}^{-1/2}}^{\infty} y^{n-k} dy \left( 1 + y^{2(n+2)} \frac{1}{(n+2)!} (L_{n})^{n+1} S_{2m}^{n+2} \left( \frac{\alpha}{4} \right)^{n+2} \right)_{1/2} \]

\[ \leq M(n) S_{2m}^{-(n+1)/2} \exp(-A S_{2m}^{-1/2} t) . \]
The result for \( t < 0 \) is achieved choosing \( \sigma = -AS_{2m}^{-1/2} \).

**REMARK.** When \( a_{2k-1} = -a_{2k} \) we have \( S_{2m}^{(1)} \geq (1/2)S_{2m}^2 \) and therefore Theorem 4.2 for \( n = 0 \) includes Lemma 2.4 of [1, p. 432]. Whenever the connection between pair is \( 0 < \theta_1 \leq |a_{2k-1}/a_{2k}| \leq \theta_2 < \infty \), where \( \theta_1, \theta_2 \) are fixed for all \( m \), we have \( S_{2m}^{(1)} \geq L_1S_{2m}L_1 > 0 \). But in case of \( n = 0 \) the restriction \( S_{2m}^{(1)} \geq L_1S_{2m} \) is not necessary as is proved by the following.

**THEOREM 4.3.** Let \( \{a_k\} \in A(2) \), then for some \( A > 0 \) we have

\[
|G_{2m}(t)| \leq MS_{2m}^{-1/2} \exp \left(-AS_{2m}^{-1/2} |t| \right).
\]

**Proof.** Following the proof of Theorem 4.2 and using Theorem 2.4 we have for \( t > 0 \) (\( t < 0 \) can be treated similarly)

\[
|G_{2m}(t)| \leq \exp \left(-AS_{2m}^{-1/2} |t| \right) \left\{ \frac{\sqrt{2}}{\pi} BS_{2m}^{-1/2} \right\}.
\]

\[
\int_{B_1S_{2m}^{-1/4}}^{\infty} dy = S_{2m}^{-1/2}(S_{2m}^{(1)})^{1/4} \int_{B_1S_{2m}^{-1/4}}^{\infty} \frac{1}{1 + y^2} dy
\]

\[
\leq 2S_{2m}^{-1/2}(S_{2m}^{(1)})^{1/4} \lim_{\xi \to \infty} \left( \arctan \frac{1 - B_1(S_{2m}^{(1)})^{1/4}}{\xi} \right) \leq B_2S_{2m}^{-1/2}.
\]

From this the proof can be easily concluded.

Lemma 3.2A of [1, p. 434] is generalized by Theorem 4.2 in case \( S_{2m}^{(2)} \geq L_2S_{2m} \) for some \( L_2 \). Case B is covered only in part. The following theorem generalizes Lemma 3.2B [1, p. 434].

**THEOREM 4.4.** Let \( a_k \in A(2), b_{2m} = 0 \) and suppose \( 0 < \theta_1 < |a_{2k}/a_{2k-1}| < \theta_2 < \infty \) where \( \theta_1, \theta_2 \) are independent of \( k \) and \( |\text{Re } a_k|/|a_k| > \eta \), then for some \( A_1 > 0 \) and \( M_1 \) we have:

\[
|G_{2m}'(t)| \leq M_1S_{2m}^{-1} \exp \left(-A_1S_{2m}^{-1/2} |t| \right).
\]

**Proof.** Let us split the proof into two cases
(a) \[ S_{2m} \geq \max_{k > m} (|a_{k-1}|^2 + |a_{2k}|^2) \geq \frac{1}{K} S_{2m} \]

and

(b) \[ S_{2m} - \max_{k > m} (|a_{2k-1}|^2 + |a_{2k}|^2) < \frac{1}{K} S_{2m} \]

In case (a) (4.13) was proved by Theorem 4.2 for any arbitrary \( K \). We shall choose \( K > 2 \). To prove (4.13) in case (b) we define \( k_0 \) by

\[ \max_{k > m} (|a_{2k-1}|^2 + |a_{2k}|^2) = |a_{2k_0-1}|^2 + |a_{2k_0}|^2. \]

(In case (b) the choice of \( k_0 \) is unique.) Define \( g_{k_0}^*(t) \) and \( G_{2m+2}(t) \) by:

\[
g_{k_0}^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ (1 - iy/a_{2k_0-1})(1 - iy/a_{2k_0}) \right]^{-1} e^{-iv't} dy.
\]

\[
G_{2m+2}^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ (1 - iy/a_{2k_0-1})(1 - iy/a_{2k_0}) \right] (E_{2m}(iy))^{-1} e^{-iv't} dy.
\]

By [9, p. 255] we have

\[ G_{2m}^*(t) = g_{k_0}^*(t) G_{2m+2}^*(t). \]

One can calculate \( g_{k_0}^*(t) \):

\[ g_{k_0}^*(t) = \begin{cases} \frac{a_{2k_0-1} a_{2k_0}}{a_{2k_0-1} - a_{2k_0}} e^{a_{2k_0} t} & t \geq 0 \\ \frac{a_{2k_0} - a_{2k_0-1}}{a_{2k_0}} e^{a_{2k_0-1} t} & t < 0 \end{cases} \]

when \( \text{Re} a_{2k_0-1} > 0, \text{Re} a_{2k_0} < 0 \).

\[ g_{k_0}^*(t) = \begin{cases} \frac{a_{2k_0-1} a_{2k_0}}{a_{2k_0} - a_{2k_0-1}} [e^{a_{2k_0-1} t} - e^{a_{2k_0} t}] & t < 0 \\ 0 & t > 0 \end{cases} \]

when \( \text{Re} a_{2k_0-1} > 0, \text{Re} a_{2k_0} > 0, a_{2k_0} \neq a_{2k_0-1} \).

\[ g_{k_0}^*(t) = \begin{cases} -a_{2k_0}^2 t e^{a_{2k_0} t} & t < 0 \\ 0 & t > 0 \end{cases} \]

when \( a_{2k_0} = a_{2k_0-1}, \text{Re} a_{2k_0} > 0 \).

Either \( g_{k_0}^*(t) \) or \( g_{k_0}^*(-t) \) is of the above form.

\[ G_{2m+2}^*(t - \text{Re} (a_{2k_0}^{-1} + a_{2k_0}^{-1})) \]

satisfies the assumptions of Theorem 4.3 with \( S_{2m+2}^* = S_{2m} - |a_{2k_0-1}|^2 - |a_{2k_0}|^2 \) and therefore
\[ |G_{2m+2}(t)| \leq M(S_{2m+2}^{*})^{1/2} \exp (-AS_{2m+2}^{*-1/2} |t + \text{Re} (a_{2k_0+1}^{-1} + a_{2k_0}^{-1})|). \]

Integrating by parts

\[
G'_{2m}(t) = \int_{-\infty}^{\infty} g_{k_0}^*(u) \frac{d}{dt} G_{2m+2}^*(t-u) du
= \left\{ \int_{-\infty}^{0} + \int_{0}^{\infty} \right\} \left( \frac{d}{du} g_{k_0}^*(u) \right) G_{2m+2}^*(t-u) du.
\]

Since

\[
\theta_1^{-2} |a_{2k_0-1}|^2 \leq |a_{2k_0}|^2 \quad \text{and} \quad |a_{2k_0}|^2 \leq \theta_2^2 |a_{2k_0-1}|^2
\]

we have

\[
(\theta_1^{-2} + 1) |a_{2k_0}|^{-2} \geq \frac{1}{2} S_{2m} \quad \text{and} \quad (\theta_2^2 + 1) |a_{2k_0-1}|^{-2} \geq \frac{1}{2} S_{2m} ;
\]

therefore

\[
\max (|a_{2k_0}|, |a_{2k_0-1}|) \leq [(2\theta_1^{-2} + 2)^{1/2} + (2\theta_2^2 + 2)^{1/2}] S_{2m}^{*-1/2} = R_1 S_{2m}^{-1/2}.
\]

By the same method \((\theta_1^{-2} + 1) |a_{2k_0}|^{-2} \leq S_{2m} \text{ and } (\theta_2^2 + 1) |a_{2k_0-1}|^2 \leq S_{2m},\)
from which we deduce

\[
|\text{Re} a_{2k_0}| \geq \gamma |a_{2k_0}| \geq \gamma (\theta_1^{-2} + 1)^{-1/2} S_{2m}^{-1/2},
\]

\[
|\text{Re} a_{2k_0-1}| \geq \gamma (\theta_2^2 + 1)^{-1/2} S_{2m}^{-1/2}
\]

and

\[
\min (|\text{Re} a_{2k_0}|, |\text{Re} a_{2k_0-1}|) \geq R_2 S_{2m}^{-1/2} > 0
\]

where

\[
R_2 = \gamma \cdot \min ((\theta_1^{-2} + 1)^{-1/2}, (\theta_2^2 + 1)^{-1/2}).
\]

One has to estimate \(G_{2m}(t)\) for different cases of \(g_{k_0}^*(t)\) of which the case where \(\text{Re} a_{2k_0} > 0, \text{Re} a_{2k_0-1} > 0 \text{ and } a_{2k_0} \neq a_{2k_0-1}\) will be done here. The other cases are similar and simpler.

\[
dg_{k_0}^*(u) \frac{d}{du} = \begin{cases} a_{2k_0-1} a_{2k_0} \exp (a_{2k_0-1} u) - a_{2k_0} \exp (a_{2k_0} u) & u < 0 \\
0 & u > 0 \end{cases}
\]

Let us recall from [8, p. 203] that if \(f'(t)\) is continuous and \(f(t)\) is complex valued, then

\[
\frac{f(a) - f(b)}{a - b} = \lambda f'(t_1) + (1 - \lambda) f'(t_2) \quad t_1, t_2 \in (a, b) \quad 0 < \lambda < 1
\]

from which it is obvious that
where \( \zeta_i = \alpha_i \zeta_1 + (1 - \alpha_i) \zeta_2, 0 \leq \alpha_i \leq 1 \) and \( i = 3, 4 \). Substituting \( f(\zeta) = \zeta e^\zeta, f'(\zeta) = e^\zeta + \zeta we^\zeta \), we obtain the following estimate for \( (d/du)g^*(u) \) when \( u < 0 \):

\[
\left| \frac{dg^*(u)}{du} \right| \leq |a_{2k_0 - 1}, a_{2k_0}| (\exp (R_{k_0} u) + \max (|a_{2k_0 - 1}|, |a_{2k_0}|) |u| \exp (R_{k_0} u))
\]

where \( R_{k_0} = \min (\Re a_{2k_0 - 1}, \Re a_{2k_0}) \). Therefore we obtain

\[
|G_{2m}'(t)| = |a_{2k_0 - 1}, a_{2k_0}| \int_{-\infty}^{0} \exp (R_{k_0} u)\{1 + |u| \max (|a_{2k_0 - 1}|, |a_{2k_0}|) \} \cdot S_{2m + 2}^*-\frac{1}{2} \exp (-AS_{2m + 2}^{*-\frac{1}{2}} |t - u + \Re (a_{2k_0 - 1} + a_{2k_0}^{-1})|) du.
\]

Using relations among \( S_{2m + 2}^*, S_{2m}, a_{2k_0} \) and \( a_{2k_0 - 1} \) one obtains

\[
\exp (-AS_{2m + 2}^{*-\frac{1}{2}} |t - u + \Re (a_{2k_0 - 1} + a_{2k_0}^{-1})|) \leq M_2 \exp (-AS_{2m + 2}^{*-\frac{1}{2}} |t - u|).
\]

Using this and the definition of \( R_1 \) and \( R_2 \) one derives

\[
|G_{2m}'(t)| \leq M_2 R_1^2 S_{2m + 1}^{-1/2} \int_{-\infty}^{0} \exp (R_{2k_0} S_{2m + 2}^{-1} u)\{1 + |u| R_1 S_{2m + 2}^{*-\frac{1}{2}} S_{2m + 2}^{-\frac{1}{2}} \cdot \exp (-AS_{2m + 2}^{*-\frac{1}{2}} |t - u|) du.
\]

We have to distinguish two cases \( t < 0 \) and \( t \geq 0 \). Let us prove first the theorem in case \( t < 0 \):

\[
|G_{2m}'(t)| \leq M_2 R_1^2 S_{2m + 1}^{-1/2} \exp (-AS_{2m + 2}^{*-\frac{1}{2}} t) \cdot \int_{-\infty}^{0} \{1 - uR_1 S_{2m + 2}^{*-\frac{1}{2}} S_{2m + 2}^{-\frac{1}{2}} \} \cdot \exp (\{R_{2k_0} S_{2m + 2}^{-1/2} + AS_{2m + 2}^{*-\frac{1}{2}} u\}) du + M_2 R_1^2 S_{2m + 1}^{-1} \exp (AS_{2m + 2}^{*-\frac{1}{2}}) t \cdot \int_{t}^{0} \{1 - uR_1 S_{2m + 2}^{*-\frac{1}{2}} S_{2m + 2}^{-\frac{1}{2}} \} \cdot \exp (\{R_{2k_0} S_{2m + 2}^{*-\frac{1}{2}} - AS_{2m + 2}^{*-\frac{1}{2}} u\}) du.
\]

Choosing \( K \) so that \( AS_{2m + 2}^{*-\frac{1}{2}} > 2R_2 S_{2m + 2}^{-1/2} \) we have

\[
|G_{2m}'(t)| \leq M_1 S_{2m + 1}^{-1} \exp (R_{2k_0} S_{2m}^{*-\frac{1}{2}}) .
\]

For \( t > 0 \)

\[
|G_{2m}'(t)| \leq M_2 R_1^2 S_{2m + 1}^{-1/2} \exp (\{R_{2k_0} S_{2m + 2}^{*-\frac{1}{2}} + AS_{2m + 2}^{*-\frac{1}{2}} u\}) \cdot \int_{-\infty}^{0} \{1 - uR_1 S_{2m + 2}^{*-\frac{1}{2}} S_{2m + 2}^{-\frac{1}{2}} \} \cdot \exp (\{R_{2k_0} S_{2m + 2}^{*-\frac{1}{2}} + AS_{2m + 2}^{*-\frac{1}{2}} u\}) du \leq M_1 S_{2m + 2}^{-1} \exp (-AS_{2m + 2}^{*-\frac{1}{2}}) \leq M_1 S_{2m + 2}^{-1} \exp (-R_{2k_0} S_{2m + 2}^{*-\frac{1}{2}}) .
\]

Estimations similar to those achieved in Theorem 4.2 for \( \{a_k\} \in B(2, \delta) \) and \( \{a_k\} \in C(2) \) are developed in the following theorems.
THEOREM 4.6. Let \( \{a_k\} \in B(2, \delta) \) and \( Q_m^{(j)} \geq L(j)Q_m \) for some \( j \), then there exist \( A > 0 \) and \( M > 0 \) (independent of \( m \)) so that for \( k \leq 2j \):

\[
|G_{2m}^{(k)}(t)| \leq MQ_m^{-k/4} \exp \left( -A \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{-1/(1+\delta)} |t|.
\]

THEOREM 4.7. Let \( \{a_k\} \in C(2) \) and \( Q_m^{(j)} \geq L(j)Q_m \) for some \( j \), then there exist \( A > 0 \) and \( M > 0 \) (independent of \( m \)) so that for \( k \leq 2j \):

\[
G_{2m}^{(k)}(t) \leq MQ_m^{-k/4} \exp \left( -AS_m^{-1/2} |t| \right).
\]

One can note that in case \( k = 0 \) no condition of the form \( Q_m^{(j)} \geq L(j)Q_m \) is needed.

Proof of Theorems 4.6 and 4.7. Using Theorems 3.5 and 3.6 (for Theorems 4.6 and 4.7 respectively) we obtain by Theorem 4.1

\[
|G_{2m}^{(k)}(t)| \leq \left| e^{-\sigma t} \int_{\sigma+i\infty}^{\sigma-i\infty} \frac{e^{\sigma t}}{E_{2m}(\sigma + iy)} dy \right|, \quad \beta(2m) < \sigma < \alpha(2m).
\]

Using the fact that \( Q_m^{-1/4} < ((1/2)S_m^{-1/2})^{-1/2} \), as \( S_m > Q_m \) (which is achieved by dropping many positive terms) and recalling that

\[
S_m^{-1/2} < \left( \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{1/(1+\delta)},
\]

we obtain

\[
Q_m^{-1/4} < \left( \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{1/(1+\delta)}.
\]

The completion of the proof is similar to the proof of Theorem 4.2.

5. Some inversion theorems. In this section we shall show that inversion formulae can be given for \( \{a_k\} \in A(2) \), \( \{a_k\} \in B(2, \delta) \) and \( \{a_k\} \in C(2) \).

THEOREM 5.1. Suppose:

(1) \( G(t) \) and \( E(s) \) are defined by (1.2) and \( \{a_k\} \in A(2) \).

(2) \( f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt \).

(3) For some \( M \) and \( K, |\varphi(t)| \leq Ke^{M|t|} \), where \( M < \min |\Re a_n| \).

(4) \( b_{2m} = o(1) \quad m \uparrow \infty \).

Then

\[
\lim_{m \to \infty} P_{2m}(D)f(x) = \lim_{m \to \infty} \exp \left( (b - b_{2m})D \right) \prod_{k=1}^{m} \left( 1 - \frac{D}{a_{2k-1}} \right) \left( 1 - \frac{D}{a_{2k}} \right).
\]
\[ \exp((\Re a_{2k-1} + a_{2k}^2)D)f(x) = \varphi(x) \text{ at any point of continuity of } \varphi(t). \]

**Proof.** By steps following those of [1; p. 433]

\[
|P_{2m}(D)f(x) - \varphi(x)| \\
\leq \sup_{|t| < \delta} |\varphi(x - t) - \varphi(x)| \int_{-\infty}^{\infty} |G_{2m}(t)| \, dt + M_0 \int_{|t| > \delta} |G_{2m}(t)| \, e^{\kappa|t|} \, dt.
\]

Using Theorem 4.3, the conditions of which are satisfied by the kernel \( G_{2m}(t + b_m) \), choosing \( m \) so big that \( |b_m| < \delta/2 \) and \( AS^{-1/2}_{2m} > 4M \), we conclude the proof of the theorem.

**Theorem 5.2.** Suppose: Assumptions (1) and (2) of Theorem 5.1 are satisfied

(3) For \( \alpha(t) = \int_0^t \varphi(u) du \) there exist positive \( M \) and \( K \) so that

\[
|\alpha(t)| \leq ke^{\beta|t|} \quad \text{where } M < \min |\Re a_k|.
\]

(4) \( b_{2m} = o(S^{(2)}_{2m}) \to \infty \).

(5) \( \int_0^h [\varphi(x + y) - \varphi(x)] dy = o(h) \quad h \to 0 \).

(6) Either \( S^{(2)}_{2m} \geq L(2)S_{2m} \) or \( 0 < \theta_1 < |a_{2k-1}/a_{2k}| < \theta_2 < \infty \) and \( |\Re a_k|/|a_k| > \eta \).

Then \( \lim_{m \to \infty} P_{2m}(D)f(x) = \varphi(x) \).

**Proof.** Integrating by parts and since \( \int_{-\infty}^{\infty} G_{2m}(t) dt = 1 \) we obtain

\[
|P_{2m}(D)f(x) - \varphi(x)| \\
\geq \int_{|x - t| \leq \delta} |\beta(t)| \cdot |G_{2m}'(x - t)| \, dt + \int_{|x - t| \geq \delta} |G_{2m}'(x - t)| \cdot |\beta(t)| \, dt,
\]

where \( \beta(t) = \int_x^t [\varphi(x + u) - \varphi(x)] du \) and therefore \( \beta(x + t) = o(t) \to 0 \) and \( |\beta(t)| \leq K e^{\beta|t|} \).

To obtain the inversion result for the case \( S^{(2)}_{2m} \geq L(2)S_{2m} \) we use the estimation from Theorem 4.2; while for \( |\Re a_k/a_k| > \eta, 0 < \theta_1 < |a_{2k-1}/a_{2k}| < \theta_2 < \infty \) we use Theorem 4.4, both are applicable to \( G_{2m}(t + b_{2m}) \).

**Remark 1.** In case \( a_{2k-1} = -a_{2k} \) (from some \( k \) onward) we can drop (5) and write instead

\[
\int_0^h [\varphi(x \pm y) - \varphi(x \pm 0)] dy = o(h) \quad h \to 0+
\]

(if the numbers \( \varphi(x \pm 0) \) exist) and then if we write \( b_{2m} = 0 \) instead of (4) and drop (6), we shall obtain
The proof is similar if we remember that \( G_{2m}(t) = G_{2m}(-t) \) and therefore \( \int_{-\infty}^{0} G_{2m}(t) dt = \frac{1}{2} \).

**Remark 2.** The condition (3) of Theorem 5.2 seems too strong since for the case where \( a_k \) are real the assumption could be dropped. We hope that at least for some classes of \( \{a_k\} \) Theorem 5.2 could be proved without (3).

**Theorem 5.3.** Suppose: (1) \( G(t) \) and \( E(s) \) are defined by (1.2) and \( \{a_k\} \in \mathcal{B}(2, \delta) \).

(2) \( f(x) = \int_{-\infty}^{\infty} G(x - t)\varphi(t)dt. \)

(3) For some \( M \) and \( K \) \( |\varphi(t)| \leq Ke^{M|t|} \) where \( M = \min |\Re a_k| \).

(4) \( \{(\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta})^{1/(1+\delta)}\}^\beta \leq KQ_m^{1/4} \) for some \( \beta \geq 1. \)

(5) \( \varphi(x) - \varphi(t) = o(\|t - x\|^{\beta-1}) \) \( t \to x. \)

Then

\[
\lim_{m \to \infty} P_{2m}(D)f(x) = \varphi(x). \]

**Proof.** We have

\[
|P_{2m}(D)f(x) - \varphi(x)| = \left| \int_{|x-t| \leq \delta} G_{2m}(x - t)[\varphi(t) - \varphi(x)]dx \right|
\leq K_1 \int_{|t| \geq \delta} |G_{2m}(x - t)| e^{M|t|} d\varepsilon + \varepsilon \int_{-\infty}^{\infty} |G_{2m}(t)| |t|^{\beta-1} dt
\leq o(1) + \varepsilon K_2 \int_{-\infty}^{\infty} |Q_m^{-1/4} |t|^{\beta-1} \exp \left(-A \left( \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{-1/(1+\delta)} |t| \right) dt
\leq o(1) + \varepsilon K_2 K \int_{-\infty}^{\infty} |u|^{\beta-1} \exp (-Au) du \leq o(1) + \varepsilon K_2 K
\]

**Theorem 5.4.** Suppose: (1) \( G(t) \) and \( E(s) \) are defined by (1.2) and \( \{a_k\} \in \mathcal{C}(2) \).

(2) \( f(x) = \int_{-\infty}^{\infty} G(x - t)\varphi(t)dt. \)

(3) For some \( K \) and \( M \) \( |\varphi(X)| \leq Ke^{M|t|} \) where \( M = \min |\Re a_k| \).

(4) \( S_{2m}^{\beta/2} \leq KQ_m^{1/4} \) for some \( \beta \geq 1. \)

(5) \( \varphi(x) - \varphi(t) = o(\|t - x\|^{\beta-1}) \) \( t \to x. \)

Then
(5.3) \[ \lim_{m \to \infty} P_{2m}(D)f(x) = \varphi(x). \]

**Proof.** Similar to that of Theorem 5.3.

**Remark.** When \( \beta \) of condition (4) of Theorem 5.4 and Theorem 5.3 is equal to one, the condition on \( \varphi(t) \) is mere continuity at a point \( t = x \).

**Lemma 5.5.** If an integer \( r \) exists such that for all \( n \), \( |a_{n+r}| > q|a_n| \) for \( q > 1 \), then

\[
\left( \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{1/1+\delta} \leq KQ_{m}^{1/4} \quad 0 \leq \delta \leq 1
\]

(for \( \delta = 1 \) we have \( S_{2m}^{1/2} \leq KQ_{m}^{1/4} \)).

**Proof.** Obvious.

**Corollary 5.5.** If the kernel is defined by \( a_{2k} = 2^{k-1}(1 + i) \) and \( a_{2k-1} = -2^{k-1}(1 + i) \) then (5.3) is valid at any point of continuity.

6. Examples, remarks and generalizations. In this section we shall show some examples of convolution transform by giving its related sequence \( \{a_k\} \). When we say \( G(t) \in A(2), B(2, \delta) \) or \( C(2) \) we mean that there is an order for which \( \{a_k\} \in A(2), B(2, \delta) \) and \( C(2) \) respectively.

**Example 6.1.** \( \{a_k\} \) defined by \( a_{2k-1} = k, a_{2k} = q^k e^{i\theta_k} \) for \( q > 1, 0 < \delta < \pi(1/2), |\theta_k - (\pi/2)| > \delta, |\theta_k + (\pi/2)| > \delta \). \( \{a_k\} \in A(2) \). The kernel \( G(t) \) related to \( \{a_k\} \) is not necessarily one of those discussed in [6]; for instance in case \( \theta_k = (2/5)\pi \) the result of Theorem 5.2 can be applied as \( S_{2m}^{(j)} \geq L(j)S_{2m} \) for all \( j \) (\( j = 2 \) is needed).

**Example 6.2.** \( G(t) \) defined by \( a_{2k-1} = (2k-1)!a_{2k} = (2k)!e^{i\theta_k} \) where \( -\pi < \theta_k < \pi, |\theta_k - (\pi/2)| > \delta, |\theta_k + (\pi/2)| > \delta \) for some \( 0 < \delta < \pi/2 \) where the \( a_k \)'s are arranged in the order of \( |a_k| \). Theorem 5.2 does not apply here as one can easily verify that \( S_{2m}^{(j)} = o(S_{2m}) \) \( m \to \infty \) for all \( j \). We can apply Theorem 5.1 and get an inversion formula.

**Example 6.3.** Let \( c_k \) be real, \( \sum c_k^{-2} < \infty \) and

\[
a_{2k-1} = c_k(\sin \theta_1)^{-1}e^{i\theta_2}, \quad a_{2k} = c_k(\sin \theta_2)^{-1}e^{-i\theta_1}
\]

where \( 0 < \theta_1, \theta_2 < \pi/2, 0 < \delta < \theta_1 + \theta_2 < \pi/2 - \delta \). (1.5) is easily veri-
fied. (1.6) is valid also since \( \sin^2 \theta_1 + \sin^2 \theta_2 - 4 \sin^2 \theta_1 \sin^2 \theta_2 \geq \eta \) and \( \cos^2 \theta_1 > \sin^2 \theta_2 \) and therefore \( \sin^2 \theta_1 + \sin^2 \theta_2 < 1 \) implies

\[
\left(1 - \frac{\eta}{2}\right)(\sin^2 \theta_1 + \sin^2 \theta_2) - 4 \sin^2 \theta_1 \sin^2 \theta_2 \geq \frac{\eta}{2}.
\]

Using \( \sin^2 \theta_1 < \cos^2 \theta_2 \) and \( \sin^2 \theta_2 < \cos^2 \theta_1 \) we get after some calculations that \( \sin^2(\theta_1 + \theta_2) \sin^2(\theta_1 - \theta_2) < \sin^2 \theta_1 + \sin^2 \theta_2 - 4 \sin^2 \theta_1 \sin^2 \theta_2 \) which implies (1.7). It should be noted that the class defined by \( a_{2k-1} = a_{2k} \) and \( \min(|\arg a_{2k}|, |\arg - a_{2k}|) \leq \pi/4 - \delta \) which includes Garder's class of transforms [5] as a very special case, is a special case of this example. Theorem 5.2 can be applied here.

**Example 6.4.** Let \( c_k \) be real, \( \sum c_k^2 < \infty \) and \( a_{2k-1} = c_k(\sin \theta_j)^{-1}e^{i\theta_j} \), \( a_{2k} = -c_k(\sin \theta_j)^{-1}e^{i\theta_j} \) where either \( 0 < \theta_1 < \pi/2, 0 < \delta, \theta_1 + \theta_2 < \pi/2 - \delta \) or \( -(\pi/2) < \theta_1, \theta_2 < 0, -(\pi/2) + \delta_2 < \theta_1 + \theta_2 < \delta_1 < 0 \).

The inequalities used in Example 6.3 for the validity of \( \{a_k\} \in A(2) \) can also be used here. It should be noted that the class of transforms defined by Dauns and Widder [1] is the case \( \theta_1 = \theta_2 \) here.

**Example 6.5.** Let \( a_{2n-1} = n^\eta(1 + i), a_{2n} = n^\eta(1 - i), \gamma > 1/2 \). In this case \( \{a_k\} \in A(2) \) (since (1.6) is not satisfied) but clearly \( \{a_k\} \in C(2) \).

In this case \( \beta \) of Theorem 5.4 is easily computed as \( S_{2m} = (1 + o(1))4^\gamma m^{-2\gamma+1}, Q_m = (1 + o(1))4^\gamma m^{-2\gamma-1} m \to \infty \) and therefore

\[
(-\gamma + 1/2)\beta \leq -\gamma + 1/4
\]

that is \( \beta \geq 1 + 1/2(2\gamma - 1) \). From this one can see easily that: (a) When \( \gamma = 1 \) it is enough to have at \( t = x \) \( \varphi(t) - \varphi(x) = o(|t - x|^{1/2}) \) for Theorem 5.4.

(b) When \( \gamma > 3/4 \) it is enough to have \( \varphi(t) - \varphi(x) = 0(t - x) \) \( t \to x \) or it is enough for \( \varphi(t) \) to have a left and right derivative at \( t = x \).

**Example 6.6.** \( a_{2n-1} = n^\eta(1 + i), a_{2n} = -n^\eta(1 + i) \). For \( \gamma > 3/4 \) \( \{a_k\} \in B(2, 1/3) \). The following remarks will constitute generalizations of the Theorems of § 5 in various directions.

**Remark 6.1.** In Theorem 5.1 \( |\varphi(t)| \leq Ke^{\pi|t|} \) can be replaced by

\[
\left|\int_0^t \varphi(t)dt\right| \leq Ke^{\pi|t|} \text{ if for every } \delta > 0 \text{ if }
\]

(6.1) \( (S_{2m}S_{2m}^{(1)}S_{2m}^{(2)})^{-1/2} \exp(-\delta S_{m}^{-1/2}) = o(1) \quad m \to \infty \).

This result can be achieved by a proper change of Theorem 4.2 that will yield now
(6.2) \( |G'_{2m}(t)| \leq M(S_{2m}S_{2m}^{(1)}S_{2m}^{(2)})^{-1/2} \exp(-AS_{2m}^{-1/2}|t|). \)

**Remark 6.2.** In Theorems 5.3 and 5.4 condition (3) can be replaced by \( \int_0^t \phi(t)dt \leq Ke^{M|t|} \) if either \( Q_m^{(1)} \geq LQ_m \) or if for all \( \eta > 0 \)

\[
(Q_m^{(1)}Q_m)^{-1/4} \exp\left(-\eta \sum_{m+1}^{\infty} |a_k|_1^{1-\delta}\right)^{-1/(1+\delta)} = o(1) \quad m \to \infty
\]

for Theorem 5.3 and \( (Q_m^{(1)}Q_m)^{-1/4} \exp(-\eta S_{2m}^{-1/2}) = o(1) \) \( m \to \infty \) for Theorem 5.4. For the above generalization slight improvements of Theorems 4.6 and 4.7 are needed in case \( Q_m^{(1)} \geq LQ_m \) is not satisfied.

**Remark 6.3.** If \( S_{2m}^{-1/2} \leq KQ_m^{1/4} \), then in Theorem 5.4 \( \phi(t) - \phi(x) = 0(1) \) \( t \to x \) can be replaced by

\[
\int_x^{x+h} [\phi(t) - \phi(x)]dt = o(h) \quad h \to 0.
\]

**Remark 6.4.** If in Theorem 5.3 (5) is replaced by

\[
\phi(t) - \phi(x+) = o(|t - x|^{\beta-1}) \quad t \to x+
\]

and

\[
\phi(t) - \phi(x-) = o(|t - x|^{\beta-1}) \quad t \to x-,
\]

then

\[
\lim_{m \to \infty} P_{zm}(D)f(x) = \frac{1}{2}[\phi(x+) + \phi(x-)].
\]

**Remark 6.5.** If in Theorem 5.3 we have

\[
\left( \sum_{m+1}^{\infty} |a_k|_1^{1-\delta}\right)^{1/(1+\delta)} \leq KQ_m,
\]

then \( \phi(t) - \phi(x) = o(1) \) \( t \to x \) can be replaced by

\[
\int_x^{x+h} [\phi(x \pm t) - \phi(x \pm 0)dt = o(h) \quad h \to 0+
\]

and then

\[
\lim_{m \to \infty} P_{zm}(D)f(x) = \frac{1}{2}[\phi(x + 0) + \phi(x - 0)].
\]
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